PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

EXISTENCE OF STRONG SOLUTIONS FOR STOCHASTIC DIFFERENTIAL EQUATIONS IN THE PLANE

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Let B be the 2-parameter Brownian motion on $D = [0, \infty] \times [0, \infty)$ and Z be a 2-parameter stochastic process defined on the boundary ∂D of D. Consider the non-Markovian stochastic differential system in 2-parameter

$$\begin{cases} dX(s, t) = \alpha(s, t, X)dB(s, t) + \beta(s, t, X)dsdt & \textbf{for} \quad (s, t) \in D\,, \\ X(s, t) = Z(s, t) & \textbf{for} \quad (s, t) \in \partial D\,. \end{cases}$$

Under the assumption that the coefficients α and β satisfy a Lipschitz condition and a growth condition and the assumption that Z has continuous sample functions and locally bounded second moment on ∂D, it is shown in this paper that the differential system has a strong solution. Pathwise uniqueness of solution is established under the assumption of the Lipschitz condition.

0. **Introduction.** Recently several papers on stochastic integrals in the plane have appeared (see [2], [3], [9], [11] and [13]). In the present paper we treat stochastic differential equations in the plane. The domain of definition of the stochastic integrals and stochastic differential equations we consider is the positive quadrant $D = [0, \infty) \times [0, \infty)$ in which a partial order $(s, t) \prec (u, v)$ for $s \leqq u$ and $t \leqq v$ is introduced. The object of our study is a stochastic differential equation of the type

$$dX(s, t) = \alpha(s, t, X)dB(s, t) + \beta(s, t, X)dsdt\,,$$

or, to be precise,

$$\begin{aligned} (0.1) \qquad & X(s, t) - X(s, 0) - X(0, t) + X(0, 0) \\ & = \int_{[0,s]\times[0,t]} \alpha(u, v, X)dB(u, v) \int_{[0,s]\times[0,t]} \beta(u, v, X)d(u, v)\,, \end{aligned}$$

where B is a 2-parameter Brownian motion on a probability space $(\Omega, \mathfrak{F}, P)$ and the domain D. The precise defininions of B and the two stochastic integrals appearing on the right side of (0.1) are given in § 2. The case in which the coefficients of the stochastic differential equation depend on X only to the extent that they depend on $X(s, t)$, i.e., the stochastic differential equation

$$dX(s, t) = a(X(s, t))dB(s, t) + b(X(s, t))dsdt$$

was treated by Cairoli [2]. The coefficients α and β in our equation

(0, 1) are real valued functions on $D \times W$ where W is the space of all real valued continuous functions on D. We impose certain measurability conditions on α and β so that $\alpha(s, t, X)$ and $\beta(s, t, X)$ depend only on that part of the sample function of X which preceeds (s, t) in the sense of the partial order $\prec$.

Let $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ be a probability space with a family of sub-σ-algebras $\mathfrak{F}_{s,t}$, $(s, t) \in D$, satisfying the usual conditions (see Definition 1.1 in § 1). For brevity we call $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ an equipped probability space. By an $\mathfrak{F}_{s,t}$ Brownian motion we mean a 2-parameter Brownian motion on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and D which is $\mathfrak{F}_{s,t}$ adapted. In § 2 we define stochastic integrals of square integrable stochastic processes with respect to an $\mathfrak{F}_{s,t}$ Brownian motion and state some well-known facts about such stochastic integrals in a way suitable for our use in treating the stochastic differntial equation (0.1).

By a boundary stochastic process we mean a 2-parameter stochastic process whose domain of definition is the boundary ∂D of D. Consider an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ with an $\mathfrak{F}_{s,t}$ Brownian motion B with $\partial B = 0$ on it. In § 3 we show that if the coefficients α and β in (0.1) satisfy the Lipschitz condition (3.3) and the growth condition (3.4) in § 3, then for every boundary stochastic process Z which is $\mathfrak{F}_{s,t}$ adapted and has continuous sample functions on ∂D and bounded second moments on every bounded subset of ∂D there exists a 2-parameter stcochastic process X which is $\mathfrak{F}_{s,t}$ adapted, has continuous sample functions on D and bounded second moments on every bounded subset of D and satisfies (0.1) and the boundary condition $\partial X = Z$. Strong solutions of stochastic differential equations were discussed in Liptzer and Shiryaev [7] and Watanabe [10]. In § 3 we define strong solutions for stochastic differential equations in the plane. In Theorem 3.12 we show that under the assumption of the conditions (3.3) and (3.4) on the coefficients α and β in (0.1) a strong solution exists. Regarding the uniqueness of the solution of (0.1), Theorem 3.8 shows that under the assumption of (3.3) on α and β, the boundary condition $\partial X = Z$ determines the solution X of (0.1) almost surely.

