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GENERALIZATION OF A THEOREM OF LANDAU

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A well known theorem of Landau asserts that

$$(1.1)\qquad \underline{\lim}_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma}$$

where $\gamma =$ Euler's constant. In this paper a generalization is obtained by focusing on

$$(1.2)\qquad G(k) = \underline{\lim}_{n\to\infty} (\log\log n)^{1/k} \max\left(\frac{\phi(n+1)}{n+1}, \cdots, \frac{\phi(n+k)}{n+1}\right).$$

Clearly, the assertion $G(1) = e^{-\gamma}$ is precisely Landau's theorem. It is proved that

$$(1.3)\qquad G(k) = e^{-\gamma/k} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1/k} \psi(k)$$

where

$$(1.4)\qquad \psi(k) = \prod_{\substack{p\mid k\\ p<k}}\left(1 - \frac{1}{p}\right)^{1/p} \prod_{\substack{p\nmid k\\ p<k}}\left(1 - \frac{1}{p}\right)^{(1/k)[k/p]+1/k}.$$

The function $\psi(k)$ satisfies $0 < \psi(k) \leqq 1$ and it is easily seen from (1.4) that

$$(1.5)\qquad \lim_{k\to\infty} \psi(k) = \prod_p \left(1 - \frac{1}{p}\right)^{1/p}.$$

2. Preliminary lemmas. The results obtained in this paper depend on the following well known theorems [1], [2], and [3].

$$(2.1)\qquad \underline{\lim}_{n\to\infty} \frac{\phi(n)\log\log n}{n} = e^{-\gamma} \qquad \text{(Landau's theorem)}$$

$$(2.2)\qquad \sum_{p\leqq x} \frac{1}{p} = \log\log x + c + O\left(\frac{1}{\log x}\right) \qquad \text{(Mertens')}$$

$$(2.3)\qquad \prod_{p\leqq x}\left(1 - \frac{1}{p}\right) = \frac{e^{-\gamma}}{\log x} + O\left(\frac{1}{\log^2 x}\right) \qquad \text{(Mertens')}$$

3. Proof of (1.3). We introduce

$$(3.1)\qquad \left(\frac{\phi(n)}{n}\right)_k = \prod_{\substack{p\mid n\\ p\geqq k}}\left(1 - \frac{1}{p}\right)$$

and

$$(3.2)\qquad f_k(n) = \prod_{\substack{p\mid n\\ p<k}}\left(1 - \frac{1}{p}\right)$$

and note that $f_k(n)$ is periodic with period $\Delta_k = \prod_{p<k} p$.

We also observe that (1.2) is clearly equivalent to

$$(3.3)\quad G(k) = \min_{1 \leqq J \leqq \Delta_k} \varliminf_{\substack{n \to \infty \\ n \equiv J (\mathrm{mod}\ '_k)}} (\log\log n)^{1/k} \max\left(\frac{\phi(n+1)}{n+1}, \cdots, \frac{\phi(n+k)}{n+k}\right).$$

On the sequence $n \equiv J (\mathrm{mod}\ \Delta_k)$

$$(3.4)\quad \left(\log\log n\right) \prod_{i=1}^{k} \frac{\phi(n+i)}{n+i} = (\log\log n) \prod_{i=1}^{k} \left(\frac{\phi(n+i)}{n+i}\right)_k f_k(J+i)\,.$$

Since a prime p divides $n+i$ and $n+j$ only if p divides $i-j$, $1 \leqq j < i \leqq k$; and the primes involved in $(\phi(n)/n)_k$ are $p \geqq k$, we have

$$\prod_{i=1}^{k} \left(\frac{\phi(n+i)}{n+i}\right)_k = \left(\frac{\phi\left[\prod_{i=1}^{k}(n+i)\right]}{\prod_{i=1}^{k}(n+i)}\right)_k .$$

This together with the result

$$\varliminf_{n \to \infty} (\log\log n)\left(\frac{\phi(n)}{n}\right)_k = e^{-\gamma} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1}$$

(which follows from Landau's theorem) yields

$$(\log\log \mathrm{n}) \prod_{i=1}^{k}\left[\frac{\phi(n+i)}{n+i}\right] \geqq (1+o(1))e^{-\gamma} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1} \prod_{i=1}^{k} f_k(J+i)\,,$$

which implies

$$(3.5)\quad \varliminf_{\substack{n \to \infty \\ n \equiv J (\mathrm{mod}\ '_k)}} (\log\log n)^{1/k} \max_{i=1,\cdots,k}\left(\frac{\phi(n+i)}{n+i}\right) \geqq e^{-\gamma/k} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1/k}\left[\prod_{i=1}^{k} f_k(J+i)\right]^{1/k} .$$

