PACIFIC JOURNAL OF MATHEMATICS
Vol. 93, No. 2, 1981

A UNIQUENESS THEOREM FOR NAVIER-STOKES EQUATIONS

CHUN-MING MA

In this paper we consider the initial boundary value problem for the Navier-Stokes equations in several types of unbounded three-dimensional domains Ω. We prove uniqueness within a class of solutions, which we call "weak class H_0 solutions", whose members satisfy the integrability conditions ∇u, $\Delta u \in L^2(0, T; L_2(\Omega))$. Moreover, the solutions are shown to depend continuously on their initial values. The results are based, primarily on establishing a simple characterization of a certain space $H_0(\Omega)$ of solenoidal functions.

For exterior domains, we have already given such a characterization of the space $H_0(\Omega)$ in Ma [14]. However, the proof given here is simpler and more direct and yields the result for "aperture domains" as well (i.e., for domains considered by Heywood [8] in studying flow through a hole in a wall).

Our uniqueness theorem should be compared with one given recently by Heywood. In [9], Heywood used our original characterization of the space H_0 to prove uniqueness in exterior domains for solutions satisfying the integrability conditions ∇u, $\Delta u \in L^2(0, T; L^2(\Omega))$ and $\nabla u_t \in L^2(\varepsilon, T; L^2(\Omega))$, for all positive $\varepsilon < T$. Here, we are able to drop the integrability condition for ∇u_t by using a technique introduced in the context of "finite energy" solutions by Prodi [15]; see also Serrin [17]. The main advantage in giving the uniqueness theorem as we do here, without Heywood's integrability condition for ∇u_t, is that one can then consider a larger class of forces. If one considers arbitrary forces, with $\nabla f \in L^2(0, T; L^2(\Omega))$, the integrability condition for ∇u_t is not known, and quite possibly does not hold; see [10]. However, for such forces, generalized solutions satisfying the conditions of our uniqueness theorem do exist. This is proved in the concluding section of the present paper.

Our results should also be compared with a remarkable new uniqueness theorem of Fabrizio [3], which appeared as we were finishing this work. This theorem (Theorem 1 in [3]) requires even less than ours in the way of integrability conditions; it is merely required that the difference of two solutions should belong to $L^s(\Omega \times (0, T))$, for some $s > 1$. On the other hand, it is apparently given only for an exterior domain (though this is not really made clear) and does not provide the continuous dependence of solutions on their initial values. Further, Fabrizio's theorem is based on several unproven preliminary results, Observation 1 and Lemma 1. In particular, Lemma 1 is the extension of L^p estimates for the non-stationary Stokes equations given by Solonnikov [18] in bounded domains, to the case of unbounded domains and nonconstant coefficients. Our preliminary results for the space H_0 are much simpler and given in detail.

The reader may also wish to consult the references [5, 16] where other uniqueness theorems are given.

2. The function space $H_0(\Omega)$. Let Ω be an open set of $R^n (n \geqq 2)$. Let $J_0(\Omega)$ denote the completion of $D(\Omega) = \{\phi: \phi \in C_0^\infty(\Omega)$ and $\nabla \cdot \phi = 0\}$ in the norm associated with the inner product

$$(\nabla\phi, \nabla\psi) = \int_\Omega \nabla\phi : \nabla\psi dx ,$$

where ϕ and ψ are R^n-valued functions, and $\nabla\phi: \Delta\psi = \sum_{i,j=1}^n \partial\phi_i/\partial x_j \, \partial\psi_i/\partial x_j$. Let $K_0(\Omega)$ be the set of all $u \in J_0(\Omega)$ such that $\int_\Omega \nabla u: \nabla\phi dx = -\int_\Omega f \cdot \phi dx$ for some $f \in D(\Omega)$ and all $\phi \in J_0(\Omega)$. Here $f \cdot \phi = \sum_{i=1}^n f_i \phi_i$. We define a map $\tilde{\Delta}: K_0(\Omega) \to J(\Omega)$ by setting $\tilde{\Delta} u = f$, where $J(\Omega)$ is the completion of $D(\Omega)$ in the norm associated with the inner product $(\phi, \psi) = \int_\Omega \phi \cdot \psi dx$. Clearly, $\tilde{\Delta}$ is well defined and closable. The space $H_0(\Omega)$ is defined as the completion of $K_0(\Omega)$ in the norm $\|\cdot\|_{H_0}$ associated with the inner product $(\nabla\phi, \nabla\psi) + (\tilde{\Delta}\phi, \tilde{\Delta}\psi)$. Note that $H_0(\Omega)$ may be regarded as a subset of $J_0(\Omega)$. The extension of $\tilde{\Delta}$ to $H_0(\Omega)$ is again denoted by $\tilde{\Delta}$. It can be shown that

$$(\nabla\phi, \nabla\psi) = -(\tilde{\Delta}\phi, \psi) \tag{1}$$

holds if $\phi \in H_0(\Omega)$ and $\psi \in J_0(\Omega) \cap L^2(\Omega)$. We refer the reader to [6] for details.

For several types of unbounded domains Ω, we shall show the space $H_0(\Omega)$ contains the set $H_0^*(\Omega) = \{u: u \in J_0(\Omega)$ and $\Delta u \in L^2(\Omega)\}$. Our proof is based on the following proposition.

PROPOSITION 1. *Let Ω be an open set of $R^n (n \geqq 2)$. Then a condition sufficient to ensure $H_0^*(\Omega) \subset H_0(\Omega)$, is that the only element w in $J_0(\Omega)$ satisfying*

$$\int_\Omega \nabla w: \nabla\phi dx = -\int_\Omega w \cdot \phi dx = \int_\Omega f \cdot \phi dx , \tag{2}$$

for some $f \in L^2(\Omega)$ and all $\phi \in D(\Omega)$, is $w \equiv 0$.

Proof. Let $u \in H_0^*(\Omega)$. Define a linear functional F on $H_0(\Omega)$ by setting $F(\phi) = (\nabla u, \nabla\phi) + (P\Delta u, \tilde{\Delta}\phi)$ for $\phi \in H_0(\Omega)$, where P is the projection of $L^2(\Omega)$ onto $J(\Omega)$. Clearly, F is bounded on $H_0(\Omega)$. Thus, the Riesz representation theorem gives a unique element v of $H_0(\Omega)$ such that

$$(\nabla v, \nabla\phi) + (\tilde{\Delta} v, \tilde{\Delta}\phi) = F(\phi) , \tag{3}$$

for all $\phi \in H_0(\Omega)$. Since $(\nabla\psi, \nabla\phi) = -(\psi, \tilde{\Delta}\phi)$ holds for all $\psi \in J_0(\Omega)$ and all $\phi \in K_0(\Omega)$, (3) implies

$$\int_\Omega [(v - u) - (\tilde{\Delta} v - P\Delta u)] \cdot \phi dx = 0$$

for all $\phi \in D(\Omega)$; remember here than $D(\Omega)$ is the image of $K_0(\Omega)$ under $\tilde{\Delta}$. Let $w = v - u$. Then, for all $\phi \in D(\Omega)$,

$$\int_\Omega w \cdot \phi dx = \int_\Omega (\tilde{\Delta} v - P\Delta u) \cdot \phi dx = -\int_\Omega \nabla w : \nabla\phi dx . \tag{4}$$

The second identity in (4) holds because (1) implies $(\tilde{\Delta} v, \phi) = -(\nabla v, \nabla\phi)$, and because, through an integration by parts, $(P\Delta u, \phi) = -(\nabla u, \nabla\phi)$. Thus, by assumption, $w \equiv 0$ and so $u \equiv v \in H_0(\Omega)$.

Before proceeding further, we recall the various Sobolev spaces to be used throughout the paper. The space $W_p^m(\Omega)$ is the set of all R^n-valued functions which belong to $L^p(\Omega)$ and possess generalized derivatives up to order m in $L^p(\Omega)$. Its norm is

$$\|u\|_{W_p^m} = \Big(\sum_{|\alpha|\leq m} \int_\Omega |D^\alpha u|^p dx\Big)^{1/p} , \tag{5}$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, and $|D^\alpha u|^p = \sum_{i=1}^n |\partial^{\alpha_1+\cdots+\alpha_n} u_i / \partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}|^p$. The space $\mathring{W}_p^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm (5). Finally, we let $J_1(\Omega)$ denote the completion of $D(\Omega)$ in the norm $\|\cdot\|_{W_2^1}$, and let $J_1^*(\Omega) = \{\phi : \phi \in \mathring{W}_2^1(\Omega)$ and $\nabla \cdot \phi = 0\}$. Also, $J_0^*(\Omega) = \{\phi : \phi \in W_0(\Omega)$ and $\nabla \cdot \phi = 0\}$, where $W_0(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in the norm $\|\nabla\phi\| = (\nabla\phi, \nabla\phi)^{1/2}$. We proved the following lemma in [14], by considering an expansion in spherical harmonics.

