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COFLAT RINGS AND MODULES

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In this paper, coflat modules are defined and it is shown that these modules are naturally dual to flat modules. A ring R is an FC ring in case it is coherent and both of its regular modules ${}_RR$ and R_R are coflat. The structure of these rings is examined with emphasis on the categorical dualities that arise. Finally, with respect to FC rings, categorial equivalence is discussed.

0. Background and notation. Throughout this paper R denotes an associative ring with identity 1. We denote the Jacobson radical of R by $J(R)$ and the right (left) socle of R by $\mathrm{Soc}(R_R)$ ($\mathrm{Soc}({}_RR)$). The pxq matrix ring over R is denoted by $\mathrm{Mat}_{pxq}(R)$. Every right (left) R-module is assumed to be unitary. We denote the endomorphism ring of a right (left) R-module, M_R (${}_RM$) by $\mathrm{End}(M_R)$ ($\mathrm{End}({}_RM)$).

The category of right (left) R-modules is denoted by CM_R (${}_RCM$) and its class of objects by $\mathscr{M}_R$ (${}_R\mathscr{M}$).

A submodule $N \leqq M$ is said to be *essential*, denoted $N \trianglelefteq M$, if $N \cap L \neq 0$ for all $0 \neq L \leqq M$.

A submodule $N \leqq M$ is said to be *superfluous*, denoted by $N \ll M$, if $K + N = M$ implies $K = M$ for all $K \leqq M$. We say

$$f: M \longrightarrow L$$

is a *superfluous homomorphism* if $N = \mathrm{Ker} f \ll M$. In particular, $J(R)$ is the largest superfluous submodule of R. A superfluous epimorphism

$$P \longrightarrow M \longrightarrow 0$$

is a *projective cover* of M if P is projective [2, Chapter 17]. Often we speak of P above as a projective cover of M. Not all modules have projective covers. A ring is *semiperfect* if every finitely generated right (left) module has a projective cover. A ring is *right* (*left*) *perfect* if every right (left) module has a projective cover. In particular, right (left) artinian rings are right (left) perfect. A ring is von Neumann regular in case $a \in aRa$ for each $a \in R$ or equivalently if every finitely generated right (left) ideal is a direct summand.

A set of tuples $\{(M_\alpha, f_\alpha)\}_{\alpha \in A}$, where $f_\alpha: M_\alpha \rightarrow N$, *generates* N *as a set*, if for each $n \in N$ there exists an f_α such that $n \in \mathrm{Im} f_\alpha$.

A module M_R is (*finitely*) *generated* by U_R in case for some (finite) index set A there is an R-epimorphism

$$U^{(A)} = \bigoplus_A U \longrightarrow M \longrightarrow 0\ .$$

If all modules in $\mathscr{M}_R$ are generated by U, then U is called a *generator*. In particular, R is a projective generator in CM_R.

Dually, a set of tuples $\{(N_\alpha, f_\alpha)\}_{\alpha \in A}$, where $f_\alpha: M \to N_\alpha$, *cogenerates* M *as a set*, if for $m_1 \neq m_2 \in M$ there exists an f_α such that $f_\alpha(m_1) \neq f_\alpha(m_2)$.

Clearly, if $\{N_\alpha\}$, M are Z-modules and the $\{f_\alpha\}$ are Z-homomorphisms, then $\{(f_\alpha, N_\alpha)\}$ cogenerates M as a set if for each $0 \neq m \in M$, there exists an f_α such that $m \notin \operatorname{Ker} f_\alpha$.

A module M_R is said to be (*finitely*) *cogenerated* by U_R in case for some (finite) index set A there is an R-monomorphism

$$0 \longrightarrow M \longrightarrow U^A = \pi_A U .$$

If all modules in $\mathscr{M}_R$ are cogenerated by U, then U is called a *cogenerator*.

1. Coflat modules. If M_R is a right R-module, then each pxq matrix $C = [[C_{ij}]]$ over R determines a unique Z-homomorphism $C: M^p \to M^q$ via the usual matrix multiplication $C: m \to mC$ for each $m = (m_1, \cdots, m_p) \in M^p$. In particular, each $a = (a_1, \cdots, a_q) \in R^q$ determines the two Z-homomorphisms $a: M \to M^q$ and $a^t: M^q \to M$ defined by $a: x \to xa = (xa_1, \cdots, xa_q)$ for all $x \in M$ and $a^t: m \to ma^t = \sum_{i=1}^q m_i a_i$ for all $m = (m_1, \cdots, m_q) \in M^q$.

In this notation, a standard nonfunctorial characterization of flatness (see, for example, [2, Lemma 19.19]) can be stated as follows

PROPOSITION 1.1. *A module M_R is flat if and only if for each $q \in N$ and for each $a \in R^q$, the kernel* $\operatorname{Ker} a^t \leqq M^q$ *is generated as a set by* $\{(M^p, C) \mid C \in \operatorname{Mat}_{pxq}(R)$ *such that* $Ca^t = 0,\ p = 1, 2, \cdots\}$.

From this characterization of flatness it is clear how to formulate a natural dual notion.

DEFINITION 1.2. A module M_R is *coflat* in case for each $p \in N$ and for each $a \in R^P$ the cokernel $M^P/\operatorname{Im} a$ is cogenerated as a set by $\{(C, M^q) \mid C \in \operatorname{Mat}_{pxq}(R)$ such that $aC = 0,\ p = 1, 2, \cdots\}$.

Clearly, another way of stating the defining condition of a coflat module M_R is that, for each $a \in R^P$, if $m \in M^P \backslash Ma$, then $mC \neq 0$ for some pxq matrix C such that $aC = 0$. In particular, one can restrict attention to the $px1$ column matrices. So also

PROPOSITION 1.3. *A module M_R is coflat if and only if for each $n \in N$, each $a \in R^n$, and each $m \in M^{(n)}$, if $m \notin Ma$, then there is a $c \in R^n$ with $ac^t = 0$ and $mc^t \neq 0$.*

That this definition of coflat is natural is supported by the following dual characterizations of modules that are flat or coflat over their endomorphism rings. The first of these, for flat modules, was given by [18, Lemma 1.3].