1. Stochastic processes and Martingales in the plane. Throughout this paper the domain of definition of a stochastic process in the plane is the set $D = [0, \infty) \times [0, \infty)$. We write ∂D for the boundary of D as a subset of $(-\infty, \infty) \times (-\infty, \infty)$ relative to the Euclidean topology. In D we introduce a partial order $\prec$ by writing

$$(s, t) \prec (s', t') \quad \text{when} \quad s \leqq s' \quad \text{and} \quad t \leqq t' .$$

We write

$$(s, t) \prec\prec (s', t') \quad \text{when} \quad s < s' \quad \text{and} \quad t < t' .$$

A transformation φ of D into a set with a partial order $\prec$ is said to be increasing if

$$(s, t) \prec (s', t') \Rightarrow \varphi(s, t) \prec \varphi(s', t') .$$

A measurable transformation of a measurable space $(S_1, \mathfrak{A}_1)$ into another $(S_2, \mathfrak{A}_2)$ will be called an $\mathfrak{A}_1/\mathfrak{A}_2$ measurable transformation. We write $\mathfrak{B}(S)$ for the σ-algebra of Borel sets in a topological space S. For the Lebesque measures on $(\boldsymbol{R}^k, \mathfrak{B}(\boldsymbol{R}^k))$ for $k = 1, 2, \cdots$ we use the generic notation m_L.

By a 2-parameter stochastic process on a probability space $(\Omega, \mathfrak{F}, P)$ we mean a transformation X of $D \times \Omega$ into $\boldsymbol{R}$ in which $(s, t, \cdot)$ is $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurable for every $(s, t) \in D$. At times we write $X_{s,t}$ for $X(s, t, \cdot)$. By a boundary stochastic process on $(\Omega, \mathfrak{F}, P)$ we mean a transformation Z of $\partial D \times \Omega$ into $\boldsymbol{R}$ in which $Z(s, t, \cdot)$ is $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurable for evey $(s, t) \in \partial D$. When X is a 2-parameter stochastic process its restriction to $\partial D \times \Omega$ is a boundary stochastic process and we write ∂X for it. For two random variables ξ and η on $(\Omega, \mathfrak{F}, P)$ we write $\xi = \eta$ when $\xi(\omega) = \eta(\omega)$ for a.e., $\omega \in \Omega$. We say that two 2-parameter stochastic processes X and Y are equivalent and write $X = Y$ when $X(\cdot, \cdot, \omega) = Y(\cdot, \cdot, \omega)$ for a.e. $\omega \in \Omega$. The equality of two boundary stochastic processes is defined in the same way.

DEFINITION 1.1. Let $\{\mathfrak{F}_{s,t}, (s, t) \in D\}$ be a system of sub-σ-algebras of $\mathfrak{F}$ in a probability space $(\Omega, \mathfrak{F}, P)$. We call

$$(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t}) \equiv (\Omega, \mathfrak{F}, P\colon \{\mathfrak{F}_{s,t}, (s, t) \in D\})$$

an equipped probability space if

1° $(\Omega, \mathfrak{F}, P)$ is a complete probability measure space,

2° $\{\mathfrak{F}_{s,t}, (s, t) \in D\}$ is an increasing system in the sense that $(s, t) \prec (s', t')$ implies $\mathfrak{F}_{s,t} \subset \mathfrak{F}_{s',t'}$,

3° $\mathfrak{F}_{0,0}$ contatins all the null sets in $(\Omega, \mathfrak{F}, P)$,

4° $\{\mathfrak{F}_{s,t}, (s, t) \in D\}$ is a right continuous system in the sense that for every $(s, t) \in D$

$$\mathfrak{F}_{s,t} = \bigcap_{(s,t) \prec\prec (s',t')} \mathfrak{F}_{s',t'} ,$$

