PACIFIC JOURNAL OF MATHEMATICS
Vol. 94, No. 2, 1981

# A SUB-ELLIPTIC ESTIMATE FOR A CLASS OF INVARIANTLY DEFINED ELLIPTIC SYSTEMS

L. M. SIBNER AND R. J. SIBNER

**We consider a certain invariantly defined nonlinear system of partial differential equations on a Riemannian manifold. Since a special case describes a steady, irrotational, compressible flow on the manifold, it is natural to refer to the (square of) the pointwise norm of the solution as the speed of the flow and to the density of the flow. Under appropriate restrictions on the density, the system is elliptic and we obtain a sub-elliptic estimate and a maximum principle for the speed of the flow in terms of the curvature of the manifold.**

**Introduction.** Let $M$ be an $n$-dimensional Riemannian manifold, and $\Lambda^p(M)$ the space of smooth $p$-forms on $M$. For $\omega \in \Lambda^p(M)$, $x \in M$, let $Q(\omega) = (\omega, \omega)(x) = {}^*(\omega \wedge {}^*\omega)(x)$ denote the pointwise norm of the form $\omega$. Let $\rho\colon C^\infty(M) \to R$ be a given bounded smooth strictly positive function which we call the density function.

In the following, we consider the invariantly defined nonlinear system of equations for $\omega \in \Lambda^p(M)$:

$$\begin{aligned} d\omega &= 0 \\ \delta(\rho(Q(\omega))\omega) &= 0\,. \end{aligned} \tag{1}$$

If $p=1$, this system describes the motion of a compressible fluid on $M$ and reduces to a single second order equation for the potential function. If the metric is Euclidean and $\rho(Q) = (1 - (\gamma - 1)/2Q)^{1/\gamma-1}$, it becomes the gas dynamics equation for polytropic flow in $R^n$. If $\rho \equiv 1$, one obtains the Laplace-Beltrami equation.

To be more explicit, if $\omega$ is a solution of (1), then it is also a solution of a homogeneous second order quasi-linear system, $A\omega = 0$. In local coordinates, let

$$\omega = \omega_{i_1\cdots i_p} dx^{i_1} \wedge \cdots \wedge dx^{i_p}\,, \quad \omega^{j_1\cdots j_p} = g^{j_1 i_1} \cdots g^{j_p i_p} \omega_{i_1\cdots i_p}\,,$$

and assume $\rho = \rho(Q(\omega))$. Then, $A\colon \Lambda^p(M) \to \Lambda^p(M)$ is given by

$$\begin{aligned} (A\omega)_{i_1\cdots i_p} = \sum_{i,j} g^{ji} \Big\{ \rho \frac{\partial^2 \omega_{i_1\cdots i_p}}{\partial x^i \partial x^j} &+ 2\rho' \omega^{j_1\cdots j_p} \sum_{k=1}^{p} \omega_{i_1\cdots i_{k-1} i i_{k+1} \cdots i_p} \frac{\partial^2 \omega_{j_1\cdots j_p}}{\partial x^{i_k} \partial s^j} \Big\} \\ &+ \text{lower order terms.} \end{aligned}$$

(We will observe the usual summation convention wherever possible.)

A computation shows that the system (1) is *elliptic* for $\omega \in \Lambda^p(M)$ if and only if

$$\frac{d}{dQ}(\rho^2 Q) > 0 \quad \text{for all} \quad x \in M . \tag{2}$$

If $M$ is compact and (1) is elliptic, there is a unique *weak* solution in each cohomology class (Sibner [7]). The question of smooth solutions is unresolved except in the case $p = 1$ ([7]) or if the metric is Euclidean (Uhlenbeck [10]).

Assuming that the system (1) is elliptic, we shall derive an inquality for the function $Q$, of the following kind:

$$LQ + B(\omega) \geqq 0 \tag{3}$$

where $\omega$ is a solution of (1), $L$ is a single second order elliptic operator with no zero order term, and $B$ is a quadratic form whose sign depends upon the curvature tensor. Such an inequality leads to a maximum principle and is perhaps a step in the direction of elliptic regularity for the system (1) (see [10]). An inequality of the form (3) and a maximum principle were previously proved by the authors for 1-forms on surfaces ($n = 2$) (see [9]).