THEOREM 1.4. *Let ${}_SM$ be a left S-module and let* $R = \mathrm{End}({}_SM)$. *Then*

(1) *M_R is flat if and only if ${}_SM$ generates all kernels of homomorphisms*

$$ {}_SM^n \longrightarrow {}_SM \qquad (n = 1, 2, \cdots) , $$

(2) *M_R is coflat if and only if ${}_SM$ cogenerates all cokernels of homomorphisms*

$$ {}_SM \longrightarrow {}_SM^n \qquad (n = 1, 2, \cdots) . $$

Proof. For (1), see [18]. We will do (2). Clearly, $\mathrm{Hom}_S(M^p, M^q)$ can be identified with $\mathrm{Mat}_{p\times q}(R)$. So M_R is coflat if and only if for each $p \in N$ and each $a \in R^p$, the image $Ma \leqq M^p$ is cogenerated by those $c \in R^p$ with $ac^t = 0$, if and only if for each $p \in N$ and each $a\colon M \to M^p$, the cokernel M^p/Ma is cogenerated by those $c^t\colon M^p \to M$ with $ac^t = 0$.

DEFINITION 1.5. A module M_R satisfies the *$\aleph$-Baer criterion* in case for every finitely generated right ideal I of R and every R-homomorphism

$$ f\colon I \longrightarrow M $$

there exists an $m \in M$ with $f(x) = mx (x \in I)$.

The $\aleph$-Baer criterion provides a characterization of coflat modules dual to the characterization of flat modules as factors of projective modules by pure submodules (see [2, Lemma 19.18]).

PROPOSITION 1.6. *A module M_R is coflat if and only if it satisfies the $\aleph$-Baer criterion.*

Proof. ($\Rightarrow$) Suppose M_R is coflat. Let $I = a_1R + \cdots + a_nR$ be a finitely generated right ideal of R and let $f\colon I \to M$ be an R-homomorphism. Set $a = (a_1, \cdots, a_n) \in R^n$ and $f(a) = (f(a_1), \cdots, f(a_n)) \in M^n$.

It will suffice to prove that $f(a) = ma$ for some $m \in M$. But if $f(a) \notin Ma$, then by 1.3 there is a $c \in R^n$ with $ac^t = 0$ and $f(ac^t) = f(a)c^t \neq 0$, a contradiction.

($\Leftarrow$) Assume that M_R satisfies the $\aleph$-Baer criterion but that it is not coflat. Then by 1.3 there exists $a \in R^n$ and $m \in M^n$ such that

$m \notin Ma$ and for all $c \in R^n$

$$ac^t = 0 \Rightarrow mc^t = 0 .$$

Then, if $I = a_1R + \cdots + a_nR$, there is an R-homomorphism $f: I \to R$ with $f(a_i) = m_i$ $(i = 1, \cdots, n)$. Therefore, by the $\aleph$-Baer criterion, there exists an $m' \in M$ such that $f(a_i) = m' \cdot a_i$ $(i = 1, \cdots, n)$ so that $m = m'a \in Ma$, a contradiction.

Since the $\aleph$-Baer criterion is simply the restriction of the general Baer criterion (i.e., The Injective Test Lemma (see [2, Lemma 18.3])) to finitely generated right ideals, Colby [4] has called modules that satisfy this criterion *$\aleph$-injective.* Since every injective module satisfies the full Baer criterion, we have

COROLLARY 1.7. *Every injective module is coflat.*

Concerning the behavior of products, coproducts and direct unions of coflat modules we have

PROPOSITION 1.8. *Let $(M_\alpha)_{\alpha \in A}$ be an indexed set of right R-modules. Then,*

(1) *$\prod_A M_\alpha$ is coflat if and only if each M_α is coflat.*

(2) *$\bigoplus_A M_\alpha$ is coflat if and only if each M_α is coflat.*

Furthermore, let $\{M_\alpha\}$ be a directed set of R-submodules of M such that $\sum M_\alpha = M$, then if each M_α is coflat, M is coflat.

Proof. 1. $(\Rightarrow)$ Let I be a finitely generated right ideal. Suppose $f_\alpha: I \to M_\alpha$ is an R-homomorphism and $i_\alpha: M_\alpha \to \prod_A M_\alpha$ is the inclusion homomorphism. Then $i_\alpha f_\alpha: I \to \prod_A M$. Since $\prod_A M_\alpha$ is coflat, there exists $(m_\alpha)_{\alpha \in A} \in \prod_A M_\alpha$ such that $i_\alpha f_\alpha: a \to (m_\alpha)_{\alpha \in A} \cdot a (a \in I)$. Let $\prod_\alpha$: $\prod_A M_\alpha \to M_\alpha$ be the projection homomorphism. Then $\prod_\alpha i_\alpha f_\alpha = f_\alpha: a \to \prod_\alpha ((m_\alpha)_{\alpha \in A}) \cdot a$.

$(\Leftarrow)$ Let $f: I \to \prod_A M_\alpha$ be an R-homomorphism. Consider $\prod_\alpha f: I \to M_\alpha$. Since each M_α is coflat, there exists m_α for each $\alpha \in A$ such that $\prod_\alpha f: a \to m_\alpha \cdot a$. Thus $f: a \to (m_\alpha)_{\alpha \in A} \cdot a$.

2. $(\Leftrightarrow)$ This proof parallels 1 completely and thus will be omitted.