LEMMA 1. *Let Ω be an exterior domain in R^n $(n > 2)$ for which $J_1^*(\Omega) = J_1(\Omega)$. If q is a function in $L_{\mathrm{loc}}^2(\Omega)$ such that $\nabla q = u + v$, where $u \in J_0^*(\Omega)$ and $v \in J(\Omega)$, then $u \in J_1(\Omega)$ and, further, $\nabla q = 0$ in Ω.*

The next lemma, due to Heywood [10], is based on a regularity theorem of Catabriga [2], or of Solonnikov and Scadilov [19] in the case of C^3 boundaries. We set $\|\nabla w\|_p = \left(\sum_{|\alpha|=1} \int_\Omega |D^\alpha w|^p dx\right)^{1/p}$ and

$\|\nu^2 w\|_p = \left(\sum_{|\alpha|=2}\int_\Omega |D^\alpha w|^p dx\right)^{1/p}$, and suppress the subscripts when $p = 2$.

LEMMA 2. *Let Ω be an open set of R^3, with boundary $\partial\Omega$ uniformly of class C^2. Suppose $w \in J_0^*(\Omega)$ satisfies $\int_\Omega \nabla w : \nabla\phi dx = \int_\Omega f \cdot \phi dx$ for some $f \in L^2(\Omega)$ and all $\phi \in D(\Omega)$. Then w possesses second order derivatives $D^2 w \in L^2(\Omega)$ and the following inequalities hold:*

$$\|D^2 w\| \leqq C_\partial(\|Pf\| + \|\nabla w\|) , \tag{6}$$

$$\|\nabla w\|_3 \leqq C_\partial(\|Pf\|^{1/2}\|\nabla w\|^{1/2} + \|\nabla w\|) , \tag{7}$$

$$\sup_{x \in \Omega} |w(x)| \leqq C_\partial(\|Pf\| + \|\nabla w\|) , \tag{8}$$

where constants C_∂ depend only on the C^2-regularity of $\partial\Omega$ (but not on the 'size' of $\partial\Omega$ or Ω).

PROPOSITION 2. *Let $\Omega \subset R^3$ be an open set with a uniformly C^2 boundary. Then the condition in Proposition 1 is necessary as well as sufficient for $H_0^*(\Omega) \subset H_0(\Omega)$.*

Proof. Let $w \in J_0(\Omega)$ satisfy (2). Lemma 2 implies $D^2 w \in L^2(\Omega)$. Integrating by parts in the first integral of (2), and remembering that $\tilde{\Delta}(K_0(\Omega)) = D(\Omega)$, one obtains

$$\int_\Omega P\Delta w \cdot \tilde{\Delta}\phi dx = \int_\Omega w \cdot \tilde{\Delta}\phi dx = -\int_\Omega \nabla w : \nabla\phi dx ,$$

for all $\phi \in K_0(\Omega)$. Since $w \in H_0^*(\Omega) \subset H_0(\Omega)$ and $K_0(\Omega)$ is dense in $H_0(\Omega)$, it follows that $P\Delta w = \tilde{\Delta} w$ and $\|\nabla w\|^2 + \|\tilde{\Delta} w\|^2 = 0$. Thus $w \equiv 0$ in Ω.

REMARK 1. It follows immediately from Lemma 2 that the inverse inclusion $H_0(\Omega) \subset H_0^*(\Omega)$ holds if $\Omega \subset R^3$ and $\partial\Omega$ is uniformly C^2. In this case, $\tilde{\Delta} u = P\Delta u$ if $u \in H_0(\Omega)$.

REMARK 2. If Ω is a domain in which Poincare's inequality holds, i.e., $\|\phi\| \leqq C_\Omega \|\nabla\phi\|$ for some constant C_Ω and all $\phi \in C_0^\infty(\Omega)$, the condition in Proposition 1 is automatically satisfied; hence $H_0^*(\Omega) \subset H_0(\Omega)$.

REMARK 3. The inclusion $H_0^*(\Omega) \subset H_0(\Omega)$ fails to hold if Ω is a two-dimensional exterior domain with a smooth boundary. Indeed, let b be an infinitely differentiable solenoidal ($\nabla \cdot b = 0$) vector field in Ω such that $b = 0$ near $\partial\Omega$ and $b = (1, 0)$ in a neighborhood of infinity. Then $b \in J_0(\Omega)$ (see Heywood [7]). Now consider the function $f = b - \nabla q - \Delta b$ where $q = x_1$. Since $f \in L^2(\Omega)$, there is a unique element $\bar{b} \in J_1(\Omega)$ such that

$$(9) \qquad \int_\Omega \nabla \bar{b} : \nabla \phi dx + \int_\Omega \bar{b} \cdot \phi dx = \int_\Omega f \cdot \phi dx ,$$

for all $\phi \in D(\Omega)$. Using regularity results of Heywood [8, p. 82] and Friedmann [4, p. 66] (see also Amick [1, p. 704]), we can show $\Delta \bar{b} \in L^2(\Omega)$. Thus $b - \bar{b} \in H_0^*(\Omega)$, and an integration by parts in (9) gives

$$\int_\Omega (b - \bar{b}) \cdot \phi dx = \int_\Omega \Delta(b - \bar{b}) \cdot \phi dx ,$$

for all $\phi \in D(\Omega)$. Now if $H_0^*(\Omega) \subset H_0(\Omega)$, we can argue as in Proposition 2 to show $b \equiv \bar{b}$, which is impossible.

If w is an element of $J_0^*(\Omega)$ satisfying (2), then $w \in C^\infty(\Omega)$ and

$$(10) \qquad \Delta w - w = \nabla p$$

holds for some harmonic function p; see for example Heywood [8]. Thus, in view of Propositions 1 and 2, the question of whether $H_0^*(\Omega) \subset H_0(\Omega)$ for a given unbounded domain Ω can be reduced to that of whether there exist nontrivial solutions w of (10) satisfying:

$$(11) \qquad \nabla \cdot w = 0 \quad \text{in} \quad \Omega ,$$

$$(12) \qquad w = 0 \quad \text{on} \quad \partial\Omega ,$$

$$(13) \qquad w(x) \longrightarrow 0 \quad \text{as} \quad |x| \longrightarrow \infty .$$

We call an element w of $J_0^*(\Omega)$, which satisfies (2) for some $f \in L^2(\Omega)$ and all $\phi \in D(\Omega)$, a *weak solution* of the problem (10)-(13). Note that, if $\Omega \subset R^n (n > 2)$, elements of $J_0^*(\Omega)$ satisfy (13) in the generalized sense $\int_\Omega |w(x)|^2/|x|^2 dx < \infty$.

The next theorem, concerning the uniqueness of problem (10)-(13) in an exterior domain Ω, is proved under the assumption that $J_1^*(\Omega) = J_1(\Omega)$. This relation, and also the relation $J_0^*(\Omega) = J_0(\Omega)$, were established by Heywood [8] for several types of domains, including exterior domains with Lipschitz boundaries. Ladyzhenskaya and Solonnikov [13] have extended these results by weakening the assumptions on the boundary regularity.

THEOREM 1. *Let Ω be an exterior domain in $R^n (n > 2)$ for which $J_1^*(\Omega) = J_1(\Omega)$. Then the only weak solution of problem* (10)-(13) *is $w \equiv 0$; hence $H_0^*(\Omega) \subset H_0(\Omega)$ by Proposition* 1.

Proof. Let w be a weak solution of (10)-(13). Since $\int_\Omega w \cdot \phi dx =$

$-\int_{\Omega} f\cdot\phi dx$ holds for some $f\in L^2(\Omega)$ and all $\phi\in D(\Omega)$, there exists a scalar function q, with $q, q_{x_i}\in L^2_{\mathrm{loc}}(\Omega)$, such that

$$w + Pf = \nabla q ;$$

see Ladyzhenskaya [12], and also Heywood [6]. Thus, it follows from Lemma 1 that w belongs to $J_1(\Omega)$. Because $D(\Omega)$ is dense in $J_1(\Omega)$, the first identity of (2) implies $\int_{\Omega}(\nabla w:\nabla w + w\cdot w)dx = 0$; thus $w\equiv 0$.

