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LOCATED SETS ON THE LINE

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Located sets are sets from which the distance of any point may be measured; they are used extensively in modern constructive analysis. Here a general method is given for the construction of all located sets on the line. It is based on a characterization of a located set in terms of the resolution of its metric complement into a union of disjoint open intervals. The characterization depends on a strong countability condition for the intervals, called the ***locating condition.*** **Included as a special case is the characterization and construction of compact sets. The techniques used are in accord with the principles of Bishop's** ***Foundations of Constructive Analysis,*** **1967.**

In many situations it is desired to measure the distance

$$\rho(x, G) \equiv \inf \{\rho(x, y): y \in G\}$$

between a point x and a set G in a metric space. However, this is *not* always possible constructively. By this we mean that a counter-example exists in the sense of Brouwer; discussions of these are found in [1] (and [4]). The italicized word "*not*" will also be used below in this sense.

Brouwer [2] introduced the concept of *located* set, for which the above distances always exist. Here the concept of located set on the line is reduced to the concept of number. The construction of an arbitrary located set is reduced to the construction of two sequences of real numbers with certain properties.

The metric complement of a located set G is the set

$$-G \equiv \{x: \rho(x, G) > 0\} .$$

Such a set is said to be *colocated.*

The characterization of a located set G on the line is obtained by means of the resolution of its metric complement $-G$ into a countable union $\bigcup_n I_n$ of disjoint open intervals, given in [4]. It is shown in [3] that only the closure of a located set G may be recovered from its metric complement $-G$. Thus we characterize *closed* located sets. Arbitrary located sets are precisely the dense subsets of these.

The characterization theorem below involves four conditions on the sequence $\{I_n\}$ of open intervals. Briefly, these are the following.

(1) The intervals are *fixative,* i.e., each is either void or *fixed*

(contains some point).

(2) The intervals are disjoint.

(3) The locating condition. This *limits* the number of intervals of a given size which intersect a given bounded segment, and *locates* them in the sequence of intervals.

(4) The representation of G as an intersection of notched lines (i.e., lines with an interval removed), complementary to the resolution of $-G$ as a union of intervals.

The essential property of located sets is expressed in the locating condition (3). For any point x on the line and for any $\varepsilon > 0$, this condition specifically locates the finitely many intervals in the sequence $\{I_n\}$ which might contain x and have length more than ε.

The set G is formed by notching out from the line the succession of open intervals. From the point of view of a given point x, the distance $\rho(x, G)$ seems to be 0 during this process, until, perhaps, x is notched out. When the notch I_n is taken, any finite endpoint of this interval is assured a permanent place in G, because under the disjointness condition no succeeding notch can remove it. Thus the exact distance is known as soon as x is notched out; however, whether this happens might not be predictable. Therefore, an estimate of the distance requires a prediction giving the specific location of those intervals which might remove x and also be large enough to produce a large distance. Thus, if finitely many intervals have been considered, and a prediction is given that the remaining intervals either stay clear of x or are of length less than ε, then $\rho(x, G)$ may be calculated to within ε.

It is easily concluded that only finitely many disjoint intervals of length greater than $1/k$ can meet $(-k, k)$, by considering their total length. However, such methods will not suffice here, for their *location* in the entire sequence $\{I_n\}$ of intervals is not determined. The locating condition (3) actually lists a finite number M_k of intervals which includes all those in question.

To construct a closed located set G on the line using the characterization below, proceed as follows.

I. First choose *any* finite number M_1 of open intervals $I_n \equiv (a_n, b_n)$ of *any* length, satisfying (1) and (2). To satisfy the fixative condition (1), decide whether (a_n, b_n) is to be void or fixed, and specify (1, 0), or ensure $a_n < b_n$, accordingly. To satisfy the disjointness condition (2), after m intervals have been chosen use step (x) in the proof to examine the finitely many segments, in one of which the next interval must be placed.

II. Next choose any finite number m_2 of open intervals in the same way, and also, to satisfy the locating condition (3), ensure that any of these m_2 intervals which meet $(-1, 1)$ are of length $\leqq 1$. Put $M_2 \equiv M_1 + m_2$.

III. Continue in this way. For example, any of the next m_3 intervals which meet $(-2, 2)$ must be of length $\leqq 1/2$.