Finally, let $(M_\alpha)_{\alpha \in A}$ be an indexed set of coflat submodules of M over A, a directed set such that $M = \sum_{\alpha \in A} M_\alpha$. Let $i_\alpha: M_\alpha \to \sum M_\alpha$ be the inclusion homomorphisms. Let $F: I \to \sum M_\alpha$ be an R-homomorphism with I, a finitely generated right ideal. Since the i_α are monomorphisms and I is finitely generated, there exists a σ such that f factors through M_σ, i.e., there exists an R-homomorphism $\bar{f}: I \to M_\sigma$ such that $i_\sigma \bar{f} = f$. Since M_σ is coflat, we are done.

COROLLARY 1.9. *Every direct union and direct sum of injective* *modules is coflat.*

In a right noetherian ring every right ideal is finitely generated. Thus the $\aleph$-Baer criterion becomes the general Baer criterion in this class of rings. In particular, every right coflat module is right injective. Conversely, if one knows every right coflat module is right injective, then by 1.9 and a theorem of Bass [2, Thm. 25.3] R is right noetherian. Summarizing we have

COROLLARY 1.10. *A ring is right noetherian if and only if every right coflat module is right injective.*

Semisimple rings can be characterized as rings over which every module is projective or, equivalently, as rings over which every module is injective. An analogous characterization holds for rings in which every module is flat or for rings over which every module is coflat.

PROPOSITION 1.11. *For a ring R, the following are equivalent:*
(a) *R is von Neumann regular.*
(b) *Every left (right) R-module is flat.*
(c) *Every left (right) R-module is coflat.*

Proof. See [5] and [6].

EXAMPLE 1.12. Let V_F be an infinite dimensional F-vector space and let $I = \{f \in \mathrm{End}(V_F) \mid \dim \mathrm{Im} f < \infty\}$. If R is the subring of $\mathrm{End}(V_F)$ generated by I, then it is easy to see that R is von Neumann regular, and ${}_R R/I$ is a simple noninjective R-module. Thus, ${}_R R/I$ is coflat but not injective.

A class of modules included in the class of coflat modules and called *FP* (for finitely presented) *injective* or *absolutely* pure has received some attention. [9, 11, 15]. Recall that a finitely generated module M_R is *finitely presented* in case every exact sequence

$$0 \longrightarrow K \longrightarrow F \longrightarrow M \longrightarrow 0$$

with F finitely generated and free, the kernel K is also finitely generated.

DEFINITION 1.13. A module M_R is *FP injective* or *absolutely pure* [15] in case for every exact sequence

$$0 \longrightarrow K_R \longrightarrow L_R \longrightarrow N_R \longrightarrow 0$$

such that N_R is finitely presented, the sequence

$$0 \longrightarrow \mathrm{Hom}_R(N_R, M_R) \longrightarrow \mathrm{Hom}_R(L_R, M_R) \longrightarrow \mathrm{Hom}_R(K_R, M_R) \longrightarrow 0$$

is exact.

Clearly, every injective module is FP injective. Conversely, however, the above example shows that FP injective modules need not be injective. However, we have

PROPOSITION 1.14. *Every right FP injective module is right coflat.*

Proof. Let I_R be a finitely generated right ideal and let M_R be right FP injective. We have the sequence

$$0 \longrightarrow I \longrightarrow R \longrightarrow R/I \longrightarrow 0 .$$

Then

$$0 \longrightarrow \mathrm{Hom}_R(R/I, M) \longrightarrow \mathrm{Hom}_R(R, M) \longrightarrow \mathrm{Hom}_R(I, M) \longrightarrow 0$$

is an exact sequence.

It is not known whether, conversely, every right coflat module is right FP injective. However, for at least one important class of rings, it is. Recall that a ring R is *right coherent* in case every finitely generated right ideal is finitely presented. In particular, right noetherian rings are right coherent.

THEOREM 1.15. *If R is a right coherent ring, then a right R-module is coflat if and only if it is FP injective.*

Proof. Stenström [16].

A useful characterization of FP injective modules is the following, given by Megibben [11].

PROPOSITION 1.16. (Megibben [11, Thm. 1]). *A module M_R is FP injective if and only if for each $p, q \in N$, each $m \in M^q$, and each $C \in \mathrm{Mat}_{pxq}(R)$, if for all $a \in R^q$*

$$Ca^t = 0 \Longrightarrow ma^t = 0$$

then there is an $\bar{m} \in M^p$ such that $m = \bar{m}C$.

It is easy to check that the characterizing condition in Megibben's result is, when $p = 1$, simply a rephrasing of the characterizing condition of coflat modules in 1.3.

Every injective submodule of a projective module is necessarily projective. We do not know whether the coflat-flat analogue holds in general. However,

COROLLARY 1.17. *Every FP injective submodule of a flat module* *is flat.*

Proof. Let M be an FP injective submodule of a flat module F. Let $a \in R^q$, $m \in M^q$ with $ma^t = 0$. Then by 1.1 there exists a $p \in N$ and $C \in \mathrm{Mat}_{p \times q}(R)$ with

$$m = xC \in F^p C .$$

So for each $b \in R^q$

$$Cb^t = 0 \Longrightarrow mb^t = xCb^t = 0 .$$

Thus by 1.16, since M is FP injective, $m \in M^p C$, so by 1.1 M is flat.

Recall that a ring R is right semihereditary if every finitely generated ideal is projective. It is clear that every regular ring is right semihereditary. We conclude this section with the following characterization of regular rings:

THEOREM 1.18. *A ring R is von Neumann regular if and only if R is right semihereditary and R_R is coflat.*

Proof. Suppose R is right semihereditary and R_R is coflat. Let I_R be a finitely generated right ideal. Then I_R is projective and thus coflat by 1.8. Let $id: I_R \to I_R$ be the identity homomorphism. This extends to an R-homomorphism $e: R \to I$. Thus I is a right direct summand.

2. FC rings—equivalence and duality.

We begin by recalling that for a ring R the following four properties are equivalent:

1° ${}_R R$ is artinian and injective.
2° ${}_R R$ is noetherian and injective.
3° R_R is artinian and injective.
4° R_R is noetherian and injective.