In the case of a half space, we prove uniqueness for problem (10)-(13) using Fourier transforms. The method is a modification of one used in [8].

THEOREM 2. *Let* $\Omega = \{x\in R^n : x_1 > 0\}$ $(n \geqq 2)$. *Then the only weak solution of problem* (10)-(13) *is* $w\equiv 0$; *hence* $H_0^*(\Omega)\subset H_0(\Omega)$ *by Proposition* 1.

Proof. A weak solution w of (10)-(13) satisfies $\int_{\Omega}\nabla w:\nabla\phi dx = -\int_{\Omega} f\cdot\phi dx$ for some $f\in L^2(\Omega)$ and all $\phi\in D(\Omega)$. Thus, letting $\Omega_\varepsilon = \{x\in\Omega: x_1 > \varepsilon\}$, Lemma 9 of [8] implies $D^2 w\in L^2(\Omega_\varepsilon)$, for every $\varepsilon > 0$. Further, since $\int_{\Omega}\nabla D^\alpha w:\nabla\phi dx = -\int_{\Omega} D^\alpha w\cdot\phi dx$ for all $\phi\in D(\Omega)$, an induction argument gives $D^\alpha w\in L^2(\Omega_\varepsilon)$, for all α with $|\alpha|\geqq 1$. Also, since $w\in J_0^*(\Omega)$, we have

$$\int_{0<x_1<a} w^2 dx \leqq a^2\int_{0<x_1<a}|\nabla w|^2 dx ,$$

for every $a > 0$.

Now equation (2) implies there exists a harmonic function p such that $\Delta w - w = \nabla p$ in Ω. Clearly, $\int_{\varepsilon<x_1<a}(\nabla p)^2 dx < \infty$ if $0<\varepsilon<a$. Hence one can take Fourier transforms with respect to $\tilde{x} = (x_2, \cdots, x_n)$ of equation $\Delta(\partial p/\partial x_i) = 0$, obtaining

$$\frac{\partial^2}{\partial x_1^2}\left(\frac{\partial p}{\partial x_i}\right)^{\wedge} - |\xi|^2\left(\frac{\partial p}{\partial x_i}\right)^{\wedge} = 0 .$$

Here $\hat{h} = \hat{h}(x_1, \xi) = (2\pi)^{-(n-1)/2}\int_{R^{n-1}} h(x_1, \tilde{x})e^{-i\tilde{x}\cdot\xi}d\tilde{x}$, where $\xi = (\xi_2, \cdots, \xi_n)$, and $|\xi|^2 = \xi_2^2 + \cdots + \xi_n^2$. The general solution of this equation is

$$\left(\frac{\partial p}{\partial x_i}\right)^{\wedge} = \alpha_i(\xi)e^{-|\xi|x_1} + \beta_i(\xi)e^{|\xi|x_1} .$$

Since $\partial^2 p/\partial x_1\partial x_i = -\partial w_1/\partial x_i + \partial/\partial x_i\Delta w_1$ and $D^3 w\in L^2(\Omega_\varepsilon)$, we have $\partial^2 p/\partial x_1\partial x_i\in L^2(\Omega_\varepsilon)$ for every $\varepsilon > 0$. Thus, in virtue of Parserval's identity

$$\int_\varepsilon^\infty \int_{R^{n-1}} \left|\left(\frac{\partial^2 p}{\partial x_1 \partial x_i}\right)^\wedge\right|^2 d\xi dx_1 = \int_{\Omega_\varepsilon} \left|\frac{\partial^2 p}{\partial x_1 \partial x_i}\right|^2 dx < \infty ,$$

we see that $\beta_i(\xi)$ $(i = 1, \cdots, n)$ vanishes almost everywhere.

For every $i = 1, 2, \cdots, n$, taking Fourier transforms of the equation $\Delta w_i - w_i = \partial p/\partial x_i$, one obtains

$$\frac{\partial^2}{\partial x_1^2}\hat{w}_i - (|\xi|^2 + 1)\hat{w}_i = \alpha_i(\xi) e^{-|\xi| x_1} . \tag{14}$$

The general solution of (14) can be found, by the method of variation of parameters, to be

$$\hat{w}_i = a_i(\xi) e^{-x_1\sqrt{|\xi|^2+1}} + b_i(\xi) e^{x_1\sqrt{|\xi|^2+1}} + \alpha_i(\xi)\gamma(\xi) e^{-|\xi| x_1} , \tag{15}$$

where

$$\gamma(\xi) = \frac{-1}{2\sqrt{|\xi|^2+1}}\left[\frac{1}{\sqrt{|\xi|^2+1}+|\xi|} + \frac{1}{\sqrt{|\xi|^2+1}-|\xi|}\right] .$$

Again, using Parserval's identity

$$\int_0^\infty \int_{R^{n-1}} \left|\frac{\partial \hat{w}_i}{\partial x_1}\right|^2 d\xi dx_1 = \int_\Omega \left|\frac{\partial w_i}{\partial x_1}\right|^2 dx < \infty ,$$

we find $b_i(\xi) = 0$.

Finally, the Fourier transform of $\nabla \cdot w = 0$ is

$$\frac{\partial \hat{w}_1}{\partial x_1} + i\xi_2 \hat{w}_2 + \cdots + i\xi_n \hat{w}_n = 0 ,$$

and so the boundary conditions $\hat{w}_i(0, \xi) = 0$ $(i = 2, \cdots, n)$ imply $\partial\hat{w}_1/\partial x_1(0, \xi)^\wedge = 0$. It follows that $a_1(\xi) = \alpha_1(\xi) = 0$. Consequently $(\partial p/\partial x_1)^\wedge = 0$ and so $\partial p/\partial x_1 = 0$. Now the function p, being harmonic in the variables $\tilde{x} \in R^{n-1}$ with $\int_{R^{n-1}} |\nabla p|^2 d\tilde{x} < \infty$, is a constant. Hence $\alpha_i(\xi) = 0$, which in turn implies $a_i(\xi) = 0$. This completes the proof.

Finally we consider aperture domains.

THEOREM 3. *Let* $\Omega = \{x \in R^n : x_1 \neq 0$ *or* $(x_2, \cdots, x_n) \in S\}$, *where* $n = 2$ *or* 3 *and* S *is a bounded open set in* $A = \{x \in R^n : x_1 = 0\}$. *Then the only weak solution* w *of problem* (10)-(13), *with* $w \in J_0(\Omega)$, *is* $w \equiv 0$; *hence* $H_0^*(\Omega) \subset H_0(\Omega)$ *by Proposition* 1.

Proof. Consider first the case $n = 3$. Suppose $w \in J_0(\Omega)$ is a weak solution of (10)-(13). Let B_R be a ball of radius R centered at the origin (assumed to lie in S) such that $S \subset B_R \cap A$. Since $w \in J_0(\Omega)$, one can show $\int_S w \cdot n ds = 0$, where n is the unit normal to the surface S; see [8, p. 93]. By Corollary 2.3 of [13], there exists a solenoidal vector v' in $W_2^1(B_R')$, where $B_R' = \{x \in B_R : x_1 > 0\}$, such that $v' = w$ on S and $v' = 0$ on $\partial B_R' \backslash S$. Likewise, there exists a solenoidal extension $v'' \in W_2^1(B_R'')$ of w on S, where $B_R'' = \{x \in B_R: x_1 < 0\}$. Let v be a vector defined in $B_R \cap \Omega$, with $v = w$ on $B_R \cap A$, such that $v = v'$ in B_R' and $v = v''$ in B_R''. Then v belongs to $J_1^*(B_R \cap \Omega)$ and thus to $J_1(B_R \cap \Omega)$; see [13], and also [8].