**1. The inequality satisfied by $Q$.** Let $I = i_1 \cdots i_p$, $J = j_1 \cdots j_p$ be multi-indices and set $g^{IJ} = g^{i_1 j_1} \cdots g^{i_p j_p}$. Then, $Q(\omega) = g^{JI}\omega_J\omega_I$ in this notation. In terms of the Riemannian metric, let $|\nabla\omega|^2 = g^{kl}g^{JI}\nabla_k\omega_J\nabla_l\omega_I$. Define the curvature form (Lichnerowicz [5, 6])

$$K(\omega) = g^{JI}\omega_J\left\{\sum_{s=1}^{p} K^t_{i_s}\omega_{i_1\cdots t\cdots i_p} + \sum_{t<s}^{1\cdots p} K^{vu}_{i_t i_s}\omega_{i_1\cdots v\cdots u\cdots i_p}\right\} .$$

Computing the Laplacian of $Q$, one obtains

$$-\frac{1}{2}\Delta Q = \frac{1}{2}g^{ji}\nabla_j\nabla_i Q = |\nabla\omega|^2 + (g^{JI}\omega_J)(g^{ji}\nabla_j\nabla_i\omega_I)$$

and using the formula for $\Delta\omega$([11]) one obtains the fundamental identity for $Q$:

$$-\frac{1}{2}\Delta Q = |\nabla\omega|^2 + K(\omega) - p!(\omega, \omega) . \tag{1.1}$$

Next, let $\omega$ be a solution of (1) and replace $\omega$ in (1.1) by $\rho\omega$. Using the fact that $\delta\rho\omega = 0$, one obtains

$$-\frac{1}{2}\Delta(\rho^2 Q) + p!(\delta d\rho\omega, \rho\omega) = \rho^2 K(\omega) + |\nabla(\rho\omega)|^2 . \tag{1.2}$$

We shall show that the left hand side is a second order differential operator on $Q$, $LQ = L_1Q + L_2Q$, and will compute its principal part.

$$L_1Q = -\frac{1}{2}\Delta(\rho^2 Q) = \frac{1}{2}(\rho^2 + 2\rho\rho' Q)g^{ji}\frac{\partial^2 Q}{\partial x^i \partial x^j} + \text{first order terms}\,.$$

Using the fact that $d\omega = 0$,

$$p!(\delta d\rho\omega)_{i_1\cdots i_p} = -g^{ji}\nabla_j\left(\rho'\omega_{i_1\cdots i_p}\frac{\partial Q}{\partial x^i} - \rho'\sum_{k=1}^{p}\omega_{i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\frac{\partial Q}{\partial x^{i_k}}\right).$$

Therefore,

$$\begin{aligned} L_2Q &= p!(\delta d\rho\omega, \rho\omega) \\ &= -g^{ji}\rho\rho'\omega_{i_1\cdots i_p}\omega^{i_1\cdots i_p}\frac{\partial^2 Q}{\partial x^i\partial x^j} + g^{ji}\rho\rho'\omega^{i_1\cdots i_p}\sum_{k=1}^{p}\omega_{i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\frac{\partial^2 Q}{\partial x^{i_k}\partial x^j} \\ &\qquad + \text{first order terms}\,. \end{aligned}$$

Combining coefficients of $\partial^2 Q/\partial x^i \partial x^j$,

$$L_2Q = \left\{-g^{ji}\rho\rho' Q + \rho\rho'\omega_{i_1\cdots i_p}\sum_{k=1}^{p} g^{ji_k i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\right\}\frac{\partial^2 Q}{\partial x^i\partial x^j} + \cdots$$

and

$$LQ = \left\{\frac{1}{2}g^{ji}\rho^2 + \rho\rho'\omega_{i_1\cdots i_p}\sum_{k=1}^{p} g^{ji_k}\omega^{i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\right\}\frac{\partial^2 Q}{\partial x^i\partial x^j} + \cdots\,.$$

PROPOSITION 1.1. *If the system* (1) *is elliptic at a solution* $\omega$, *then* $L$ *is an elliptic operator.*

*Proof.* The principal symbol of $L_1$ on a cotangent vector $\pi = (\pi_1, \cdots, \pi_n) \neq 0$ is

$$\frac{1}{2}(\rho^2 + 2\rho\rho' Q)g^{ji}\pi_i\pi_j > 0$$

using the ellipticity condition (2). Therefore, $L_1$ is elliptic.

Choose geodesic normal coordinates at a point, in which case $g^{ij} = \delta^{ij}$, $\omega^I = \omega_I$, and $Q(\omega) = \sum_I(\omega_I)^2$. The principal symbol of $L_2$ in these coordinates becomes:

$$\begin{aligned} &-\rho\rho'\left(Q\delta^{ij} - \sum_{k=1}^{p}\omega_{i_1\cdots i_{k-1}ji_{k+1}i_p}\omega_{i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\right)\pi_i\pi_j \\ &\qquad = -\rho\rho'\left\{(\textstyle\sum(\omega_{i_1\cdots i_p})^2)(\sum\pi_i^2) - \left(\displaystyle\sum_{k=1}^{p}\omega_{i_1\cdots i_{k-1}ii_{k+1}\cdots i_p}\pi_i\right)^2\right\}. \end{aligned}$$

If $\rho' \leqq 0$, one sees immediately from the Schwartz inequality that this expression is nonnegative. Therefore, if $\rho' \leqq 0$, the principal symbol of $L = L_1 + L_2$ is positive definite and $L$ is elliptic.