Moreover, we recall that a ring R is *quasi-Frobenius* (abbreviated, QF) in case it satisfies these equivalence conditions. In this section we focus on a class of rings, called FC rings, that generalize the class of QF rings. These rings are a slight variation of a class of rings first introduced by Colby [4], called IF rings.

DEFINITION 2.1. A ring R is *left FC* (*right FC*) in case R is left (right) coherent and ${}_R R$ (R_R) is coflat. A ring R is an *FC ring* in case it is both left and right FC.

Since every QF ring is both left and right noetherian and both left and right self injective, we have

PROPOSITION 2.2. *If R is a QF ring, then R is an FC ring.*

We readily observe that FC rings need not be QF. For indeed, if K is a field, then for any set A, the product K^A is a commutative regular ring that is noetherian if and only if A is finite. Moreover, we shall see (Example 2.8) that left FC rings need not be FC.

In spite of this last fact, the class of FC rings, with their built in left-right symmetry, displays many properties analogous to those of the smaller class of QF rings. Before stating some of them we recall a few things.

If R is a ring and if $X \subseteq R$, then we denote the left and right annihilators of X, respectively, by

$$\mathrm{Ann}_l(X) = \{a \in R \mid ax = 0 \forall x \in X\}$$
$$\mathrm{Ann}_r(X) = \{a \in R \mid xa = 0 \forall x \in X\} .$$

DEFINITION 2.3. Let $\mathscr{L}$ and $\mathscr{R}$ be sets of left and right ideals of R, respectively. Then R has the *double annihilator property* for $\mathscr{L}$ and $\mathscr{R}$ in case

$$I \in \mathscr{L} \Longrightarrow \mathrm{Ann}_r(I) \in \mathscr{R} \quad \text{and} \quad \mathrm{Ann}_l \mathrm{Ann}_r(I) = I$$

and

$$I \in \mathscr{R} \Longrightarrow \mathrm{Ann}_l(I) \in \mathscr{L} \quad \text{and} \quad \mathrm{Ann}_r \mathrm{Ann}_l(I) = I .$$

In this connection we recall that a left noetherian ring R is QF if and only if R has the double annihilator property for the classes of all left and all right ideals.

If R is a ring, then there are two contravariant functors (see [2, Chapter 20])

$$\mathrm{Hom}_R(\_\_, {}_RR): {}_RCM \longrightarrow CM_R$$

and

$$\mathrm{Hom}_R(\_\_, R_R): CM_R \longrightarrow {}_RCM ,$$

called the *R-dual* functors. For each R-module M we denote its R-dual, that is, its image under the appropriate R-dual functor, by M^*, so

$$M^* = \mathrm{Hom}_R(M, R) .$$

Also for each R homomorphism

$$f: M \longrightarrow N$$

we denote its R-dual by f^*, so

$$f^*: N^* \longrightarrow M^* .$$

Now in each case, left or right, there is a natural transformation from the identity functor to the double dual. For example, for each left R-module ${}_RM$,

$$\sigma_M: M \longrightarrow M^{**}$$

is defined via

$$\sigma_M(x)(f) = f(x) .$$

A module M is *R-reflexive* in case σ_M is an isomorphism.

If $\mathscr{A}$ and $\mathscr{B}$ are full subcategories of ${}_RCM$ and CM_R respectively, then $\mathscr{A}$ and $\mathscr{B}$ are *R-dual* in case restricted to $\mathscr{A}$ and $\mathscr{B}$, the R-dual is a functor

$$*: \mathscr{A} \longrightarrow \mathscr{B} \quad \text{and} \quad *: \mathscr{B} \longrightarrow \mathscr{A} ,$$

and each M in $\mathscr{A}$ and each N in $\mathscr{B}$ is R-reflexive.

In this connection, we recall that a ring R is QF if and only if the categories ${}_RFM$ and FM_R of finitely generated left and right modules are R-dual. Consequently, a noetherian ring is QF if and only if it cogenerates every finitely generated module.

These various characterizations of QF rings serve as models for analogous characterizations of FC rings. We give these in

THEOREM 2.4. *For a ring R, the following statements are equivalent*:

(a) *R is FC.*

(b) *Every left (right) coflat module is left (right) flat.*

(c) *R is coherent and cogenerates every finitely presented left (right) R-module.*

(d) *R is coherent and every left (right) flat module is left (right) coflat.*

(e) *The categories of finitely presented left and right R-modules are R-dual.*

(f) *R is coherent and has the double annihilator property for finitely generated left and right ideals.*

(g) *R is left coherent and the classes of left flat and left coflat modules are the same.*

Proof. (a) $\Leftrightarrow$ (b) Colby [4, Theorem 2], 1.15 and 1.17.

(b) $\Leftrightarrow$ (c) Colby [4, Theorem 1], 1.7, 1.15 and 1.17.

(d) $\Rightarrow$ (a) Clear.

(a) $\Rightarrow$ (d) Every flat module is a direct limit of finitely generated projective R-modules [6, Proposition 11.32] which are coflat by 1.8.

Now use Stenström [15, Proposition 4.2].

(a) $\Leftrightarrow$ (e) Stenström [15, Theorem 4.9].

(b) $\Leftrightarrow$ (f) Colby [4, Corollary 1].

(b) $\Leftrightarrow$ (d) $\Rightarrow$ (g) Clear.

(g) $\Rightarrow$ (b) Colby [4, Theorem 1].

Recall that two rings R and S are *(Morita) equivalent*, abbreviated

$$R \approx S$$

in case there is an equivalence between the categories CM_R and CM_S.

LEMMA 2.5. *Let R and S be Morita equivalent rings, via inverse equivalences*

$$F: CM_R \longrightarrow CM_S \quad \textit{and} \quad G: CM_S \longrightarrow CM_R .$$

Then $M \in \mathscr{M}_R$ is coflat if and only if $F(M) \in \mathscr{M}_S$ is coflat.