We extend v' to Ω', where $\Omega' = \{x \in \Omega: x_1 > 0\}$, by setting it equal to zero outside of B_R'. For every $\phi \in J_1(\Omega')$, let $F(\phi) = -\int_{\Omega'} (v' \cdot \phi + \nabla v' : \nabla \phi) dx$. Then F defines a bounded linear functional on $J_1(\Omega')$ and so the Riesz representation theorem gives a unique vector u' in $J_1(\Omega')$ such that

$$\int_{\Omega'} (u' \cdot \phi + \nabla u' : \nabla \phi) dx = F(\phi) ,$$

for all $\phi \in J_1(\Omega')$. Let $\bar{w} = w' - (u' + v')$, where w' is the restriction of w to Ω'. We can easily show $\bar{w} \in W_0(\Omega')$, $\nabla \cdot \bar{w} = 0$ and

$$\int_{\Omega'} \nabla \bar{w} : \nabla \phi dx = -\int_{\Omega'} \bar{w} \cdot \phi dx = \int_{\Omega'} (f + u' + v') \cdot \phi dx ,$$

for all $\phi \in D(\Omega')$. Thus Theorem 2 implies $\bar{w} = 0$ in Ω', and so $w' = u' + v'$ in Ω'. Similarly, if w'' is the restriction of w to Ω'', where $\Omega'' = \{x \in \Omega: x_1 < 0\}$, then $w'' = u'' + v''$ for some $u'' \in J_1(\Omega'')$. Hence $w = w' + w'' = u' + u'' + v$ belongs to $J_1(\Omega)$, because $J_1(\Omega')$, $J_1(\Omega'')$ and $J_1(B_R \cap \Omega)$ are all subspaces of $J_1(\Omega)$. By letting ϕ tend to w in (2), we obtain $\int_\Omega (w \cdot w + \nabla w : \nabla w) dx = 0$, which implies $w \equiv 0$.

For $n = 2$, let B' be a bounded open subset of the right half space Ω' such that $\partial B' \cap A = S$ and $\partial B'$ is smooth. Using a method given in [12, p. 27], one can construct a solenoidal vector $v' \in W_2^1(B')$ which equals w on S and zero on $\partial B' \backslash S$. Similarly, construct B'' and v'' in the left half space Ω''. Proceeding now as in the three-dimensional case, we can show $w \equiv 0$.

3. Uniqueness and continuous dependence. Let Ω be a domain in $R^n (n \geqq 2)$. The initial boundary value problem for the Navier-Stokes equations in the space-time cylinder $\Omega \times (0, T)$ is to find a pair of functions u, p which satisfies

$$u_t + u \cdot \nabla u = -\nabla p + \nu \Delta u + f \quad \text{in} \quad \Omega \times (0, T) , \tag{16a}$$

$$\nabla \cdot u = 0 \quad \text{in} \quad \Omega \times (0, T) , \tag{16b}$$

$$u(x, 0) = a(x) \quad \text{for} \quad x \in \Omega , \tag{16c}$$

(16d) $\quad u(x, t) = b^*(x, t) \qquad \text{for} \quad (x, t) \in \partial\Omega \times (0, T)$,

(16e) if Ω is bounded, $u(x, t) \to b_\infty^*(t)$ as $|x| \to \infty$ for $t \in (0, T)$,

(16f) if Ω is a domain for which $J_0^*(\Omega) \neq J_0(\Omega)$, auxiliary conditions in the sense given in [8] are imposed to determine, for every t, a specific coset of $J_0^*(\Omega)/J_0(\Omega)$.

Here, u represents the velocity vector, p the pressure, f the external force, and ν the constant kinematic viscosity. We assume the boundary values b^* can be extended continuously into $\Omega \times (0, T)$ as a solenoidal function $b \in C^2(\bar{\Omega} \times [0, T])$ which satisfies (16d), (16e), (16f) and the following conditions:

$$\sup_{t \in [0,T]} ||\nabla b(t)|| < \infty , \quad \int_0^t ||D_x^2 b(\tau)||^2 d\tau < \infty \quad \text{for} \quad t \in (0, T) , \tag{17a}$$

$$\begin{gathered} b_t(x, t) \longrightarrow b_{\infty t}^*(t) \quad \text{as} \quad |x| \longrightarrow \infty , \\ \int_0^t ||\nabla b_t(\tau)||^2 d\tau < \infty \quad \text{for} \quad t \in (0, T) , \end{gathered} \tag{17b}$$

$$\int_0^t \sup_\Omega |b(\tau)|^2 d\tau < \infty \quad \text{for} \quad t \in (0, T) , \tag{17c}$$

$$a - b(\cdot, 0) \in J_0(\Omega) , \tag{17d}$$

(17e) the forcing term $g \equiv f - b_t + \nu \Delta b - b \cdot \nabla b$ admits the decomposition $g = f_1 + f_2 + \nabla q$, where $f_1 \in L^2(0, t; J_0(\Omega))$ and $f_2 \in L^2(0, t; L^2(\Omega))$ for all $t \in (0, T)$, and where $q, q_{x_i} \in L^2_{\text{loc}}(\Omega \times (0, T))$.

Any such extension b of the boundary values is said to be *admissible*.

We note, if $\partial\Omega$ is of class C^2, that condition (17a) implies

$$\int_0^t ||\nabla b(\tau)||_3^2 d\tau < \infty \quad \text{for} \quad t \in (0, T) . \tag{17f}$$

This is proved by substituting ∇b for ϕ in the Sobolev inequality

$$||\phi||_3 \leqq C_\partial(||\nabla\phi||^{1/2}||\phi||^{1/2} + ||\phi||) ,$$

which is valid for all $\phi \in W_2^1(\Omega)$ with a constant C_∂ dependent only on the C^2-regularity of $\partial\Omega$; see Friedman [4, p. 27]. If, furthermore, Ω is an exterior domain in R^3 with a class C^2 boundary, then

(i) (17a) implies (17c) provided $\int_0^T |b_\infty^*(t)|^2 dt < \infty$, and

(ii) every vector field $g(x)$, with $\int_\Omega (\nabla g)^2 dx < \infty$, can be expressed as a sum described in (17e) above;

see [9] and [10].

The solution of problem (16) is sought in the form $u = v + b$, where b is an admissible extension and v satisfies

$$v \in L^2(0, T; H_0(\Omega)), \tag{18a}$$

$$\sup_{t \in (0,T)} \|\nabla v\| < \infty, \tag{18b}$$

$$\begin{aligned} &\int_0^T (\nabla v, \nabla \phi_t)dt - \nu \int_0^T (\tilde{\Delta} v, \tilde{\Delta}\phi)dt + \int_0^T (v \cdot \nabla v + b \cdot \nabla v + v \cdot \nabla b, \tilde{\Delta}\phi)dt \\ &\qquad = -(\nabla(a - b(0)), \nabla\phi(0)) + \int_0^T (g, \tilde{\Delta}\phi)dt \end{aligned} \tag{18c}$$

for all $\phi \in S_T$, where $S_T = \{\phi\colon \phi, \phi_t \in L^2(0, T; K_0(\Omega)),\ \phi(\cdot, 0) \in K_0(\Omega)$ and $\phi(\cdot, T) \equiv 0\}$.

We call such a function u a *weak class* H_0 solution of problem (16).

Equation (18c) is obtained formally by multiplying (16a) by $\tilde{\Delta}\phi$, integrating over $\Omega \times (0, T)$ and performing several integration by parts. All integrals in (18c) make sense because $\tilde{\Delta}\phi$ has compact support in Ω. Conversely, (18c) implies (16a) holds with some scalar function p such that $p, p_{x_i} \in L^2_{\mathrm{loc}}(\Omega \times (0, T))$, provided v satisfies the conditions (18) and also $v_t \in L^2(\varepsilon, T; J_0(\Omega))$ for all positive $\varepsilon < T$.

In what follows, we only consider domains $\Omega \subset R^3$ for which $H_0^*(\Omega) \subset H_0(\Omega)$ and for which $\partial\Omega$ is regular enough (say, uniformly C^2) so that the estimates in Lemma 2 hold.

PROPOSITION 3. *Suppose $u = v + b$ is a weak class H_0 solution of problem* (16), *where b is an admissible extension and v satisfies* (18). *Then v, after redefinition on a set of t-measure zero, satisfies the identity*

$$\begin{aligned} &\int_0^t (\nabla v, \nabla \phi_t)d\tau - \nu \int_0^t (\tilde{\Delta} v, \tilde{\Delta}\phi)d\tau + \int_0^t (v \cdot \nabla v + b \cdot \nabla v + v \cdot \nabla b, \tilde{\Delta}\phi)d\tau \\ &\qquad = (\nabla v(t), \nabla\phi(t)) - (\nabla(a - b(0)), \nabla\phi(0)) + \int_0^t (g, \tilde{\Delta}\phi)d\tau, \end{aligned} \tag{19}$$

for all $\phi \in S_T$ and all $t \in (0, T)$.