If $\rho' \geqq 0$, the principal symbol of $L$ in geodesic normal coordinates is

$$\frac{1}{2}\rho^2 \sum \pi_i^2 + \rho\rho'(\sum \omega_{i_1 \cdots i_{k-1} i i_{k+1} \cdots i_p} \pi_i)^2 > 0$$

using the fact that $\rho$ is positive. Hence, $L$ is elliptic.

Summarizing the results of this section, we have shown

PROPOSITION 1.2. *Let $\omega \in \Lambda^p(M)$ be a solution of the homogeneous elliptic system* (1). *Then $Q(\omega)$ is a solution of the single scalar equation*

$$LQ = \rho^2 K(\omega) + |\nabla(\rho\omega)|^2 \tag{1.3}$$

*where $L$ is a second order elliptic operator having no zero order terms.*

**2. The maximum principle.** Our main result is the following.

THEOREM. *Let $\omega$ be a solution of the elliptic system* (1). *Then,*

(a) *$Q = (\omega, \omega)$ cannot have a relative maximum at a point $x_0$ where $K(\omega)|_{x_0} > 0$.*

(b) *If $Q$ has a relative maximum at a point $x_0$, in a neighborhood $N$ of which, $K(\omega) \geqq 0$ then*

( i ) *$Q$ is constant on $N$.*

(ii) *$K(\omega) \equiv 0$ in $N$.*

(iii) *$\nabla\omega \equiv 0$ on $N$.*

*Proof.* As in [9], statements (a) and (i) of (b) follow from Proposition 1.2 and the Hopf Maximum Principle. But if $Q$ is constant in $N$, then $LQ \equiv 0$ in $N$ which gives, again by Proposition 1.2, statements (ii) and (iii).

If $p = 1$, the curvature expression $K(\omega)$ reduces to $R^{ij}\omega_i\omega_j$ where $R^{ij}$ is the Ricci curvature tensor. One speaks of $R^{ij}\omega_i\omega_j$ as the Ricci curvature *in the direction* $\omega$. In the language of gas dynamics we have the

COROLLARY. *A subsonic compressible flow on $M$ cannot assume its maximum speed at a point where the Ricci curvature is positive in the direction of the flow. If the maximum speed is attained at a point of a region $N$ in which the Ricci curvature in the flow direction is nonnegative, then the curvature must in fact be zero, the speed $Q$ must be constant, and the flow parallel in $N$* (i.e., *the covariant derivatives, $\nabla_i\omega_j \equiv 0$). If it is further known that* *the Ricci curvature is positive definite, then* $Q \equiv 0$ *and hence* $\omega \equiv 0$ *in* $N$.

*Added in proof.* The authors, with P. D. Smith, have obtained a regularity theorem for the system (1) using the estimate (3). It will appear in a forthcoming paper.

## References

1. L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics,* Wiley, New York, 1958.
2. S. Bochner, *Vector fields and Ricci curvature,* Bull. Amer. Math. Soc., **52** (1946), 776-797.
3. ———, *Curvature and Betti numbers*, Annals of Math., **49** (1948), 379-390.
4. E. Hopf, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen zweiter Ordnung vom elliptischen Typus,* Sitzber. Preuss. Akad. Wiss. Physik-math. Kl, **19** (1927),147-152.
5. A. Lichnerowicz, *Courbure et nombres de Betti d'une varieté riemannienne compacte,* Compt. Rend. Acad. Sci. Paris, **226** (1948), 1678-1680.
6. ———, *Courbure, nombres de Betti espaces symmetriques,* Proc. Intern. Congr. of Math., **2** (1952), 216-223.
7. L. M. Sibner and R. J. Sibner, *A nonlinear Hodge-de Rham theorem*, Acta Math., **125** (1970), 57-73.
8. ———, *Nonlinear Hodge theory: Applications,* Advances in Math., **31** (1979), 1-16.
9. ———, *A maximum principle for compressible flow,* Proc. Amer. Math. Soc., **71** (1978), 103-108.
10. K. Uhlenbeck, *Regularity for a class of nonlinear elliptic systems,* Acta Math., **138** (1977), 219-240.
11. K. Yano, *Integral formulas in Riemannian Geometry,* Marcel Dekker, New York, 1970.

Received May 8, 1980. Research of the first author was partially supported by NSF Grant MCS78-03276 and research of the second author was partially supported by NSF Grant MCS78-03268.

Polytechnic Institute of New York
Brooklyn, NY 11201
and
City University of New York
Brooklin College
Brooklin, NY 11210