Proof. An exact sequence

$$0 \longrightarrow K \longrightarrow P \longrightarrow N \longrightarrow 0$$

is a finite presentation of N in CM_S if and only if

$$0 \longrightarrow G(K) \longrightarrow G(P) \longrightarrow G(N) \longrightarrow 0$$

is a finite presentation of $G(N)$ in CM_R. Thus the exactness of

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_R (G(N), M) &\longrightarrow \operatorname{Hom}_R (G(P), M) \\ \longrightarrow \operatorname{Hom}_R (G(K), M) &\longrightarrow 0 \end{aligned}$$

implies the exactness of

$$\begin{aligned} 0 \longrightarrow \operatorname{Hom}_S (N, F(M)) &\longrightarrow \operatorname{Hom}_S (P, F(M)) \\ \longrightarrow \operatorname{Hom}_S (K, F(M)) &\longrightarrow 0 \end{aligned}$$

so if M is coflat, then $F(M)$ is coflat. The converse follows since $GF(M)$ is naturally isomorphic to M.

THEOREM 2.6. *Let R and S be Morita equivalent rings. Then R is FC if and only if S is FC.*

Proof. The ring R is right coherent if and only if every product of flat right R-modules is flat (see [3]). Thus, since flat is a categorical property, so is coherence. So applying 2.5 and 2.4(g) we have that FC is categorical.

Although we have seen that FC rings need not be QF, the following examples are more interesting.

EXAMPLE 2.7. (Colby [4, Example 1]). Let R be the ring with underlying group

$$R = Z \oplus Q/Z$$

and with multiplication

$$(n_1, q_1)\cdot(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1) .$$

Then it is easy to see that R is a commutative coherent ring with Jacobson radical

$$J(R) = \{(n, q) \mid n = 0\} .$$

Moreover, it is clear that every finitely generated ideal of R is principal. Thus, R is FC but $R/J(R)$ is not FC.

EXAMPLE 2.8. (Colby [4]). Let R be an algebra over a field F with basis

$$\{1, e_0, e_1, e_2, \cdots, x_1, x_2, x_3, \cdots\}$$

where for all i, j

$$\begin{aligned} e_i e_j &= \delta_{i,j} e_j \\ x_i e_j &= \delta_{i,j+1} x_i \\ e_i x_j &= \delta_{i,j} x_j \\ x_i \cdot x_j &= 0 . \end{aligned}$$

It is easy to see both that R is left coherent and that every R-homomorphism

$$f\colon {}_R I \longrightarrow {}_R R$$

extends to one over R. Thus, R is left coflat. However, R is not right coflat since the homomorphism

$$x_1 R \longrightarrow e_0 R$$

via

$$x_1 r \longrightarrow e_0 r$$

cannot be extended over R.

Thus R is a left FC ring but not a right FC ring.

Since von Neumann regular rings are coherent and since every module is coflat, we have

PROPOSITION 2.9. *If R is a von Neumann regular ring, then R is an FC ring.*

Stenström [15] has proved that both left perfect, left FC rings and left perfect, left coherent, right FC rings are QF. There are no known counterexamples to the claim that a left perfect, right FC ring is QF. Specifically, we have

THEOREM 2.10. (Stenström [15, Theorem 4.4]). *Let R be left perfect, then the following statements are equivalent*:

(a) *R is left FC.*

(b) *R is left coherent and right FC.*

(c) *R is QF.*

However, for semiperfect rings, FC need not imply QF.

EXAMPLE 2.11. ([7]). Let p be a prime number. Then let R be the ring with the underlying group

$$R = Z_{p^\infty} \oplus \mathrm{End}\,({}_Z Z_{p^\infty})$$

and with multiplication

$$(n_1, q_1)\cdot(n_2, q_2) = (n_1 n_2,\, n_1 q_2 + n_2 q_1)\;.$$

Osofsky has shown that R is a nonartinian, injective cogenerator with linearly ordered ideals. As in 2.7, one can show R is coherent. Thus, R is a semiperfect FC ring.

It is well known that if R is a ring and G a finite group, then the group ring $R[G]$ is QF if and only if R is QF (see [7]). Colby has proved in [4] the analogous result for FC rings and locally finite groups (i.e., groups in which every finite subset generates a finite subgroup).

THEOREM 2.12. (Colby [4, Theorem 3]). *Let R be a ring and G a group. Then the group $R[G]$ is an FC ring if and only if R is an FC ring and G is a locally finite group.*

EXAMPLE 2.13. Let S be the group of all permutations of $N = \{1, 2, \cdots\}$ that fix all but finitely many elements. Then S is locally finite, so $F[S]$ is FC for all fields F.

It is well known that if R is left or right self injective, then $R/J(R)$ is von Neumann regular [7, Theorem 19.27]. On the other hand an FC ring need not be regular modulo its radical. Indeed, we have already seen that for the FC ring R of Example 2.7, we have

$$R/J(R) \cong Z\;.$$

A somewhat more interesting example of this phenomenon is the following

EXAMPLE 2.14. Let F be a finite field and let S be the locally finite group of Example 2.13. Now S has a subgroup of order $p = \text{char}\, F$. Thus (see [10]) the FC ring $R = F[S]$ is not von Neumann regular. Finally, since Formanek has shown [8, Theorem 1] that the nil and Jacobson radicals are the same, $J(R) = 0$ (see [13]). Thus R is a nonregular FC ring with zero radical.

We have already seen in 2.10 that a left artinian, left or right FC ring is QF. Since Hopkins theorem [2, Theorem 15.20] tells us that every left artinian ring is left noetherian, we extend this result by

THEOREM 2.15. *Let R be a left noetherian ring. Then the following statements are equivalent*:

(a) *R is left FC.*

(b) *R and $R/J(R)$ are right FC rings.*

(c) *R is right FC with essential left socle.*

(d) *R is QF.*

Proof. (a) $\Leftrightarrow$ (d) 1.10.

