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NORM ATTAINING OPERATORS ON LEBESGUE SPACES

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Let X and Y be Lebesgue spaces (AL-spaces). Then the norm attaining operators mapping X to Y are dense in the space of all linear bounded operators from X to Y.

For any two real Banach spaces X and Y by $B(X, Y)$ we denote the Banach space of all bounded linear operators from X to Y. In [7] Uhl proved that for any strictly convex Banach space Y the norm attaining operators are (norm) dense in $B(L^1[0, 1], Y)$ if and only if Y has the Radon-Nikodym property. The question of whether the norm attaining operators are dense in $B(L^1[0, 1], L^1[0, 1])$ has remained unsolved (cf. [7], p. 299). Here we answer this question in the affirmative. In fact we prove a slightly more general result.

First we introduce some notations. Let I stand for the unit interval. For any function μ defined on the product algebra in $I \times I$ by $\mu^i (i = 1, 2)$ we denote the corresponding marginal functions defined on the Borel subsets of I:

$$\mu^1(A) = \mu(A \times I) ,$$
$$\mu^2(B) = \mu(I \times B) .$$

The vector lattice of all finite signed Borel measures on $I \times I$ will be denoted by M. Given any two finite positive Borel measures m_1, m_2 on I we write $M(m_1, m_2)$ for the set of all measures μ in M such that $|\mu|^i$ is absolutely continuous with respect to $m_i (i = 1, 2)$ and

$$\frac{d|\mu|^1}{dm_1} \in L^\infty(m_1) .$$

The measures m_1 and m_2 will be fixed throughout the rest of the paper.

Let us recall that $B(L^1(m_1), L^1(m_2))$ is a Banach lattice under its canonical order (see [5], IV Theorem 1.5 (ii)).

The forthcoming theorem establishes an isomorphism between $M(m_1, m_2)$ and $B(L^1(m_1), L^1(m_2))$, and extends a corresponding result of J. R. Brown on doubly stochastic operators ([1], p. 18). As was kindly indicated by the referee, our Theorem 1 is also related to N. J. Kalton's representation of the endomorphisms from L^p to L^p for $0 < p \leqq 1$ ([3], Theorem 3.1).

By $\langle \cdot, \cdot \rangle$ we denote the canonical bilinear form on $L^\infty(m_2)^* \times L^\infty(m_2)$.

THEOREM 1. *The space* $M(m_1, m_2)$ *is a vector lattice ideal in* M *and to each* $\mu \in M(m_1, m_2)$ *there corresponds a unique operator* $T_\mu \in B(L^1(m_1), L^1(m_2))$ *such that*

$$\langle T_\mu f, h \rangle = \int f(x)h(y)d\mu(x, y)$$

for all $f \in L^1(m_1)$ *and* $h \in L^\infty(m_2)$. *Moreover, the mapping* $\mu \rightarrow T_\mu$ *is a vector lattice isomorphism of* $M(m_1, m_2)$ *onto* $B(L^1(m_1), L^1(m_2))$ *and*

$$\| T_\mu \| = \left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty$$

for every $\mu \in M(m_1, m_2)$.

Proof. First we note that $M(m_1, m_2)$ is a vector subspace of M. Since $\nu \in M(m_1, m_2)$ whenever $0 \leqq \nu \in M$ and $\nu \leqq \mu \in M(m_1, m_2)$, we observe that $M(m_1, m_2)$ is a lattice ideal (and clearly a sublattice) in M. If $\mu \in M(m_1, m_2)$ then it is easy to see that the bilinear form

$$[f, h] = \int f(x)h(y)d\mu(x, y)$$

is well-defined and continuous on $L^1(m_1) \times L^\infty(m_2)$. Therefore there exists a unique operator $T_\mu \in B(L^1(m_1), L^\infty(m_2)^*)$ such that

$$[f, h] = \langle T_1, f, h \rangle$$

(see e.g., [5], IV § 2). Clearly the mapping $\mu \rightarrow T_\mu$ is one-to-one and $\mu \geqq 0$ if and only if T_μ is a positive operator in the Banach lattice sense. Moreover, for an arbitrary $\nu \geqq 0$ in $M(m_1, m_2)$ and for any $h \in L^\infty(m_2)$ we have $\langle T_\nu 1, h \rangle = \int h d\nu^2$, so

$$T_\nu 1 = \frac{d\nu^2}{dm_2} \in L^1(m_2) ,$$

whence $T_\nu f \in L^1(m_2)$ for any $f \in L^\infty(m_1)$. Consequently, $T_\nu \in B(L^1(m_1), L^1(m_2))$ by continuity. Since every $\mu \in M(m_1, m_2)$ is a difference of two positive measures in $M(m_1, m_2)$ and $\mu \rightarrow T_\mu$ is a linear map, we have $T_\mu \in B(L^1(m_1), L^1(m_2))$ for all $\mu \in M(m_1, m_2)$.