Proof. Let $\phi \in S_T$. For any fixed $t_0 \in (0, T)$ and every $\delta \in (0, T - t_0)$, let

$$\phi_\delta(t) = \begin{cases} \phi(t) & \text{for } 0 \leqq t \leqq t_0 \\ \delta^{-1}(t_0 + \delta - t)\phi(t_0) & \text{for } t_0 \leqq t \leqq t_0 + \delta \\ 0 & \text{for } t_0 + \delta \leqq t \leqq T. \end{cases}$$

Then $\phi_\delta \in S_T$ and can be substituted for ϕ in (18c). Note that

$$\begin{aligned}\int_0^T (\nabla v, \nabla\phi_{\delta t})dt &= \int_0^{t_0} (\nabla v, \nabla\phi_t)dt - \delta^{-1}\int_{t_0}^{t_0+\delta} (\nabla v(t), \nabla\phi(t_0))dt \\ &= \int_0^{t_0} (\nabla v, \nabla\phi_t)dt - \left(\delta^{-1}\int_{t_0}^{t_0+\delta} \nabla v(t)dt, \nabla\phi(t_0)\right).\end{aligned} \tag{20}$$

The last term in (20) converges to $(\nabla v(t_0), \nabla\phi(t_0))$ as $\delta \to 0$ if t_0 belongs to the Lebesque set M of v, i.e., $M = \{\tau: \lim_{\delta\to 0}\delta^{-1}\int_\tau^{\tau+\delta} v(t)dt = v(\tau)$ strongly in $J_0(\Omega)\}$. The set M has the property that $\operatorname{mes}((0, T) - M) = 0$; see for example [11, p. 88]. Next, it is easy to see, as $\delta \to 0$, that

$$\int_0^T (\tilde\Delta v, \tilde\Delta\phi_\delta)dt \longrightarrow \int_0^{t_0} (\tilde\Delta v, \tilde\Delta\phi)dt\,,$$

$$\int_0^T (G(v), \tilde\Delta\phi_\delta)dt \longrightarrow \int_0^{t_0} (G(v), \tilde\Delta\phi)dt\,,$$

$$\int_0^T (g, \tilde\Delta\phi_\delta)dt \longrightarrow \int_0^{t_0} (g, \tilde\Delta\phi)dt\,,$$

where we have denoted the term $v\cdot\nabla v + b\cdot\nabla v + v\cdot\nabla b$ by $G(v)$. Thus, if $t_0 \in M$, (19) holds for all $\phi \in S_T$.

Now let t_0 be an arbitrary instant of time in $[0, T)$ and let $\{t_j\} \subset M$ converge to t_0. Then there exists a subsequence $\{t_{j_k}\}$ and a function $V(x, t_0) \in J_0(\Omega)$ such that $v(t_{j_k}) \to V(t_0)$ weakly in $J_0(\Omega)$ as $k \to \infty$. Letting $t = t_{j_k}$ and $k \to \infty$ in (19), we obtain

$$\begin{aligned}\int_0^{t_0} \{(\nabla v, \nabla\phi_t) - \nu(\tilde\Delta v, \tilde\Delta\phi) + (G(v), \tilde\Delta\phi) - (g, \tilde\Delta\phi)\}dt \\ = (\nabla V(t_0), \nabla\phi(t_0)) - (\nabla(a - b(0)), \nabla\phi(0))\,,\end{aligned}$$

for all $\phi \in S_T$. Clearly, if $t_0 \in M$, $(\nabla v(t_0), \nabla\phi(t_0)) = (\nabla V(t_0), \nabla\phi(t_0))$ holds for all $\phi \in S_T$, and in particular, for all $\phi \in D(\Omega \times [0, T))$. It follows that $V(x, t_0) = v(x, t_0)$ for $x \in \Omega$. Note also $V(x, 0) = a(x) - b(x, 0)$. Thus, if we redefine $v(x, t)$ by setting $v(x, t) \equiv V(x, t)$ for $t \notin M$, then (19) holds for all $t \in (0, T)$.

REMARK. The redefined function $v(x, t)$ is weakly continuous in $J_0(\Omega)$ as a function of t. Indeed, it suffices to observe that (19) implies $(\nabla v(t), \nabla\phi) \to (\nabla v(t_0), \nabla\phi)$ as $t \to t_0$, for all $\phi \in D(\Omega)$.

LEMMA 3. *Suppose u is a weak class H_0 solution of problem* (16), *say $u = v + b$, where b is an admissible extension of the boundary values and together v and b satisfy* (18) *with initial value a. Let $\bar b$ be any other admissible extension and set $\bar v = u - \bar b$. Then together $\bar v$ and $\bar b$ satisfy* (18) *with initial value a.*

Proof. Clearly $\bar v = v + b - \bar b$. Hence if $b - \bar b$ satisfies (18a) and

(18b), so does $\bar{v}$. The assumptions on b and $\bar{b}$ imply $b - \bar{b} \in C^2(\bar{\Omega} \times [0, T])$, $\nabla \cdot (b - \bar{b}) = 0$ in $\Omega \times (0, T)$, $\sup_{t\in(0,T)} \|\nabla(b - \bar{b})(t)\| < \infty$, $b - \bar{b} = 0$ on $\partial\Omega \times (0, T)$ and $(b - \bar{b})(x, t) \to 0$ continuously as $|x| \to \infty$ for $t \in (0, T)$. Thus $(b - \bar{b})(\cdot, t) \in J_0^*(\Omega)$ for $t \in (0, T)$ by Lemma 4 of [8]. Since b and $\bar{b}$ both satisfy the auxiliary condition (16f), it follows that $(b - \bar{b})(\cdot, t) \in J_0(\Omega)$ for every $t \in (0, T)$. Also $\int_0^T \|\Delta(b - \bar{b})(t)\|^2 dt < \infty$; so $b - \bar{b} \in L^2(0, T; H_0^*(\Omega)) \subset L^2(0, T; H_0(\Omega))$.

Arguing as above, we show $(b - \bar{b})_t(\cdot, t) \in J_0(\Omega)$ for every $t \in (0, T)$. It follows that $(\nabla(b - \bar{b})_t, \nabla\phi) = -((b - \bar{b})_t, \tilde{\Delta}\phi)$ for all $\phi \in K_0(\Omega)$. Thus, integrating by parts with respect to t, we obtain

$$\int_0^T (\nabla\bar{v}, \nabla\phi_t)dt = \int_0^T (\nabla v, \nabla\phi_t)dt + \int_0^T ((b - \bar{b})_t, \tilde{\Delta}\phi)dt - (\nabla(b(0) - \bar{b}(0)), \nabla\phi(0))$$

for all $\phi \in S_T$. Also, a simple calculation gives

$$\bar{v} \cdot \nabla\bar{v} + \bar{b} \cdot \nabla\bar{v} + \bar{v} \cdot \nabla\bar{b} = v \cdot \nabla v + b \cdot \nabla v + v \cdot \nabla b + (b \cdot \nabla b - \bar{b} \cdot \nabla\bar{b}) .$$

It follows easily now that $\bar{v}$ and $\bar{b}$ satisfy a similar identity to (18c) with $\bar{g} = f - \bar{b}_t + \nu\Delta\bar{b} - \bar{b} \cdot \nabla\bar{b}$.

We now proceed to prove the continuous dependence and uniqueness of solutions of problem (16). Suppose u and $\bar{u}$ are two weak class H_0 solutions of (16), having the same boundary values but possibly different initial values a with $\bar{a}$ respectively. Let $a - \bar{a} \in J_0(\Omega)$ and let b be an admissible extension of the boundary values, with $a - b(\cdot, 0) \in J_0(\Omega)$. Then $\bar{a} - b(\cdot, 0) = \bar{a} - a + a - b(\cdot, 0)$ also belongs to $J_0(\Omega)$. By Lemma 3, $v = u - b$ and $\bar{v} = \bar{u} - b$ both satisfy conditions (18) with the same extension b, and with initial values a and $\bar{a}$ respectively. Our first objective is to show that the difference $w = u - \bar{u}$ satisfies the identity

$$\begin{aligned} \frac{1}{2}\|\nabla w(t)\|^2 + \nu\int_0^t \|\tilde{\Delta}w(\tau)\|^2 d\tau &= \frac{1}{2}\|\nabla(a - \bar{a})\|^2 \\ &\quad + \int_0^t (v \cdot \nabla v - \bar{v} \cdot \nabla\bar{v}, \tilde{\Delta}w)d\tau \\ &\quad + \int_0^t (b \cdot \nabla w + w \cdot \nabla b, \tilde{\Delta}w)d\tau , \end{aligned} \tag{21}$$

for every $t \in (0, T)$.