(d) $\Rightarrow$ (b) Clear.

(b) $\Rightarrow$ (c) By Theorem 2.4(c) $R/J(R)$ contains an isomorphic copy of each simple left $R/J(R)$ module. Thus, each simple left $R/J(R)$ module is projective and $R/J(R)$ is semisimple. Extending a result of Faith [7, Lemma 24.19] on QF rings, it is clear that there exists an $n \in N$ such that $J(R)^n = 0$. R is thus left artinian [2, Theorem 15.20].

(c) $\Rightarrow$ (d) By Theorem 2.4(c) every finitely generated left R-module is cogenerated by R. Moreover, Colby has shown that these modules are actually finitely cogenerated by R [4, Theorem 1]. So every left R-module has nonempty socle and R is right perfect. Now 2.10 finishes the proof.

We know of no example of a left noetherian, right FC ring that is not QF.

Recall that the left singular ideal of R is

$$Z_l(R) = \{x \in R \mid \text{Ann}_l\,(x) \trianglelefteq {}_R R\}$$

and the right singular ideal

$$Z_r(R) = \{x \in R \mid \text{Ann}_r\,(x) \trianglelefteq R_R\}\,.$$

In general, these are not equal and are unrelated to $J(R)$.

PROPOSITION 2.16. *If R is an FC ring, then $Z_r(R) = Z_l(R) = J(R)$.*

Proof. Let $x \in Z_r(R)$, then

$$\mathrm{Ann}_r(x) = \mathrm{Ann}_r(Rx)$$

is essential. Since R is cyclic, it will suffice to show if $Rx + {}_RK = R$ with K finitely generated, then $K = R$. So suppose ${}_RK$ is finitely generated and

$$Rx + K = R .$$

Then

$$\mathrm{Ann}_r(Rx) \cap \mathrm{Ann}_r(K) = 0 .$$

Thus $\mathrm{Ann}_r(K) = 0$. By Theorem 2.4(f), this implies $K = R$. Hence $Z_r(R) \leqq J(R)$.

Next suppose $x \in J(R)$. We claim $\mathrm{Ann}_r(Rx) \trianglelefteq R_R$. By Theorem 2.4(f), it will suffice to show $\mathrm{Ann}_r(Rx) \cap \mathrm{Ann}_r(K) \neq 0$ for all nonzero finitely generated ${}_RK \neq R$. So suppose

$$\mathrm{Ann}_r(Rx) \cap \mathrm{Ann}_r(K) = 0$$

for some nonzero finitely generated ${}_RK \leqq R$. Then $Rx + K = R$ implies $K = R$ and hence $\mathrm{Ann}_r(K) = 0$. Symmetry finishes the result.

It is of interest to note that in on FC ring R, $J(R)$ need not equal $N(R)$, the nilpotent radical, though this is true in QF and regular rings (see [13]).

In general, FC rings need not be von Neumann regular (see for Example 2.7). However,

PROPOSITION 2.17. *If R is an FC ring with no nonzero nilpotent elements, then R is von Neumann regular.*

Proof. Let $x \in R$. Then we claim $Rx \cap \mathrm{Ann}_l(xR) = 0$. For suppose $rxx = 0$. Then $xrxxrx = 0$, so since R has no nonzero nilpotent elements, $xrx = 0$. Likewise, $rxrx = 0$ implies $rx = 0$. So $Rx \cap \mathrm{Ann}_l(xR) = 0$. Therefore, by Theorem 2.4(f)

$$\mathrm{Ann}_r(Rx) + xR = R .$$

Let $1 = n + xs$ where $n \in \mathrm{Ann}_r(Rx)$ and $0 \neq s \in R$. Then $x = nx + xsx$. But $nxnx = 0$, so $nx = 0$ and $x = xsx$.

COROLLARY 2.18. *If R is a commutative FC ring with $J(R) = 0$, then R is von Neumann regular.*

It is of interest that this property of commutative FC rings does not carry over to FC rings with a polynomial identity as shown by the following.

EXAMPLE 2.19. Let F be a field of characteristic 2 and let D_{p_∞}, p an odd prime, be the locally finite group given by generators and relations

$$\langle Z_{p^\infty}, y\colon xyx = y,\ y^2 = 1\ \forall x \in Z_{p^\infty}\rangle .$$

By 2.12 $F[D_{p^\infty}]$ is an FC ring. Since D_{p^∞} has an abelian subgroup of finite index, $F[D_{p^\infty}]$ satisfies a polynomial identity (see [13, Theorem 3.9]). Moreover $F[D_{p^\infty}]$ can be easily shown to have $J(F[D_{p^\infty}]) = 0$ since D_{p^∞} has no finite normal subgroups with order divisible by 2 ([13]). However, $F[D_{p^\infty}]$ is not regular by a theorem of Connell ([13, Theorem 1.5]).

In a future paper, we will examine conditions that assure that an FC ring with a polynomial identity and $J(R) = 0$ is von Neumann regular.

Even though an FC ring R with $J(R) = 0$ need not be von Neumann regular (see 2.14), these rings have some regular-like properties.

PROPOSITION 2.20. *If R is an FC ring with $J(R) = 0$, then*

(1) *R has a von Neumann regular, right (left) self injective maximal right (left) ring of quotients.*

(2) *There is a unique largest two sided ideal I that contains no nonzero nilpotent elements. Moreover,*

$$\mathrm{Ann}_l\, \mathrm{Ann}_r\,(I) = \mathrm{Ann}_r\, \mathrm{Ann}_l\,(I) = I .$$

Proof. (1) Follows from 2.16 since $Z_r(R) = Z_l(R) = 0$ (see [10]).