In (3.5), taking the minimum over J, $1 \leqq J \leqq \Delta_k$, and using (3.3) yields

$$(3.6)\quad G(k) \geqq e^{-\gamma/k} \prod_{p<k}\left(1 - \frac{1}{p}\right)^{-1/k}\left[\min_{1 \leqq J \leqq\ '_k} \prod_{i=1}^{k} f_k(J+i)\right]^{1/k} .$$

Choose J^* such that

$$\left[\min_{1 \leqq J \leqq \Delta_k} \prod_{i=1}^{k} f_k(J+i)\right]^{1/k} = \left[\prod_{i=1}^{k} f_k(J^*+i)\right]^{1/k} .$$

We next observe that for the $\psi(k)$ given in (1.4) we have

$$(3.7)\quad \left[\prod_{i=1}^{k} f_k(J^*+i)\right]^{1/k} = \psi(k)\,.$$

To see this note first that the left side of (3.7) equals

$$\min_{1\leqq J\leqq \varDelta_k}\left[\prod_{\substack{p\mid J+1\\ p<k}}\left(1-\frac{1}{p}\right)\prod_{\substack{p\mid J+2\\ p<k}}\left(1-\frac{1}{p}\right)\cdots\prod_{\substack{p\mid J+k\\ p<k}}\left(1-\frac{1}{p}\right)\right]^{1/k}. \tag{3.8}$$

Since each of the factors $(1-1/p)<1$, the minimum of the product in (3.8) is achieved for that value of J for which each prime $p<k$ divides as many of the k integers $J+1,\cdots,J+k$ as possible. Since $p<k$, $k=pt+r$, $t=[k/p]$, $0\leqq r<p$. If $r=0$, i.e., $p\mid k$, then the k integers $J+1,\cdots,J+k$ can be broken up into exactly t complete residue systems modulo p and in each system we have one integer $\equiv 0(\text{mod}\, p)$; this situation is independent of the choice of J. If $r>0$ then the k integers $J+1,\cdots,J+k$ form t complete residue classes modulo p together with $r<p$ remaining integers. In each of the complete residue classes there is one integer $\equiv 0(\text{mod}\, p)$. We would like to show that it can be arranged that for each $p<k$, $p\nmid k$, one of the r remaining integers is $\equiv 0(\text{mod}\, p)$, and thus we have $[k/p]+1$ integers divisible by p. Since $1\leqq J\leqq \varDelta_k$ where $\varDelta_k=\prod_{p<k}p$, we can choose $J=\varDelta_k-1$; then every $p<k$ divides $J+1$. Hence for $p\nmid k$, the $[k/p]+1$ integers $J+1+\tau p$, $0\leqq\tau\leqq t$ are divisible by p as desired, and (3.7) follows.

From (3.6) and (3.7) we see that

$$G(k)\geqq e^{-\gamma/k}\prod_{p<k}\left(1-\frac{1}{p}\right)^{-1/k}\psi(k)\ ; \tag{3.9}$$

and it remains to prove the reverse inequality. This is achieved by showing that there exists an infinite sequence $n\equiv J^*(\text{mod}\,\varDelta_k)$ on which

$$\varliminf_{\substack{n\to\infty\\ n\equiv J^*(\text{mod}\,\varDelta_k)}}(\log\log n)^{1/k}\max_{i=1,\cdots,k}\left(\frac{\phi(n+i)}{n+i}\right) \leqq e^{-\gamma/k}\prod_{p<k}\left(1-\frac{1}{p}\right)^{-1/k}\psi(k)\ . \tag{3.10}$$

This is done by producing a sequence $n\equiv J^*(\text{mod}\,\varDelta_k)$ for which

$$(\log\log n)^{1/k}\max_{i=1,\cdots,k}\left(\frac{\phi(n+i)}{n+i}\right)_k\sim e^{-\gamma/k}\prod_{p<k}\left(1-\frac{1}{p}\right)^{-1/k}\lambda_i \tag{3.11}$$

where for all $i=1,\cdots,k$,

$$\lambda_i=\frac{\psi(k)}{f_k(J^*+i)}\ .$$

On this sequence

$$(\log\log n)^{1/k}\max_{i=1,\cdots,k}\left(\frac{\phi(n+i)}{n+i}\right)\sim e^{-\gamma/k}\prod_{p<k}\left(1-\frac{1}{p}\right)^{-1/k}\max_{i=1,\cdots,k}(\lambda_i f_k(J^*+i))$$

$$\sim e^{-\gamma/k} \prod_{p<k} \left(1 - \frac{1}{p}\right)^{-1/k} \psi(k) ,$$

which gives the reverse inequality to (3.9) and establishes (1.3).