We now show that $\mu \rightarrow T_\mu$ is an "onto" mapping. Since $B(L^1(m_1), L^1(m_2))$ is a Banach lattice, it suffices to prove that every positive operator $T \in B(L^1(m_1), L^1(m_2))$ is of the form T_μ. Given any such T we define a set function

$$\mu(A \times B) = \langle T\chi_A, \chi_B \rangle$$

on all Borel rectangles in $I \times I$. Evidently μ extends uniquely to a finitely additive positive measure (denoted also by μ) on the product algebra. The marginal measures $\mu^1(A) = \int_A T^*1dm_1$ and $\mu^2(B) = \int_B T1dm_2$ are finite, positive, and countably additive, so they are compact by the classical result of Ulam. Since μ is a nondirect product of μ^1 and μ^2, it is countably additive by Theorem 1 (i) in [4]. The unique extension of μ to a finite positive (countably additive) Borel measure on $I \times I$ is again denoted by μ. By a standard approximation argument,

$$\int f(x)h(y)d\mu(x, y) = \langle Tf, h \rangle$$

for all $f \in L^1(m_1)$ and $h \in L^\infty(m_2)$. Therefore $T = T_\mu$. Finally, we note that for every $\mu \in M(m_1, m_2)$

$$\begin{aligned} \| T_\mu \| = \| T_{|\mu|} \| &= \sup \| T_{|\mu|} f \|_1 = \sup \langle T_{|\mu|} f, 1 \rangle \\ &= \sup \int f(x) d|\mu|^1(x) = \sup \int f(x) \frac{d|\mu|^1}{dm_1}(x) dm_1(x) \\ &= \left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty , \end{aligned}$$

where the suprema are taken over all nonnegative functions $f \in L^1(m_1)$ with $\|f\|_1 \leqq 1$.

COROLLARY 1. *Let $\nu \in M(m_1, m_2)$. If there exists a function $g \in L^\infty(m_2)$ with $|g| = 1$ such that the Radon-Nikodym derivative of the marginal measure $(g(y)d\nu(x, y))^1$ with respect to m_1 equals*

$$\left\| \frac{d|\nu|^1}{dm_1} \right\|_\infty$$

on a set B of positive m_1 measure, then the operator T_ν attains its norm on the unit ball in $L^1(m_1)$.

Proof. We put $d\lambda(x, y) = g(y)d\nu(x, y)$. Then

$$\begin{aligned} \langle T_\nu(\chi_B/m_1(B)), g \rangle &= \frac{1}{m_1(B)} \int \chi_B(x) d\lambda(x, y) \\ &= \frac{1}{m_1(B)} \int_B \frac{d\lambda^1}{dm_1} dm_1 = \left\| \frac{d|\nu|^1}{dm_1} \right\|_\infty , \end{aligned}$$

implying $\| T_\nu(\chi_B/m_1(B))\|_1 = \| T_\nu \|$ by Theorem 1.

The algebra of sets generated by all dyadic-rational rectangles in $I \times I$ will be denoted by $\mathscr{A}$. The σ-algebra generated by $\mathscr{A}$ coincides with the Borel algebra in $I \times I$.

THEOREM 2. *The norm attaining operators are dense in* $B(L^1(m_1), L^1(m_2))$.

Proof. Let $T \in B(L^1(m_1), L^1(m_2))$. By Theorem 1 we have $T = T_\mu$ for some measure μ in $M(m_1, m_2)$. Without any loss of generality we may assume

$$\left\| \frac{d|\mu|^1}{dm_1} \right\|_\infty = 1 .$$

Given $0 < \varepsilon < 1$, the set

$$D = \left\{ x \in I \colon \frac{d|\mu|^1}{dm_1}(x) > 1 - \frac{\varepsilon}{4} \right\}$$

is of positive m_1 measure, say, $m_1(D) = \delta > 0$. Now let P, $(I \times I) - P$ be the Hahn decomposition for μ with μ^+ concentrated on P (see [2], § 29 Theorem A). Since P is a Borel set, there exists $\tilde{P} \in \mathscr{A}$ such that $|\mu|(P \Delta \tilde{P}) < \delta\varepsilon/4$ ([2], § 13 Theorem D). We define a new measure $\tilde{\mu}$ by

$$d\tilde{\mu} = \chi_{\tilde{P}} d\mu^+ - \chi_{(I \times I) - \tilde{P}} d\mu^- .$$

Evidently $\tilde{P}, (I \times I) - \tilde{P}$ is the Hahn decomposition for $\tilde{\mu}$ and $d|\mu - \tilde{\mu}| = \chi_{P \Delta \tilde{P}} d|\mu|$. Since $|\mu - \tilde{\mu}|(I \times I) < \delta\varepsilon/4$, the Radon-Nikodym derivative of $|\mu - \tilde{\mu}|^1$ with respect to m_1 must be less than $\varepsilon/4$ on some set $C \subset D$ of positive m_1 measure. As $\tilde{P} \in \mathscr{A}$, there exists a natural number n such that $\tilde{P}$ is a union of finitely many squares corresponding to the dyadic partition of I into 2^n subintervals of equal length. Let I_0 be any such open subinterval intersecting C on a set $B = C \cap I_0$ of positive m_1 measure. We let

$$d\nu(x, y) = \chi_B(x) \left(\frac{d|\mu|^1}{dm_1} \right)^{-1}(x) d\tilde{\mu}(x, y) + \chi_{I-B}(x) d\mu(x, y) .$$