(2) Let $I = \sum_{\alpha \in A} I_\alpha$ where $\{I_\alpha \mid \alpha \in A\}$ is the family of all two sided ideals of R that contain no nonzero nilpotent elements. Clearly,

$$I \leqq \mathrm{Ann}_l\, \mathrm{Ann}_r\,(I)$$

is a two sided ideal. To complete this proof it will suffice to show that $\mathrm{Ann}_r\, \mathrm{Ann}_l\,(I)$ contains no nonzero nilpotent elements. So suppose $0 \neq x \in \mathrm{Ann}_l\, \mathrm{Ann}_r\,(I)$ such that $x^2 = 0$. Now suppose $Rx \cap I_\alpha = 0$ for all $\alpha \in A$. Then $I_\alpha \cdot Rx = 0$ for all $\alpha \in A$, and hence $IRx = 0$. Thus,

$$RxRx \leqq \mathrm{Ann}_l\, \mathrm{Ann}_r\,(I) \cdot \mathrm{Ann}_r\,(I) = 0$$

and $RxRx = 0$. But $J(R) = 0$, so $Rx = 0$ and $x = 0$. Thus a contradiction.

Therefore, there is an $\alpha \in A$ and an $0 \neq a \in R$ such that

$$0 \neq Ra \leqq Rx \cap I_\alpha .$$

So Ra has no nonzero nilpotent elements. We claim $Ra \cap \mathrm{Ann}_l(aR) = 0$. For suppose $saa = 0$. Then $0 = assasa \in Ra$, but $Ra \leqq I_\alpha$, so $asa = 0$. Likewise $0 = sasa \in Ra$, so $sa = 0$.

Now by Theorem 2.4(f)

$$\mathrm{Ann}_r(Ra) + aR = R .$$

Thus $1 = n + ar$ for some $r \in R$, $n \in \mathrm{Ann}_r(Ra)$. So $a = na + ara$. But $0 = nana \in Ra \leqq I_\alpha$ implies $na = 0$. Thus, $a = ara$ and $Ra = Re$ where e is idempotent. So $Re \leqq Rx$. Therefore $e = cs$. Note $xe = xcx$ implies $xexe = xcxxcx = 0$ which implies $0 = xe \in Re$. Finally, $e = e^2 = cxe = 0$, a contradiction.

We conclude with the surprising result that every FC ring is its own classical ring of quotients, denoted by $Q(R)$ (see [10]).

THEOREM 2.21. *If R is an FC ring, then every regular element (i.e., nonzero divisor) is invertible. In particular, $R = Q(R)$, its classical ring of right (left) quotients.*

Proof. Let $0 \neq x$ be a regular element. Then $\mathrm{Ann}_l(xR) = 0$. Thus by Theorem 2.4(f)

$$xR = \mathrm{Ann}_r \mathrm{Ann}_l(xR) = \mathrm{Ann}_r(0) = R .$$

Hence, x is right invertible. Symmetry now finishes the proof.

3. Endomorphism rings of projective modules over FC rings.

Let P_R be a finitely generated projective R-module and let

$$S = \mathrm{End}(P_R) .$$

Set

$$P^* = \mathrm{Hom}_R(P, R) .$$

Then P and P^* are bimodules

$${}_SP_R \quad \text{and} \quad {}_RP^*_S ,$$

and there are natural functors

$$F_P \colon M \longrightarrow P \otimes_R M$$
$$G_P \colon N \longrightarrow P^* \otimes_S N$$

between the categories ${}_RCM$ and ${}_SCM$. Of course, if P_R is a generator, the Morita theorem [2, Theorem 22.1] implies that F_P and G_P are inverse equivalences. When P_R is not a generator, F_P and G_P are not equivalences; still, for many projective modules P, a considerable amount of information is often available about S. (See, for example

[1], [18]). Here, we focus primarily on the cases where R is an FC ring. But first,

PROPOSITION 3.1. *If R is right coherent and if ${}_SP$ is flat, then S is right coherent.*

Proof. Let $I_S \leqq S_S$ be finitely generated, let $n \geqq 1$ and let

$$0 \longrightarrow K \longrightarrow S^{(n)} \longrightarrow I \longrightarrow 0$$

be exact in CM_S. It will suffice to show that K is finitely generated. Since ${}_SP$ is flat,

$$0 \longrightarrow K \otimes {}_SP \longrightarrow P^{(n)} \longrightarrow I_S \otimes P \longrightarrow 0$$

is exact in CM_R. Also, since ${}_SP$ if flat, there is a monomorphism

$$0 \longrightarrow I \otimes {}_SP_R \longrightarrow P_R \cong S \otimes {}_SP_R \ .$$

But since I_S and P_R are finitely generated, $I \otimes {}_SP_R$ is a finitely generated R-module. Thus, since R is right coherent, $I \otimes {}_SP_R$ is finitely presented, and hence $K \otimes {}_SP_R$ is finitely generated.

Since ${}_SP$ is flat by Theorem 1.4, with an appropriate interchange of the rings R and S, there exists an $m \geqq 1$ such that

$$P_R^{(m)} \longrightarrow K \otimes {}_SP_R \longrightarrow 0 \ .$$

Thus,

$$P_R^{(m)} \otimes {}_RP_S^* \longrightarrow K \otimes {}_SP_R \otimes {}_RP_S^* \longrightarrow 0 \ .$$

But $P \otimes {}_RP_S^* \cong S_S$ (see [2, Chapter 20]), hence

$$S^{(m)} \longrightarrow K \longrightarrow 0$$

is exact and K is finitely generated.

THEOREM 3.2. *If R is an FC ring and if*

$${}_SP \quad \textit{and} \quad P_S^*$$

are flat, then S is also an FC ring.

Proof. By Proposition 3.1, S is right coherent. But $P \simeq P^{**}$ [2, Prop. 20.17] and $S = \mathrm{End}\,({}_RP^*)$, so S is also left coherent.