To construct the sequence $n \equiv J^*(\text{mod } \Delta_k)$ which satisfies (3.8) let

$$B_1 = \prod_{k \leqq p < \exp(c_1 \log x)} p ,$$
$$B_i = \prod_{\exp((c_{i-1})(\log x)^{i-1}) \leqq p < \exp(c_i (\log x)^i)} p , \qquad i = 2, \cdots, k ;$$

where $c_k = 1$, and for $i = 0, \cdots, k-1$, c_i is determined by

$$\frac{c_{i-1}}{c_i} = e^{-\gamma/k} \prod_{p<k} \left(1 - \frac{1}{p}\right)^{-1/k} \lambda_i .$$

Since $\prod_{i=1}^k \lambda_i = 1$ it follows that $c_0 = e^{-\gamma} \prod_{p<k} (1 - 1/p)$. As the B_i, $i = 1, \cdots, k$ are k integers made up of primes $p \geqq k$ and are relatively prime in pairs, as well as each relatively prime to Δ_k, by the Chinese Remainder Theorem the system

$$\begin{aligned} y + 1 &\equiv O(\text{mod } B_1) \\ y + 2 &\equiv O(\text{mod } B_2) \\ &\vdots \\ y + k &\equiv O(\text{mod } B_k) \\ y &\equiv J^*(\text{mod } \Delta_k) \end{aligned} \tag{3.13}$$

has a solution $y = n^*$, $0 < n^* < \Delta_k \prod_{i=1}^k B_i$ which is unique modulo $\Delta_k \prod_{i=1}^k B_i$.

For this integer $n^* \equiv J^*(\text{mod } \Delta_k)$ we have for $i = 1, \cdots, k$

$$\begin{aligned} \left(\frac{\phi(n^* + i)}{n^* + i}\right)_k &= \prod_{\substack{p \mid n^* + i \\ p \geqq k}} \left(1 - \frac{1}{p}\right) \leqq \prod_{p \mid B_i} \left(1 - \frac{1}{p}\right) \\ &\leqq \frac{c_{i-1}}{c_i}\left(\frac{1}{\log x}\right) + O\left(\frac{1}{\log^2 x}\right) , \end{aligned}$$

(note that the value obtained for c_0 validates this for $i = 1$). Then

$$\begin{aligned} &\left(\frac{\phi(n^* + i)}{n^* + i}\right)_k f_k(J^* + i) \\ &\leqq \frac{\lambda_i e^{-\gamma/k} \prod_{p<k} \left(1 - \dfrac{1}{p}\right)^{-1/k}}{\log x} f_k(J^* + i)(1 + o(1)) . \end{aligned} \tag{3.14}$$

But from the Prime Number Theorem since

$$n^* < \Delta_k \prod_{i=1}^k B_i = \prod_{p < \exp\{\log x\}^k} p , \qquad (c_k = 1) ,$$

it follows that

$$\log n^* \leqq \sum_{p<\exp\{\log x\}^k} \log p = O(e^{(\log x)^k})$$

so that

$$\log\log n^* \leqq (\log x)^k + O(1) \ . \tag{3.15}$$

Since (3.14) holds for all $i = 1, \cdots, k$, it certainly holds for the maximum of these functions. Thus inserting (3.15) in (3.14) yields

$$\begin{aligned}(\log\log n^*)^{1/k} \max_{i=1,\cdots,k} \left(\frac{\phi(n^*+i)}{n^*+i}\right)_k f_k(J^*+i) \\ \leqq (1+o(1))e^{-\gamma/k} \prod_{p<k}\left(1-\frac{1}{p}\right)^{1/k} \psi(k) \ .\end{aligned} \tag{3.16}$$

Clearly as x tends to infinity the n^* (which depends on x) also tends to infinity, so that (3.16) yields

$$G(k) \leqq e^{-\gamma/k} \prod_{p<k}\left(1-\frac{1}{p}\right)^{-1/k} \psi(k) \tag{3.17}$$

which completes the proof of (1.3).

References

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