PACIFIC JOURNAL OF MATHEMATICS
Vol. 83, No. 2, 1979

ON A THEOREM OF HAYMAN CONCERNING THE DERIVATIVE OF A FUNCTION OF BOUNDED CHARACTERISTIC

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W. Hayman [On Nevanlinna's second theorem and extensions, Rend. Circ. Mat. Palermo, Ser. II, II (1953).] has given sufficient conditions on a function, f, of bounded characteristic in the unit disc, in order that f' also have bounded characteristic. In this paper it is shown that one of these conditions is also necessary for the conclusion of the theorem to hold.

Let U be the open unit disc in the complex plane and let T be its boundary. It is well known that there are functions f, that are bounded and holomorphic in U, such that $f' \notin N(U)$. Here $N(U)$ is the Nevanlinna class. In fact, O. Frostman, [1, Théoreme IX], has shown that there are Blaschke products with some degree of "smoothness" whose derivatives fail to lie in $N(U)$. More precisely, he shows that there is a Blaschke product B, whose zeros $\{a_n\}$ satisfy the condition,

$$\sum_n (1 - |a_n|)^\alpha < \infty \,, \quad \text{for all} \quad \alpha > \frac{1}{2} \,,$$

but $B' \notin N(U)$. In Frostman's example, every point of T is a limit point of the sequence $\{a_n\}$.

W. Hayman, [2, Theorem IV], has proved a result in the positive direction. A function f, that is holomorphic in a bounded domain D, is said to be of order K if, for every complex number a, the number of solutions of the equation, $f(z) = a$, that are at a distance of at least ε from the boundary of D is at most $C\varepsilon^{-K}$, for some constant C. C may depend on a but not on ε. We say f has finite order if it has order K for some K. Now let D be a bounded open set such that $U \subseteqq D$, and let $D \cap T = \bigcup_n I_n$, where $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$.

THEOREM A (*Hayman*). *Suppose that*

(i) (a) $\sum_n (\beta_n - \alpha_n) = 2\pi$

(b) $\sum_n (\beta_n - \alpha_n) \log 1/(\beta_n - \alpha_n) < \infty$.

(ii) *there are constants* $\varepsilon, C > 0$ *such that if* $\alpha_n < \theta < \beta_n$, *then*

$$\operatorname{dist}(e^{i\theta}, \partial D) \geqq \varepsilon(|\theta - \alpha_n|\,|\theta - \beta_n|)^C \,.$$

(iii) *f is holomorphic and of finite order in D and $f \in N(U)$.*

Then $f^{(k)} \in N(U)$ *for* $k = 1, 2, 3, \cdots$.

The conditions (i)(a) and (i)(b) just mean that the set $E = T \backslash \bigcup_n I_n$ is what is usually called a Carleson set.

In [4], P. Kennedy investigates the necessity of condition (i)(b). He shows that if (i)(a) holds but

$$(*) \qquad \overline{\lim_{n\to\infty}} \left(\sum_{j=n}^{\infty} (\beta_j - \alpha_j)\right) \log \frac{1}{(\beta_n - \alpha_n)} = \infty ,$$

then there is a bounded open set $D \supseteqq U$ such that $D \cap T = \bigcup_n I_n$, $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$, and a function f that is bounded and holomorphic in D such that $f' \notin N(U)$. He observes that condition (*) does not follow from the condition

$$\sum_n (\beta_n - \alpha_n) \log \frac{1}{\beta_n - \alpha_n} = \infty ,$$

and writes that "there is still a gap between the positive information given by Hayman's theorem and the negative information" given by his example.

In this note we close the gap by showing that condition (i)(b) is the right one. Our example is a Blaschke product that retains the same degree of smoothness as the one of Frostman's example.

THEOREM. *To each sequence of arcs* $\{I_n\}$, $I_n = \{e^{i\theta}: \alpha_n < \theta < \beta_n\}$, *that satisfies* (i)(a) *but not* (i)(b), *there corresponds a Blaschke product,* B, *whose zero sequence,* $\{a_n\}$, *clusters only on* $T \backslash \bigcup_n I_n$, *such that* $B' \notin N(U)$ *and* $\sum (1 - |a_n|)^\alpha < \infty$ *for all* $\alpha > 1/2$. *Moreover, there is a bounded open set* D, *such that* $D \supseteqq U$, $D \cap T = \bigcup_n I_n$, D *satisfies condition* (i)(c) *with* $C = 2$, *and* B *extends to be bounded and of order* 1 *in* D.