IV. The set G obtained is the set of all points x such that, for each n, either $x \leqq a_n$ or $x \geqq b_n$.

An *open interval* is a set of the form $(a, b) \equiv \{x \in R: a < x < b\}$. To include the unbounded intervals, a and b are taken to be extended real numbers. To include also the case of an open interval for which it may be unknown whether it is bounded or unbounded, we use the system R^∞ of extended real numbers constructed in [4, § 4]. Thus it may be unknown whether an endpoint is finite or infinite.

Allowing $+\infty$ as a distance, we consider the void set to be located. Furthermore, the extended real numbers in R^∞ will be used as distances. Thus it may be unknown whether a given located set G is void or fixed, since it may be unknown whether $\rho(x, G)$ is infinite or finite. For example, if $\{a_n\}$ is an increasing sequence of zeros and ones for which it is unknown whether or not some $a_n = 1$, then the set $\{n: a_n = 1\}$ is *not* fixative; yet it is located in the sense of distances measured with extended real numbers. Although the distances $\rho(x, G)$ used in the study of located sets G are extended real numbers, the points of G and the points x considered are always finite real numbers.

A set on the line is compact if and only if it is closed, located, bounded, and fixed. Thus, compact sets are characterized by the conditions of the theorem together with the following additional conditions:

(5) (bounded) Some $a_n = -\infty$ and some $b_k = +\infty$.

(6) (fixed) For each n, either $a_n > -\infty$ or $b_n < +\infty$. (See [5, Theorem 3].)

An example in which the locating condition (3) does *not* hold, and thus where the set G defined by (4) is *not* located, is easily derived from [3, Example 2]. The question raised by that example, of when a countable union of fixative disjoint open intervals is colocated, is answered by the locating condition in the corollary below.

Only constructive properties of the real numbers and extended real numbers, such as are found in [1] and [4], will be admitted. We shall need to perform certain limited algebraic operations in R^∞. If $s \equiv \{s_n\}$ is an extended real number (where $\{s_n\}$ is an extended Cauchy sequence of real numbers), and a is a finite number, then $s \pm a$ and $a - s$ are easily defined in R^∞; for example, $s + a \equiv \{s_n + a\}$. Furthermore, although in general extended real numbers

a and b can *not* be subtracted, if $a < b$ then $b-a$ is easily defined. Thus we define the *length* $l(I) \equiv b - a$ of any fixed open interval $I \equiv (a, b)$.

THEOREM. *For any closed located set G on the line, there exist sequences $\{a_n\}$ and $\{b_n\}$ of extended real numbers such that*

(1) $a_n \neq b_n$, *for all* n.

(2) $b_n \leqq a_k$ *or* $b_k \leqq a_n$, *whenever* $n \neq k$ *and* $a_n < b_n$ *and* $a_k < b_k$.

(3) *There exists a sequence $\{M_k\}$ of positive integers such that $n \leqq M_k$ whenever (a_n, b_n) meets $(-k, k)$ and $b_n - a_n > 1/k$.*

(4) $G = \bigcap_n ((-\infty, a_n] \cup [b_n, +\infty))$.

Conversely, whenever $\{a_n\}$ and $\{b_n\}$ are sequences of extended real numbers satisfying (1), (2) *and* (3), *then the set G defined by* (4) *is closed and located.*

Proof. Necessity. Let G be a closed located set.

(i) *Definitions.* Let

$$-G = \bigcup_n (a_n, b_n)$$

be the resolution of the colocated set $-G$ into a countable union of fixative disjoint open intervals, as obtained in [4]. The intervals (a_n, b_n) are called *notches*. When the notch (a_n, b_n) is fixed, the numbers a_n and b_n are called *endpoints*. Clearly, all finite endpoints lie in G. For each n, the set

$$H_n \equiv (-\infty, a_n] \cup [b_n, +\infty)$$

is called a *notched* line. Clearly, each set H_n is located and for any $x \in R$,

$$\rho(x, H_n) = ((x - a_n) \wedge (b_n - x)) \vee 0 .$$

For each m, the set

$$G_m \equiv \bigcap_{n=1}^{m} H_n$$

is called a *finitely notched line.* The theorem expresses a located set as an *infinitely notched line.*

(ii) *Condition* (1) *is satisfied.*

For any n, if (a_n, b_n) is fixed, then $a_n < b_n$, while if (a_n, b_n) is void, then redefine $a_n \equiv 1$ and $b_n \equiv 0$.