Since $\bar{v} \in L^2(0, T; H_0(\Omega))$, there exists a sequence $\{\bar{v}^k\}$ in $L^2(0, T; K_0(\Omega))$ such that

$$\int_0^T (\|\nabla\bar{v}^k - \nabla\bar{v}\|^2 + \|\tilde{\Delta}\bar{v}^k - \tilde{\Delta}\bar{v}\|^2)dt \longrightarrow 0$$

as $k \to \infty$. For fixed $t \in (0, T)$ and every k, let

$$\bar{v}_\rho^k(x, \tau) = \int_0^t \omega_\rho(\tau - \eta)\bar{v}^k(x, \eta)d\eta ,$$

where $\omega_\rho \in C_0^\infty(R)$ is a kernel with support contained in $\{\tau: |\tau| < \rho < \min(t, |\tau - t|)\}$ such that $\omega_\rho(\tau) = \omega_\rho(-\tau)$ and $\int_{-\infty}^\infty \omega_\rho(\tau)d\tau = 1$. Clearly every $\bar{v}_\rho^k$ belongs to S_T, and can therefore be substituted for ϕ in (19). Since $\omega'_\rho(\tau) = -\omega'_\rho(-\tau)$, an application of Fubini's theorem yields

$$\int_0^t (\nabla v(\tau), \nabla \bar{v}_{\rho t}^k(\tau))d\tau = -\int_0^t (\nabla v_{\rho t}(\tau), \nabla \bar{v}^k(\tau))d\tau , \tag{22}$$

where $v_\rho(x, \tau) = \int_0^t \omega_\rho(\tau - \eta)v(x, \eta)d\eta$. The right side of (22) converges to $-\int_0^t (\nabla v_{\rho t}, \nabla \bar{v})d\tau = \int_0^t (\nabla v, \nabla \bar{v}_{\rho t})d\tau$, as $k \to \infty$. Using conditions (17) and (18a), we can show $v \cdot \nabla v$, $b \cdot \nabla v$, $v \cdot \nabla b \in L^2(0, T; L^2(\Omega))$. For instance, applying Holder's inequality, Lemma 2, and the well-known Sobolev inequality

$$\|v\|_6 \leqq C\|\nabla v\| , \tag{23}$$

valid for $v \in C_0^\infty(R^3)$, we have

$$\begin{aligned} \|v \cdot \nabla v\| \leqq \|v\|_6\|\nabla v\|_3 &\leqq C\|\nabla v\|(\|\nabla v\|^{1/2}\|\tilde{\Delta} v\|^{1/2} + \|\nabla v\|) \\ &\leqq C(\|\nabla v\|^2 + \|\tilde{\Delta} v\|^2) . \end{aligned}$$

Here, the term $\|\nabla v\|^{3/2}\|\tilde{\Delta} v\|^{1/2}$ was estimated using Young's inequality: $ab \leqq a^p/p + b^q/q$ with $p = 4/3$ and $q = 4$. Now $G(v) = v \cdot \nabla v + b \cdot \nabla v + v \cdot \nabla b \in L^2(0, T; L^2(\Omega))$ and it follows that

$$\int_0^t (G(v), \tilde{\Delta}\bar{v}_\rho^k)d\tau \longrightarrow \int_0^t (G(v), \tilde{\Delta}\bar{v}_\rho)d\tau$$

as $k \to \infty$. Thus the following identity is obtained by letting $\phi = \bar{v}_\rho^k$ and $k \to \infty$ in (19):

$$\begin{aligned} \int_0^t (\nabla v, \nabla \bar{v}_{\rho t})d\tau - \nu\int_0^t (\tilde{\Delta} v, \tilde{\Delta}\bar{v}_\rho)d\tau + \int_0^t (G(v), \tilde{\Delta}\bar{v}_\rho)d\tau \\ = (\nabla v(t), \nabla \bar{v}_\rho(t)) - (\nabla(a - b(0)), \nabla \bar{v}_\rho(0)) \\ + \int_0^t \{-(\nabla f_1, \nabla \bar{v}_\rho) + (f_2, \tilde{\Delta}\bar{v}_\rho)\}d\tau . \end{aligned}$$

If we add this identity to a similar one with the roles of v and $\bar{v}$ interchanged, and note that

$$\int_0^t (\nabla v, \nabla \bar{v}_{\rho t})d\tau = -\int_0^t (\nabla \bar{v}, \nabla v_{\rho t})d\tau ,$$

the result is

$$
\begin{aligned}
&-\nu\int_0^t\{(\tilde{\Delta}v,\ \tilde{\Delta}\bar{v}_\rho)+(\tilde{\Delta}\bar{v},\ \tilde{\Delta}v_\rho)\}d\tau+\int_0^t\{(G(v),\ \tilde{\Delta}\bar{v}_\rho)+(G(\bar{v}),\ \tilde{\Delta}v_\rho)\}d\tau\\
&\quad=(\nabla v(t),\ \nabla\bar{v}_\rho(t))+(\nabla\bar{v}(t),\ \nabla v_\rho(t))-(\nabla(a-b(0)),\ \nabla\bar{v}_\rho(0))\\
&\qquad-(\nabla(\bar{a}-b(0)),\ \nabla v_\rho(0))+\int_0^t\{-(\nabla f_1,\ \nabla v_\rho+\nabla\bar{v}_\rho)+(f_2,\ \tilde{\Delta}v_\rho+\tilde{\Delta}\bar{v}_\rho)\}d\tau\ .
\end{aligned}
\tag{24}
$$

We study the behavior of each of the integrals in (24) as $\rho\to 0$. Using the fact that $\lim_{\sigma\to 0}\int_0^t\|\bar{v}(\tau+\sigma)-\bar{v}(\tau)\|_{H_0}^2d\tau=0$ (c.f. [11, p. 86]), one can show $\lim_{\rho\to 0}\int_0^t\|\bar{v}_\rho-\bar{v}\|_{H_0}^2d\tau=0$. This implies $\int_0^t(\tilde{\Delta}v,\ \tilde{\Delta}\bar{v}_\rho)d\tau\to\int_0^t(\tilde{\Delta}v,\ \tilde{\Delta}\bar{v})d\tau$ as $\rho\to 0$, and also the convergence of all other integrals with respect to t in (24). Next, to show

$$
(\nabla v(t),\ \nabla\bar{v}_\rho(t))\longrightarrow\frac{1}{2}(\nabla v(t),\ \nabla\bar{v}(t))\quad\text{as}\quad\rho\longrightarrow 0\ ,
\tag{25}
$$

we recall that $\bar{v}$ is weakly continuous in $J_0(\Omega)$ as a function of t. Thus,

$$
(\nabla v(t),\ \nabla\bar{v}_\rho(t))=\int_0^\rho\omega_\rho(\eta)[(\nabla v(t),\ \nabla\bar{v}(t))+\varepsilon(\eta)]d\eta\ ,
$$

where $\varepsilon(\eta)\to 0$ as $\eta\to 0$. This implies (25), since $\int_0^\rho\omega_\rho(\eta)d\eta=1/2$. The convergence of the remaining terms in (24), as $\rho\to 0$, is handled similarly. Now, letting $\rho\to 0$ in (24), we obtain

$$
\begin{aligned}
&-2\nu\int_0^t(\tilde{\Delta}v,\ \tilde{\Delta}\bar{v})d\tau+\int_0^t\{(G(v),\ \tilde{\Delta}\bar{v})+(G(\bar{v}),\ \tilde{\Delta}v)\}d\tau\\
&\quad=(\nabla v(t),\ \nabla\bar{v}(t))-(\nabla(a-b(0)),\ \nabla(\bar{a}-b(0)))\\
&\qquad+\int_0^t\{-(\nabla f_1,\ \nabla(v+\bar{v}))+(f_2,\ \tilde{\Delta}(v+\bar{v}))\}d\tau\ .
\end{aligned}
\tag{26}
$$

Replacing $\bar{v}$ and $\bar{a}$ in (26) by v and a, respectively, and vice versa, we find

$$
\begin{aligned}
&-\nu\int_0^t\|\tilde{\Delta}v(\tau)\|^2d\tau+\int_0^t(G(v),\ \tilde{\Delta}v)d\tau\\
&\quad=\frac{1}{2}\|\nabla v(t)\|^2-\frac{1}{2}\|\nabla(a-b(0))\|^2\\
&\qquad+\int_0^t\{-(\nabla f_1,\ \nabla v)+(f_2,\ \tilde{\Delta}v)\}d\tau\ ,
\end{aligned}
$$

and a similar identity for $\bar{v}$. Adding these identities for v and $\bar{v}$, and subtracting (26), gives the identity (21).