Proof. Let $\varepsilon_n = \beta_n - \alpha_n$. We are assuming that $\sum_n \varepsilon_n \log (1/\varepsilon_n) = \infty$. We may choose numbers δ_n, $0 < \delta_n < 1$, such that $\lim_{n\to\infty} \delta_n = 0$, and $\sum_n \delta_n \varepsilon_n \log 1/\varepsilon_n = \infty$. Now define $d_n = \varepsilon_n^{2-\delta_n}$ and $c_n = (1 - d_n)e^{i\alpha_n}$ and $\gamma_n = (1 - d_n)e^{i\beta_n}$. Let B be the Blascke product whose zeros are $\{c_n\} \cup \{\gamma_n\}$. The zeros of B cluster only on the set $E = T \backslash \bigcup_n I_n$ so B is holomorphic on I_n for every n. We calculate that

$$B'(z) = B(z)\left\{\sum_n \frac{1 - |c_n|^2}{(z - c_n)(1 - \bar{c}_n z)} + \sum_n \frac{1 - |\gamma_n|^2}{(z - \gamma_n)(1 - \bar{\gamma}_n z)}\right\}$$

so that when $e^{i\theta} \in I_n$ we get

$$e^{i\theta} B'(e^{i\theta}) = B(e^{i\theta})\left\{\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} + \sum_k \frac{1 - |\gamma_k|^2}{|e^{i\theta} - \gamma_k|^2}\right\} .$$

If B' were in $N(U)$ it would follow that

$$\sum_n \int_{I_n} \log^+ \left(\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} \right) d\theta < \infty .$$

Now,

$$\begin{aligned} |e^{i\theta} - c_n|^2 &= (1 - |c_n|)^2 + 4|c_n| \sin^2\left(\frac{\theta - \alpha_n}{2}\right) \\ &\leqq d_n^2 + (\theta - \alpha_n)^2 \end{aligned}$$

and hence

$$\frac{1 - |c_n|^2}{|e^{i\theta} - c_n|^2} \geqq \frac{d_n}{d_n^2 + (\theta - \alpha_n)^2} .$$

If $e^{i\theta} \in I_n$, then

$$\begin{aligned} \log^+\left(\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} \right) &\geqq \log \left(\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} \right) \\ &\geqq \log \frac{1 - |c_n|}{|e^{i\theta} - c_n|^2} \geqq \log \frac{d_n}{d_n^2 + (\theta - \alpha_n)^2} . \end{aligned}$$

So we see that

$$\begin{aligned} \sum_n \int_{I_n} \log^+\left(\sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} \right) &\geqq \sum_n \int_{I_n} \log \frac{d_n}{d_n^2 + (\theta - \alpha_n)^2} d\theta \\ &\geqq \sum_n \varepsilon_n \log \frac{d_n}{d_n^2 + \varepsilon_n^2} . \end{aligned}$$

Since $\delta_n < 1$, we see that $d_n = \varepsilon_n^{2-\delta_n} \leqq \varepsilon_n$ (assuming $\varepsilon_n < 1$), so

$$\log \frac{d_n}{d_n^2 + \varepsilon_n^2} \geqq \log \frac{d_n}{2\varepsilon_n^2} = \log \frac{1}{2\varepsilon_n^{\delta_n}} = \log \frac{1}{2} + \delta_n \log \frac{1}{\varepsilon_n} .$$

Hence,

$$\sum_n \int_{I_n} \log^+ \left\{ \sum_k \frac{1 - |c_k|^2}{|e^{i\theta} - c_k|^2} \right\} d\theta \geqq 2\pi \log \frac{1}{2} + \sum_n \delta_n \varepsilon_n \log \frac{1}{\varepsilon_n} = \infty .$$

So $B' \notin N(U)$. Also we see that

$$\sum_n (1 - |a_n|)^\alpha = 2 \sum_n d_n^\alpha = \sum \varepsilon_n^{(2-\delta_n)\alpha} < \infty$$

if $\alpha > 1/2$ because $(2 - \delta_n)\alpha \geqq 1$ for all sufficiently large n.

It remains to construct the domain D. We have the inequality,

$$|B(re^{i\theta})|^2 \geqq 1 - (1 - r^2)\left\{\sum_k \frac{1 - |c_k|^2}{|1 - re^{i\theta}\bar{c}_k|^2} + \sum_k \frac{1 - |\gamma_k|^2}{|1 - re^{i\theta}\bar{\gamma}_k|^2}\right\}$$

$$\geqq 1 - 4(1 - r^2)\left\{\sum_k \frac{1 - |c_k|^2}{\left|re^{i\theta} - \dfrac{1}{\bar{c}_k}\right|^2} + \sum_k \frac{1 - |\gamma_k|^2}{\left|re^{i\theta} - \dfrac{1}{\bar{\gamma}_k}\right|^2}\right\}.$$

(We may assume $|c_k| \geqq 1/2$, $|\gamma_k| \geqq 1/2$.)
Now suppose $\alpha_n \leqq \theta \leqq (\alpha_n + \beta_n)/2$ and $|z| \leqq 1$, then

$$|B(re^{i\theta})|^2 \geqq 1 - \frac{4(1 - r^2)}{|re^{i\theta} - e^{i\alpha_n}|^2}\Big(\sum_k (1 - |c_k|^2) + \sum_k (1 - |\gamma_k|^2)\Big).$$