(iii) *Condition* (2) *is satisfied.*

Either $b_n < b_k$ or $b_n > a_k$. It follows from the disjointness of the intervals that in the first case $b_n \leqq a_k$ and in the second case $b_k \leqq a_n$.

(iv) *Condition* (4) *is satisfied, and thus*

$$G = \bigcap_n H_n = \bigcap_m G_m .$$

Since $-H_n \subseteqq -G$ it follows that $G \subseteqq H_n$ for all n. Now let $x \in \bigcap_n H_n$. From $x \in -G$ it would follow that $x \in -H_n$ for some n; hence $x \in G$.

(v) *Each set* G_m *is located, with*

$$\rho(x, G_m) = \bigvee_{n=1}^{m} \rho(x, H_n)$$

for all $x \in R$.

Let ρ be the indicated maximum. Clearly ρ is a lower bound for the set of numbers $|x - y|$, with $y \in G_m$. To show that ρ is the infimum of this set it must be shown that whenever $\rho < \sigma$ there exists $y \in G_m$ such that $|x - y| < \sigma$. Equivalently, when $\rho < +\infty$ and $\varepsilon > 0$ we must construct $y \in G_m$ such that $|x - y| < \rho + \varepsilon$. Either $\rho(x, G) < \varepsilon$ or $\rho(x, G) > 0$. In the first case construct $y \in G$ so that $|x - y| < \varepsilon$. In the second case choose n so that $a_n < x < b_n$. In the subcase $n > m$, put $y \equiv x$. In the subcase $n \leqq m$, then $\rho(x, G) = \rho(x, H_n) = \rho$; construct $y \in G$ such that $|x - y| < p + \varepsilon$.

(vi) *For any finite interval* J *and any* $\varepsilon > 0$ *there exists* m *such that*

$$|\rho(x, G) - \rho(x, G_m)| < \varepsilon$$

for all x *in* J. *In this way a located set is approximated by finitely notched lines.*

First consider the case in which J consists of a single point x. Either $\rho(x, G) < \varepsilon$ or $\rho(x, G) > 0$. In the first case $\rho(x, G_1) < \varepsilon$ because $G \subseteqq G_1$, so it suffices to put $m \equiv 1$. In the second case choose m so that $x \in (a_m, b_m)$. Then $\rho(x, G) = \rho(x, G_m)$. Thus, in either case, $|\rho(x, G) - \rho(x, G_m)| < \varepsilon$. Note that although in general extended real numbers s and t can *not* be subtracted, even when $s \leqq t$, the indicated difference here is meaningful because the numbers involved are either finite or equal.

We may assume J is fixed, and construct a finite $\varepsilon/3$ approximation A to J. For each a in A construct m_a so that $|\rho(a, G) - \rho(a, G_{m_a})| < \varepsilon/3$. Put $m \equiv \max\{m_a: a \in A\}$. To show that the finitely notched set G_m approximates G to within ε on J, consider any point x in J and construct a in A such that $|x - a| < \varepsilon/3$. It suffices to consider the second of the cases $\rho(x, G) < \varepsilon$ and $\rho(x, G) > 2\varepsilon/3$. Choose n so that $x \in (a_n, b_n)$. Then also $a \in (a_n, b_n)$ and $\rho(a, G) > \varepsilon/3$; hence $\rho(a, G_{m_a}) > 0$. Thus $n \leqq m_a$, so $n \leqq m$ and $\rho(x, G) = \rho(x, G_m)$.

(vii) *The locating condition* (3) *obtains.*

Consider any k and choose M_k so that

$$|\rho(x, G) - \rho(x, G_{M_k})| < 1/2k$$

for all x in $(-k-1,\ k+1)$. Let (a_n, b_n) meet $(-k, k)$ with $b_n - a_n > 1/k$. Since the length of the interval $I \equiv (a_n, b_n) \cap (-k-1,\ k+1)$ is greater than $1/k$, the midpoint x of I has a distance $\rho(x, G) > 1/2k$. It follows that $n \leqq M_k$.

Sufficiency. Now let the sequences $\{a_n\}$ and $\{b_n\}$ satisfy conditions (1), (2) and (3), and let G be defined by (4).

(viii) *Definitions.* Define the sets H_n and G_m as in (i) above. It follows that $G \equiv \bigcap_n H_n = \bigcap_m G_m$ and that all finite endpoints lie in G.