Note first that

$$\begin{aligned} d|\nu - \mu| = \Big| \chi_B(x) \left(\frac{d|\mu|^1}{dm_1} \right)^{-1}(x) \, |d(\tilde{\mu} - \mu)(x, y) \\ + \left(1 - \frac{d|\mu|^1}{dm_1}(x) \right) d\mu(x, y) \Big| \leqq 2\chi_C(x) d|\tilde{\mu} - \mu|(x, y) + \frac{\varepsilon}{2} d|\mu|(x, y) . \end{aligned}$$

Therefore

$$\frac{d|\nu - \mu|^1}{dm_1} < 2\frac{\varepsilon}{4} + \frac{\varepsilon}{2} = \varepsilon ,$$

whence $\| T_\nu - T_\mu \| = \| T_{\nu-\mu} \| \leqq \varepsilon$. Moreover,

$$\frac{d|\mu|^1}{dm_1} = 1 \text{ on } B \text{ and } \leqq 1 \text{ elsewhere.}$$

The set $(I_0 \times I) \cap \tilde{P}$ is a finite union of squares of the form $I_0 \times I_k (k = 1, \cdots, m)$, where each I_k is an element of the dyadic partition of I into 2^n subintervals of equal length. Therefore $(B \times I) \cap \tilde{P}$ is the finite union of the Borel rectangles $B \times I_k$. We define a function $g \in L^\infty(m_2)$ as follows

$$g(y) = \begin{cases} 1 \text{ if } y \in \cup I_k , \\ -1 \text{ otherwise.} \end{cases}$$

Clearly the Radon-Nikodym derivative of the marginal measure $(g(y) d\nu(x, y))^1$ coincides with

$$\frac{d|\nu|^1}{dm_1} = 1$$

on B. Therefore, by Corollary 1, T_ν attains its norm and the proof is completed.

By the known representation theorems for Lebesgue spaces (see e.g., [5], II 8.5 Corollary and [2], § 41 Theorem C, or [6], 26.4.9 Exercise (C)), every separable Lebesgue space (i.e., separable AL-space in terms of [5]) is Banach lattice isomorphic with $L^1(m)$ for some finite positive Borel measure m on I. Therefore we obtain the following corollary to our result:

COROLLARY 2. *Let X and Y be separable Lebesgue spaces. Then the norm attaining operators are dense in $B(X, Y)$.*

After the paper was accepted for publication, the last corollary has been generalized to arbitrary (nonseparable) Lebesgue spaces as a result of the author's conversations with Professors J. Bourgain and H. P. Lotz. The proof is outlined below:

Theorem 1 remains true if we replace (I, m_i) by (J_i, m_i) with J_i compact Hausdorff and m_i a finite regular (compact) positive measure on the Borel σ-algebra $\mathscr{B}_i$, and with M being the space of all finite signed measures on the product σ-alglebra $\mathscr{B}_1 \times \mathscr{B}_2$. Indeed, the marginal measures $\int_A T^* 1 dm_1$, $\int_B T1 dm_2$ are compact since the measures m_i are regular, and so Theorem 1 (i) of [4] is still applicable. The rest of the proof remains unchanged.

Theorem 2 is valid for the general spaces $L^1(J_i, m_i)$ with essentially the same proof as before, $\mathscr{A}$ being replaced now by the algebra of all finite unions of Borel rectangles in $J_1 \times J_2$.

Now if X_1, X_2 are arbitrary Lebesgue spaces then every $T \in B(X_1, X_2)$ can be approximated by norm attaining operators. Indeed, let (x_n) be a sequence in X_1 such that $\|x_n\| \leqq 1$ and $\lim \|Tx_n\| = \|T\|$. The Banach lattice ideal Y_1 spanned by (x_n) is a Lebesgue subspace with a weak order unit. Also the image TY_1 is contained in a Lebesgue subspace $Y_2 \subset X_2$ with a weak order unit. By the Kakutani representation theorem there exist compact spaces J_i with finite regular positive measures m_i such that $Y_i = L^1(J_i, m_i)$. By the above, the restriction T_1 of T to Y_1 can be approximated within a given $\varepsilon > 0$ by a norm attaining operator $T_0 \in B(Y_1, Y_2)$ satisfying $\|T_0\| = \|T\|$. If P denotes the canonical band projection of X_1 onto Y_1 then it is easy to see that $T_0P + T(I - P)$ has norm $\|T_0\|$, is norm attaining, and approximates T within ε.

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