So, $|B(re^{i\theta})|^2 \geqq 1/4$ if

$$\frac{1 - r^2}{|re^{i\theta} - e^{i\alpha_n}|^2} \leqq \frac{3}{16}\frac{1}{\sum (1 - |c_k|^2) + \sum (1 - |\gamma_k|^2)} = C.$$

Note that C is independent of θ and n. Similarly we see that if $(\alpha_n + \beta_n)/2 \leqq \theta \leqq \beta_n$ and

$$\frac{1 - r^2}{|re^{i\theta} - e^{i\beta_n}|^2} \leqq \frac{3}{16}\frac{1}{\sum_k (1 - |c_k|^2) + \sum (1 - |\gamma_k|^2)} = C,$$

then $|B(re^{i\theta})|^2 \geqq 1/4$. We may calculate that, for $C > 0$,

$$\left\{re^{i\theta}: \frac{1 - r^2}{|re^{i\theta} - e^{i\lambda}|^2} < C\right\} = \{re^{i\theta}: |re^{i\theta} - \rho e^{i\lambda}| > 1 - \rho\},$$

where $\rho = C/(1 + C)$.

So, if

$$\Delta_n = \{re^{i\theta}: r \leqq 1, \alpha_n < \theta < \beta_n, |re^{i\theta} - \rho e^{i\alpha_n}| > 1 - \rho, \text{ and } |re^{i\theta} - \rho e^{i\beta_n}| > 1 - \rho\} \text{ and } \Delta = \textstyle\bigcup_n \Delta_n, \quad \text{then } |B(z)| \geqq 1/2, z \in \Delta.$$

Now for $|z| > 1$, $B(z) = 1/\overline{B(1/\bar{z})}$, so $|B(z)| \leqq 2$ if $1/\bar{z} \in \Delta$. Assuming, as we may, that $C < 1$, we see that $\Gamma_n = \{z: 1/\bar{z} \in \Delta_n\} = \{z: |z| \geqq 1, |z + \delta e^{i\alpha_n}| < 1 + \delta$ and $|z + \delta e^{i\beta_n}| < 1 + \delta\}$, where $\delta = C/(1 - C)$. Finally, if we let $\mathcal{O} = U \cup \bigcup_n \Gamma_n$ then $\mathcal{O}$ is an open set and $|B(z)| \leqq 2$ for $z \in \mathcal{O}$.

Now we define a function

$$\psi(\theta) = \begin{cases} (\theta - \alpha_n)^2(\theta - \beta_n)^2 & \text{if } \alpha_n < \theta < \beta_n \text{ for some } n \\ 0 & \text{otherwise}. \end{cases}$$

We check that $\psi'(\theta)$ exists for all θ and that there is a constant K such that

$$|\psi'(\theta_1) - \psi'(\theta_2)| \leqq K|\theta_1 - \theta_2| .$$

(See [4, Lemma 1] for a similar calculation.) For $\varepsilon > 0$ we define $D_\varepsilon = \{re^{i\theta}: r < e^{\varepsilon\psi(\theta)}\}$. Then D_ε satisfies condition (ii). of Theorem A with $C = 2$. (Again, see [4, Lemma 2], for a similar calculation.) Also, it is not hard to that $D_\varepsilon \subseteqq \mathcal{O}$ for all sufficiently small $\varepsilon > 0$. So we fix some $\varepsilon > 0$ such that $D_\varepsilon \subseteqq \mathcal{O}$ and let $D = D_\varepsilon$. Since $D \subseteqq \mathcal{O}$, B is bounded in D. It remains to show that B has order 1 in D. Let $\varphi: D \to U$ be a conformal map. Since ψ' satisfies a Lipschitz condition it follows from a theorem of Kellogg [3], that φ' extends to be continuous and nonvanishing on $\bar{D}$. From this we can conclude that there is a $\delta > 0$ such that $1 - |\varphi(z)| \geqq \delta \operatorname{dist}(z, \partial D)$ for all $z \in D$. Fix $a \in \boldsymbol{C}$ and let $f = B - a$ and let $\{a_n\}$ be the zero sequence of f. Then $\{\varphi(a_n)\}$ is the zero sequence of the bounded function $f \circ \varphi^{-1}$ so $\sum_n (1 - |\varphi(a_n)|) < \infty$ and hence $\sum_n \operatorname{dist}(a_n, \partial D) < \infty$. From this we may conclude that B has order 1 in D.

As a final remark we point out that we may choose the arcs I_n in such a way that $E = T \backslash \bigcup_n I_n$ is a countable set with only one limit point, and such that (i)(b) fails. If we apply the theorem to this situation we get a Blaschke product B whose zeros converge to a single point such that $B' \notin N(U)$, while the zeros sequence, $\{a_n\}$, satisfies $\sum (1 - |a_n|)^\alpha < \infty$ for all $\alpha > 1/2$.

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Received January 1, 1979.
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