(ix) *The set G is closed.*

Let $\{x_k\}$ be a sequence in G with $x_k \to x$. To show $x \in G$, it suffices, because of condition (1), to consider a fixed notch (a_n, b_n). It suffices to consider the first of the cases $x < b_n$ and $x > a_n$. Then $x_k < b_n$ eventually; it follows that $x_k \leqq a_n$ eventually, and thus $x \leqq a_n$. Thus $x \in G$.

(x) *For any m, the indexing of the q fixed notches (if any) among the first m may be rearranged so that*

$$a_1 < b_1 \leqq a_2 < b_2 \leqq \cdots \leqq a_q < b_q\ .$$

We may assume that $q \geqq 2$ and that the intervals (a_n, b_n) with $n \leqq q$ are the fixed intervals; let b be the minimum of their right endpoints b_n. It follows that b is finite. Choose $\varepsilon > 0$ so that $\varepsilon < b_n - a_n$ for all $n \leqq q$. Since some b_n is less than $b + \varepsilon$, we may by reindexing assume $b_1 < b + \varepsilon$. It follows from condition (2) that $a_1 < b_1 \leqq a_n$ for $2 \leqq n \leqq q$. An induction completes the proof.

(xi) *If $a \in R^\infty$ and $\varepsilon > 0$, then for any $x \in R$ either $x > a$ or $x < a + \varepsilon$.*

Of the cases $a < x$ and $a > -\infty$ we need consider only the second. Of the subcases $a < +\infty$ and $a > x$ we need consider only the first, and now a is finite.

(xii) *Consider any m and the rearrangement of* (x), *with $q \geqq 1$. For any $x \in R$ and any $\varepsilon > 0$, one of the following holds*

(a) $x < a_1 + \varepsilon$.

(b) $a_n < x < b_n$ *for some* $n \leqq q$.

(c) $b_n - \varepsilon < x < a_{n+1} + \varepsilon$ *for some* $n < q$.

(d) $b_q - \varepsilon < x$.

By (xi), either $x < a_1 + \varepsilon$ or $a_1 < x$. An induction completes the proof.

(xiii) *Each set G_m is located, with*

$$\rho(x, G_m) = \bigvee_{n=1}^{m} \rho(x, H_n)$$

for all x in R.

Let ρ be the indicated maximum. For any $y \in G_m$, clearly $\rho \leqq |x - y|$. Now let $\rho < +\infty$ and $\varepsilon > 0$; we must construct $y \in G_m$ so that $|x - y| < \rho + \varepsilon$. Either $\rho > 0$ or $\rho < \varepsilon$. In the first case choose $n \leqq m$ so that $\rho(x, H_n) > 0$. Since $\rho(x, H_n) < \rho + \varepsilon$ one of the endpoints of (a_n, b_n) is finite and suffices for y. In the second case, we may assume $q \geqq 1$ in (x), and according to the alternatives in (xii), define y as follows.

(a) $y \equiv x \wedge a_1$.

(b) $y \equiv a_n$ or $y \equiv b_n$, as suitable.

(c) $y \equiv b_n \vee x \wedge a_{n+1}$.

(d) $y \equiv x \vee b_q$.

(xiv) *Let A be a set of extended real numbers. If for every $\varepsilon > 0$ there exists t in A such that $t + \varepsilon$ is an upper bound for A, then A has a supremum in R^∞.*

For each n, construct t_n in A such that $a \leqq t_n + 1/n$ for all a in A. To show that $\{t_n\}$ converges in R^∞ we apply [4, Theorem 3]. First we construct the auxiliary sequence $\{\sigma_n\}$. Note that if some $t_n < +\infty$ then clearly all $t_n < +\infty$. Thus we construct an increasing sequence $\{\sigma_n\}$ of 0's and 1's such that $t_n > n + 1$ when $\sigma_n = 0$ and $t_n < +\infty$ when $\sigma_n = 1$. Now consider any n. If $\sigma_n = 0$ then $n + 1 < t_n \leqq t_k + 1/k$ and thus $t_k > n$ for all k. If $\sigma_n = 1$ and $k \geqq n$ then $t_k \leqq t_n + 1/n$ and $t_n \leqq t_k + 1/k$; thus $|t_k - t_n| \leqq 1/n$. Thus we define $s \equiv \lim t_n$ and it follows that $s = \sup A$.

(xv) *The set G is located, with*

$$\rho(x, G) = \bigvee_{n} \rho(x, H_n)$$

for all $x \in R$.