Since $v\cdot\nabla v-\bar{v}\cdot\nabla\bar{v}=w\cdot\nabla w+w\cdot\nabla v+v\cdot\nabla w$, (21) can be rewritten as

$$\frac{1}{2}||\nabla w(t)||^2+\nu\int_0^t||\tilde{\Delta}w(\tau)||^2d\tau = \frac{1}{2}||\nabla(a-\bar{a})||^2+\int_0^t(w\cdot\nabla w,\ \tilde{\Delta}w)d\tau$$
$$+\int_0^t((v+b)\cdot\nabla w + w\cdot\nabla(v+b),\ \tilde{\Delta}w)d\tau\ . \tag{27}$$

The second term on the right side of (27) can be estimated using Holder's ineguality, (7), (23) and Young's inequality:

$$\begin{aligned}|(w\cdot\nabla w,\ \tilde{\Delta}w)| &\leqq ||w||_6||\nabla w||_3||\tilde{\Delta}w||\\ &\leqq C_\partial||\nabla w||(||\nabla w||^{1/2}||\tilde{\Delta}w||^{1/2}+||\nabla w||)||\tilde{\Delta}w||\\ &\leqq \alpha||\tilde{\Delta}w||^2+C_{\partial,\alpha}||\nabla w||^6+C_{\partial,\alpha}||\nabla w||^4\\ &\leqq \alpha||\tilde{\Delta}w||^2+C_{\partial,\alpha}||\nabla w||^6+C_{\partial,\alpha}||\nabla w||^2\ .\end{aligned}$$

Here, $C_{\partial,\alpha}$ denotes a constant dependent only on $\alpha>0$ and the regularity of $\partial\Omega$. The last term on the right side of (27) can also be estimated using Holder's inequality, (7), (8), (23) and Young's inequality:

$$\begin{aligned}((v+b)\cdot\nabla w,\ \tilde{\Delta}w) &\leqq \sup_\Omega|v+b|\,||\nabla w||\,||\tilde{\Delta}w||\\ &\leqq \alpha||\tilde{\Delta}w||^2+C_{\partial,\alpha}(||\tilde{\Delta}v||^2+||\nabla v||^2+\sup_\Omega|b|^2)||\nabla w||^2\ ,\end{aligned} \tag{29}$$

$$\begin{aligned}(w\cdot\nabla(v+b),\ \tilde{\Delta}w) &\leqq ||w||_6(||\nabla v||_3+||\nabla b||_3)||\tilde{\Delta}w||\\ &\leqq \alpha||\tilde{\Delta}w||^2+C_{\partial,\alpha}(||\tilde{\Delta}v||^2+||\nabla v||^2+||\nabla b||_3^2)||\nabla w||^2\ .\end{aligned} \tag{30}$$

Setting $\alpha=\nu/3$, we combine these estimates for terms on the right side of (27) obtaining

$$||\nabla w(t)||^2 \leqq ||\nabla(a-\bar{a})||^2+\int_0^t(h(\tau)||\nabla w(\tau)||^2+\sigma||\nabla w(\tau)||^6)d\tau\ , \tag{31}$$

where $h(\tau)=\sigma(||\tilde{\Delta}v||^2+||\nabla v||^2+\sup_\Omega|b|^2+||\nabla b||_3^2)$, and σ is a constant dependent only on ν and the regularity of $\partial\Omega$.

The function $R(t)\equiv||\nabla(a-\bar{a})||^2+\int_0^t(h(\tau)||\nabla w(\tau)||^2+\sigma||\nabla w(\tau)||^6)d\tau$ is continuous on $[0, T]$ and absolutely continuous on $(0, T)$, as a function of t. Using (31), we obtain

$$\begin{aligned}\frac{d}{dt}R(t) &= h(t)||\nabla w(t)||^2+\sigma||\nabla w(t)||^6\\ &\leqq (h(t)+\sigma||\nabla w(t)||^4)R(t)\ .\end{aligned}$$

It follows that

$$R(t)\leqq||\nabla(a-\bar{a})||^2\exp\int_0^t(h(\tau)+\sigma||\nabla w(\tau)||^4)d\tau\ , \tag{32}$$

for all $t\in[0, T]$. Now, suppose $R(0)=||\nabla(a-\bar{a})||^2<A$ for a given positive number A, and let $[0, T_A]$ be the largest subinterval of

$[0, T]$ on which $R(t) \leqq A$. Then, for all $t \in (0, T_A)$, we have $\|\nabla w(t)\|^2 \leqq R(t) \leqq A$ and, by (32),

$$R(t) \leqq \|\nabla(a - \bar{a})\|^2 \exp(\sigma A^2 t + H(t)) ,$$

where $H(t) = \int_0^t h(\tau)d\tau$. Choose $A = T^{-1/2}$ and suppose $\|\nabla(a - \bar{a})\|^2 \leqq Ae^{-\sigma - H(T)}$. Then $R(T) \leqq A$, which implies $T_A = T$. We have proved

THEOREM 4. *Let $\Omega \subset R^3$ be a domain with a uniformly C^2 boundary, for which $H_0^*(\Omega) \subset H_0(\Omega)$. Let u and $\bar{u}$ be two weak class H_0 solutions of problem* (16) *with the same prescribed boundary values and forces b^*, b^*_∞, f, but with possibly different initial values a and $\bar{a}$ respectively. Let b be any admissible extension of the boundary values, with $a - b(0) \in J_0(\Omega)$. Suppose $a - \bar{a} \in J_0(\Omega)$ and*

$$\|\nabla(a - \bar{a})\|^2 \leqq T^{-1/2} \exp(-\sigma - H(T)) ,$$

where σ is a constant dependent only on ν and $\partial\Omega$, and

$$H(t) = \sigma \int_0^t (\|\tilde{\Delta}(u - b)\|^2 + \|\nabla(u - b)\|^2 + \sup_\Omega |b|^2 + \|\nabla b\|_3^2)d\tau .$$

Then, for all $t \in [0, T)$,

$$\|\nabla u(t) - \nabla\bar{u}(t)\|^2 \leqq \|\nabla a - \nabla\bar{a}\|^2 \exp\left(\frac{\sigma t}{T} + H(t)\right) .$$

In particular, $u \equiv \bar{u}$ if $a = \bar{a}$.

4. Existence. In this section, we prove a local existence theorem for weak class H_0 solutions of the nonstationary problem (16), if the prescribed data satisfy conditions (17). Since weak class H_0 solutions need not possess a time derivative, we will not need the rather unnatural assumption that the force f_1 in (17e) vanishes for some initial time interval $(0, \varepsilon)$. This assumption was required in a related existence theorem of Heywood [9].

We seek solutions of problem (16) in the form $u = v + b$, where b is an admissible extension and v satisfies (18). We use Galerkin's method, taking as basis functions, a sequence $\{a^k\}$ in $K_0(\Omega)$, which is complete in $H_0(\Omega)$ and orthonormal in $J_0(\Omega)$. Let

$$v^n(x, t) = \sum_{k=1}^n C_{kn}(t)a^k(x)$$

be the solution of the initial value problem for the system

$$(33) \qquad \begin{aligned} &(\nabla v_t^n, \nabla a^l) + \nu(\tilde{\Delta}v^n, \tilde{\Delta}a^l) - (v^n \cdot \nabla v^n + b \cdot \nabla v^n + v^n \cdot \nabla b, \tilde{\Delta}a^l) \\ &\quad = (\nabla f_1, \nabla a^l) - (f_2, \tilde{\Delta}a^l) , \end{aligned}$$

for $t \geqq 0$ and $l = 1, 2, \cdots, n$, with initial conditions $C_{kn}(0) = (\nabla(a - b(0)), \nabla a^k)$, $k = 1, 2, \cdots, n$.