We now claim that

$$S_S = \mathrm{Hom}_R\,({}_SP_R,\, P_R)$$

is coflat. So let I_S be a finitely generated right ideal of S. Then, since ${}_SP$ is flat,

$$0 \longrightarrow I \otimes {}_S P_R \longrightarrow P_R \simeq S \otimes {}_S P_R$$

is exact. By [5] $I \otimes {}_S P_R$ is finitely presented. So by Theorem 1.15, the bottom row of the commuting diagram (see [2, Proposition 20.6])

$$\begin{array}{ccccc}
\mathrm{Hom}_S(S_S, \mathrm{Hom}_R({}_SP_R, P_R)) & \longrightarrow & \mathrm{Hom}_S(I_S, \mathrm{Hom}_R({}_SP_R, P_R)) & \longrightarrow & 0 \\
\wr\| & & \wr\| & & \\
\mathrm{Hom}_R(S \otimes {}_SP_R, P_R) & \longrightarrow & \mathrm{Hom}_R(I \otimes {}_SP_R, P_R) & \longrightarrow & 0
\end{array}$$

is exact. Hence the top row is exact, and so S_S is coflat. By symmetry ${}_SS$ is coflat.

Recall that for a right R-module M_R, its *trace* is the ideal

$$T(M) = \Sigma\{f(M) \mid f \in \mathrm{Hom}_R(M, R)\} .$$

COROLLARY 3.3. *If R is an FC ring and if both*

$$T(P)_R \quad and \quad {}_RT(P) = {}_RT(P^*)$$

are flat, then S is also an FC ring.

Proof. Anderson [1, Theorem 2.2] has proved that if $T(P)_R$ is flat, then ${}_SP$ is flat. Now apply Theorem 3.2.

COROLLARY 3.4. *If R is an FC ring and if both*

$$(R/T(P)_R) \quad and \quad R(R/T(P))$$

are flat, then S is also an FC ring.

Proof. By a theorem of Zimmermann-Huisgen [18, Theorem 2.4] P_R and ${}_RP^*$ are self generators. So by Theorem 1.4, ${}_SP$ and P^*_S are flat. Now apply Theorem 3.2.

COROLLARY 3.5. *If R is an FC ring such that R is self injective and if*

$${}_SP \quad and \quad P^*_S$$

are flat, then S is a self injective FC ring.

Proof. Since P_R is an injective R-module, and since ${}_SP$ is flat, $S_S = \mathrm{Hom}_R({}_SP_R, {}_SP)$ is injective (see [6, Proposition 11.35]). We have, by symmetry that ${}_SS$ is injective. Now apply Theorem 3.2.

Of course, if $e \in R$ is a nonzero idempotent, then eR is a finitely generated projective, and

$$eRe \simeq \mathrm{End}_R(eR) .$$

In [14] Rosenberg and Zelinsky give an example of a quasi-Frobenius, hence FC, ring R and a nonzero idempotent $e \in R$ such that eRe is not QF. Thus, since eRe is artinian, it is not FC.

In [1] Anderson called P_R an *injector* in case for each injective module ${}_RM$,

$$ {}_SP \otimes {}_RM $$

is injective as an S-module. Moreover, he obtained several characterizations of these injectors. We shall add to this list.

DEFINITION 3.6. The finitely generated projective module P_R is an *FP injector* in case for every FP injective module ${}_RM$, the S-module

$$ {}_SP \otimes {}_RM $$

is FP injective.

THEOREM 3.7. *The finitely generated projective module P_R is an injector if and only if it is an FP injector.*

Proof. Anderson proved that P_R is an injector if and only if P_S^* is flat. Let M be an FP injective right R-module and let I be a finitely generated left ideal of S. Consider the commutative diagram ([1])

$$\begin{array}{ccccc}
\mathrm{Hom}_S({}_SS, {}_SP \otimes {}_RM) & \longrightarrow & \mathrm{Hom}_S({}_SI, {}_SP \otimes {}_RM) & \longrightarrow & 0 \\
\Vert\wr & & \Vert\wr & & \\
\mathrm{Hom}_R({}_RP^* \otimes {}_SS, {}_RM) & \longrightarrow & \mathrm{Hom}_R({}_RP^* \otimes {}_SI, {}_RM) & \longrightarrow & 0 .
\end{array}$$

Since every injective is FP injective, the bottom row is exact for all FP injective ${}_RM$ if and only if P_S^* is flat. But the top row is exact if and only if ${}_SP \otimes {}_RM$ is FP injective.

Miller [12] defined a *flatjector* to be a finitely generated projective P_R such that

$$ {}_SP \otimes {}_RM $$

is flat over S for each flat ${}_RM$. Dually,

DEFINITION 3.8. The finitely generated projective P_R is a *coflatjector* if

$$ {}_SP \otimes {}_RM $$

is coflat over S for each coflat ${}_RM$.

THEOREM 3.9. *If R is an FC ring, then the following statements are equivalent*:

(a) P_R *and* ${}_RP^*$ *are coflatjectors.*

(b) P_R *and* ${}_RP^*$ *are flatjectors.*

Moreover, if R is an FC ring and if (a) *or* (b) *holds, then S is an FC ring where*

$$S = \text{End}\,(R_R)\ .$$

Proof. Since, over an FC ring coflats and flats are the same, it will suffice to prove the final assertion.

Assume (a). Since R is coherent, coflatjectors are precisely the FP injectors, and hence, by (3.7), precisely the injectors. But Anderson [1, Theorem 2.1] proved that P_R and ${}_RP^*$ are injectors if and only if ${}_SP$ and P_S^* are flat.

Applying (3.2), we are done.

Assume (b). Miller [12, Theorem 2.3] proved that P_R and ${}_RP^*$ are flatjectors if and only if P_S^* and ${}_SP$ are flat. By 3.2, S is an FC ring.

Note, however, that in general the equivalence of (a) and (b) does not imply that R is an FC ring. Indeed, if R is a coherent commutative ring, then (a) and (b) are equivalent.

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