We first show that for any $x \in R$ the indicated supremum exists in R^∞. By (xiii) it suffices to show that the distances $\rho(x, G_n)$ have a supremum. To verify the condition of (xiv), let $\varepsilon > 0$ and choose k so that $x \in (-k, k)$ and $1/k < \varepsilon$. Using the locating parameter M_k from condition (3), put $t \equiv \rho(x, G_{M_k})$. We must show that $\rho(x, G_n) \leqq t + \varepsilon$ for all n. For this it suffices to show $\rho(x, H_n) \leqq t + \varepsilon$ for all n. Consider any n. It suffices to consider the second of the cases $\rho(x, H_n) < \varepsilon$ and $\rho(x, H_n) > \varepsilon/2$. Then $x \in (a_n, b_n)$ and $b_n - a_n > \varepsilon$. It follows from the locating condition (3) that $n \leqq M_k$, and thus $\rho(x, H_n) \leqq t$.

Thus the supremum exists; denote it by ρ. To show that $\rho \leqq |x - y|$ for every $y \in G$, let $\varepsilon > 0$; we must show $\rho < |x - y| + \varepsilon$. It suffices to consider the second of the cases $\rho < \varepsilon$ and $\rho > 0$.

Choose m so that $\rho(x, H_m) > 0$. Then $\rho = \rho(x, H_m) \leqq |x - y|$.

Now let $\rho < +\infty$ and $\varepsilon > 0$. We must construct $y \in G$ such that $|x - y| \leqq \rho + \varepsilon$. Either $\rho > 0$ or $\rho < \varepsilon$. In the first case choose m so that $\rho(x, H_m) > 0$. Since $\rho(x, H_m) < \rho + \varepsilon$, we obtain $|x - y| < \rho + \varepsilon$, where y is a suitably chosen endpoint of (a_m, b_m), and thus $y \in G$.

In the second case, where $\rho < \varepsilon$, we shall construct a point y in G such that $|x - y| \leqq \varepsilon$. First construct an increasing sequence $\{\tau_k\}$ of 0's, and 1's, with $\tau_1 \equiv 0$, such that $\rho < \varepsilon/k$ when $\tau_k = 0$ and $\rho > 0$ when $\tau_k = 1$. Now construct a sequence $\{y_k\}$ of real numbers as follows. Consider any k. In the case $\tau_k = 0$, we have $\rho(x, G_k) < \varepsilon/k$; construct $y_k \in G_k$ such that $|x - y_k| < \varepsilon/k$. In the case $\tau_k = 1$ first consider the subcase in which k is the least integer, denoted j, of such integers k. Choose n so that $a_n < x < b_n$. Since $\tau_{j-1} = 0$ we have $\rho < \varepsilon/(j - 1)$. Put $y_j \equiv a_n$ or $y_j \equiv b_n$ so that $|x - y_j| < \varepsilon/(j - 1)$. Since then y_j is a finite endpoint, $y_j \in G$. When $k > j$ put $y_k \equiv y_j$. This defines the sequence $\{y_k\}$ with $|x - y_k| < \varepsilon$ and $y_k \in G_k$ for each k.

To show that $\{y_k\}$ is a Cauchy sequence, let $n > k$. Three cases result, depending on the values of τ_k and τ_n. First, when both are 0, we have $|x - y_n| < \varepsilon/n$ and $|x - y_k| < \varepsilon/k$, and thus $|y_n - y_k| < 2\varepsilon/k$. Second, when both are 1, then $y_n = y_k$. Finally, when $\tau_k = 0$ and $\tau_n = 1$, then $|x - y_k| < \varepsilon/k$ and $|x - y_n| < \varepsilon/(j - 1) \leqq \varepsilon/k$. Thus $|y_n - y_k| < 2\varepsilon/k$. Hence $\{y_k\}$ is a Cauchy sequence converging to a point y with $|x - y| \leqq \varepsilon$. From (ix) it follows that the sets G_k are closed; thus $y \in G$. □

COROLLARY. *A countable union* $\bigcup_n I_n$ *of fixative disjoint open intervals is colocated if and only if there exists a sequence* $\{M_k\}$ *of positive integers such that* $n \leqq M_k$ *whenever* I_n *meets* $(-k, k)$ *and* $\ell(I_n) > 1/k$.

Some applications of the theorem will be given in [5], where those located sets which have suprema in R^∞, and those which are fixed, are characterized by means of numerical conditions on the endpoints of the notches.

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