Multiplying (33) through by $C_{ln}(t)$ and summing $\sum_{l=1}^{n}$, we obtain

$$(34)\qquad \frac{1}{2}\frac{d}{dt}\|\nabla v^n\|^2 + \nu\|\tilde{\Delta} v^n\|^2 = (v^n \cdot \nabla v^n + b \cdot \nabla v^n + v^n \cdot \nabla b, \tilde{\Delta} v^n) + (\nabla f_1, \nabla v^n) - (f_2, \tilde{\Delta} v^n)\ .$$

For simplicity, we shall suppress the superscript n. The first term on the right side of (34) can be estimated similarly to (28)-(30). Thus, for any $\alpha > 0$,

$$|(v \cdot \nabla v, \tilde{\Delta} v)| \leqq \alpha\|\tilde{\Delta} v\|^2 + C_{\partial,\alpha}\|\nabla v\|^4 + C_{\partial,\alpha}\|\nabla v\|^6\ ,$$

$$|(b \cdot \nabla v, \tilde{\Delta} v)| \leqq \alpha\|\tilde{\Delta} v\|^2 + C_{\partial,\alpha}\sup_{\Omega}|b|^2\|\nabla v\|^2\ ,$$

$$|(v \cdot \nabla b, \tilde{\Delta} v)| \leqq \alpha\|\tilde{\Delta} v\|^2 + C_{\partial,\alpha}\|\nabla b\|_3^2\|\nabla v\|^2\ .$$

The last two terms of (34) can be estimated using Holder's inequality:

$$|(\nabla f_1, \nabla v)| \leqq \frac{1}{2}\|\nabla f_1\|^2 + \frac{1}{2}\|\nabla v\|^2\ ,$$

$$|(f_2, \tilde{\Delta} v)| \leqq \alpha\|\tilde{\Delta} v\|^2 + \frac{1}{4\alpha}\|f_2\|^2\ .$$

Combining these estimates for terms on the right side of (34) and setting $\alpha = \nu/8$, we obtain

$$\frac{d}{dt}\|\nabla v\|^2 + \nu\|\tilde{\Delta} v\|^2 \leqq C_{\partial,\alpha,b}\|\nabla v\|^2 + C_{\partial,\alpha}\|\nabla v\|^4 + C_{\partial,\alpha}\|\nabla v\|^6 + \|\nabla f_1\|^2 + 4/\nu\|f_2\|^2\ .$$

This differential inequality can be integrated to give

$$(35)\qquad \|\nabla v(t)\|^2 \leqq F(t) \quad\text{and}\quad \int_0^t \|\tilde{\Delta} v(\tau)\|^2 d\tau \leqq \tilde{F}(t)\ , \qquad t \in [0, T^*)\ ,$$

where $F(t)$ and $\tilde{F}(t)$ are continuous functions in $[0, T^*)$ with $F(0) = \|\nabla(a - b(0))\|^2$, and T^* depends on $\|\nabla(a - b(0))\|$, ν, $\partial\Omega$, f_1 and f_2; see for example [10, Lemmas 3 and 4]. Thus one can choose a subsequence of $\{v^n\}$, again denoted by $\{v^n\}$, which converges weakly in $L^2(0, T^*; H_0(\Omega))$ to a function v. The limit v can be taken to satisfy the estimates in (35).

Let $\phi^m(x, t) = \sum_{l=1}^{m} C_l(t)a^l(x)$, where the coefficients $C_l(t)$'s are continuously differentiable in $[0, T^*)$ with $C_l(T^*) = 0$, $l = 1, 2, \cdots, m$. Multiplying (33) through by $C_l(t)$, summing $\sum_{l=1}^{m}$, and integrating over $(0, T^*)$, we obtain

$$
(36)\quad \begin{aligned} &\int_0^{T^*}\{(\nabla v_t^n, \nabla\phi^m) + \nu(\tilde{\Delta} v^n, \tilde{\Delta}\phi^m) - (v^n\cdot\nabla v^n + b\cdot\nabla v^n + v^n\cdot\nabla b, \tilde{\Delta}\phi^m)\}dt \\ &\qquad = \int_0^{T^*}\{(\nabla f_1, \nabla\phi^m) - (f_2, \tilde{\Delta}\phi^m)\}dt\,. \end{aligned}
$$

In the first integral of (36), one can integrate by parts with respect to t to get

$$
(37)\quad \int_0^{T^*}(\nabla v_t^n, \nabla\phi^m)dt = -\int_0^{T^*}(\nabla v^n, \nabla\phi_t^m)dt - (\nabla v^n(0), \nabla\phi^m(0))\,.
$$

If $n \geqq m$, then $(\nabla v^n(0), \nabla\phi^m(0)) = (\nabla(a - b(0)), \nabla\phi^m(0))$ because $(\nabla v^n(0), \nabla a^l) = (\nabla(a - b(0)), \nabla a^l)$ for all $n \geqq l$. Thus, letting $n \to \infty$ in (36) yields

$$
(38)\quad \begin{aligned} &\int_0^{T^*}(\nabla v,\nabla\phi_t^m)dt - \nu\int_0^{T^*}(\tilde{\Delta} v, \tilde{\Delta}\phi^m)dt + \int_0^{T^*}(v\cdot\nabla v + b\cdot\nabla v + v\cdot\nabla b, \tilde{\Delta}\phi^m)dt \\ &\qquad = -(\nabla(a - b(0)), \nabla\phi^m(0)) + \int_0^{T^*}\{-(\nabla f_1, \nabla\phi^m) + (f_2, \tilde{\Delta}\phi^m)\}dt\,. \end{aligned}
$$

It can be shown that every ϕ in S_{T^*} can be approximated by functions of the form ϕ^m in such a way that

$$
\|\phi^m(t) - \phi(t)\|_{H_0} + \|\phi_t^m(t) - \phi_t(t)\|_{H_0} \longrightarrow 0\,,
$$

uniformly in $t \in [0, T^*]$ as $m \to \infty$. It follows that (38) holds for all $\phi \in S_{T^*}$. This establishes the existence of a local solution of problem (16).

It is not difficult to show the solution $u = v + b$ just constructed satisfies the initial condition in the sense that $\lim_{t\to 0^+}\|\nabla u(t) - \nabla a\| = 0$. Since v satisfies the first estimate in (35), we have $\limsup_{t\to 0^+}\|\nabla v(t)\| \leqq \|\nabla(a - b(0))\|$. Therefore, since $v(t) \to a - b(0)$ weakly in $J_0(\Omega)$ as $t \to 0^+$, it follows that $v(t) \to a - b(0)$ strongly in $J_0(\Omega)$ as $t \to 0^+$. Our assertion follows because

$$
\begin{aligned} \|\nabla(v(t) + b(t)) - \nabla a\| &\leqq \|\nabla v(t) - \nabla(a - b(0))\| + \|\nabla b(t) - \nabla b(0)\| \\ &\leqq \|\nabla v(t) - \nabla(a - b(0))\| + t^{1/2}\Big(\int_0^t \|\nabla b_t(\tau)\|^2 d\tau\Big)^{1/2}\,. \end{aligned}
$$

Finally, we note the function v possesses second order derivatives with respect to x in $L^2(0, T^*; L^2(\Omega))$.

THEOREM 5. *Let $\Omega \subset R^3$ be a domain with a uniformly C^3 boundary. Suppose, for the initial boundary value problem* (16), *that the prescribed data permit the boundary values to be extended into $\bar{\Omega}\times[0, T]$ as a solenoidal function b satisfying conditions* (17). *Then there exists a weak class H_0 solution $u = v + b$ on some interval $(0, T^*)$, with $0 < T^* \leqq T$, such that $\nabla\cdot u = 0$ and $\nabla u, D^2u \in L^2(0, T^*;$*

$L^2(\Omega)$). The time interval $(0, T^*)$ depends only on $\|\nabla(a - b(0))\|$, ν, the C^2-regularity of $\partial\Omega$, and the functions $\int_0^t \|\nabla f_1\|^2 d\tau$ and $\int_0^t \|f_2\|^2 d\tau$. Further, $\lim_{t\to 0^+} \|\nabla u(t) - \nabla a\| = 0$.

ACKNOWLEDGEMENT. I am indebted to professor John G. Heywood for his support and advice in the preparation of this paper.

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Received June 8, 1979.

THE UNIVERSITY OF BRITISH COLUMBIA
VANCOUVER, CANADA