5° for every $(s, t) \in D$, $\mathfrak{F}_{s,\cdot} \equiv \sigma(\bigcup_{v \in [0,\infty)} \mathfrak{F}_{s,v})$ and $\mathfrak{F}_{\cdot,t} \equiv \sigma(\bigcup_{u \in [0,\infty)} \mathfrak{F}_{u,t})$ are conditionally independent relative to $\mathfrak{F}_{s,t}$. □

A 2-parameter stochastic process X on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ is said to be $\mathfrak{F}_{s,t}$ adapted if for every $(s, t) \in D$, $X(s, t, \cdot)$ is $\mathfrak{F}_{s,t}/\mathfrak{B}(\boldsymbol{R})$ measurable. Similarly a boundary stochastic process Z is said to be $\mathfrak{F}_{s,t}$ adapted if for every $(s, t) \in \partial D$, $Z(s, t, \cdot)$ is $\mathfrak{F}_{s,t}/\mathfrak{B}(\boldsymbol{R})$ measurable.

DEFINITION 1.2. A 2-parameter stochastic process X on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ is called a 2-parameter martingale relative to $\{\mathfrak{F}_{s,t}, (s, t) \in D\}$ if

1° X is $\mathfrak{F}_{s,t}$ adapted,

2° $E[|X_{s,t}|] < \infty$ for every $(s, t) \in D$,

3° $E[X_{s',t'} | \mathfrak{F}_{s,t}] = X_{s,t}$ whenever $(s, t) \prec (s', t')$.

A 2-parameter martingale X is said to be square integrable if

4° $E[X_{s,t}^2] < \infty$ for every $(s, t) \in D$.

It is said to be right continuous if

5° $\lim_{\substack{(s',t') \to (s,t) \\ (s,t) \prec (s',t')}} X_{s',t'}(\omega) = X_{s,t}(\omega)$ for all $(s, t) \in D$ for $(s, t) \prec (s', t')$ a.e. $\omega \in \Omega$. □

DEFINITION 1.3. Given an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$, we write $\mathfrak{M}_2(\mathfrak{F}_{s,t})$ for the linear space of equivalence classes of right continuous square integrable martingales X on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ with $\partial X = 0$. We write $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ for the collection of $X \in \mathfrak{M}_2(\mathfrak{F}_{s,t})$ such that $(\cdot, \cdot, \omega)$ is continuous on D for a.e. $\omega \in \Omega$. □

DEFINITION 1.4. For $X \in \mathfrak{M}_2(\mathfrak{F}_{s,t})$, we write

$$|X|_{(s,t)} = \{E[X_{s,t}^2]\}^{1/2} \quad \text{for} \quad (s, t) \in D \,, \tag{1}$$

$$|X|_T = |X|_{(T,T)} \quad \text{for} \quad T \in [0, \infty) \,, \tag{2}$$

$$|X| = \sum_{k=1}^{\infty} 2^{-k}\{|X|_k \wedge 1\} \,. \tag{3}$$ □

PROPOSITION 1.5. *$\mathfrak{M}_2(\mathfrak{F}_{s,t})$ is a Banach space relative to the norm $|\cdot|$ and $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ is a closed linear subspace of $\mathfrak{M}_2(\mathfrak{F}_{s,t})$.* □

PROPOSITION 1.6. *If a sequence $\{X^{(n)}, n = 1, 2, \cdots\}$ in $\mathfrak{M}_2(\mathfrak{F}_{s,t})$ converges to some X in $\mathfrak{M}_2(\mathfrak{F}_{s,t})$, then for every $T > 0$*

$$P - \lim_{n \to \infty} \{ \sup_{(s,t) \prec (T,T)} |X_{s,t}^{(n)} - X_{s,t}|\} = 0$$

and there exist a subsequence $\{m\}$ of $\{n\}$ and a null set N in $(\Omega, \mathfrak{F}, P)$ such that

$$\lim_{m \to \infty} X^{(m)}(s, t, \omega) = X(s, t, \omega)$$

uniformly on every bounded subset of D when $\omega \in N^c$. □

2. Stochastic integrals in the plane.

DEFINITION 2.1. Given an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$, by an $\mathfrak{F}_{s,t}$ Brownian motion on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ we mean a 2-parameter stochastic process B on $(\Omega, \mathfrak{F}, P)$ and D which satisfies the following conditions:

1° Every sample function of B is continuous on D.

2° For every finite rectangle of the type $\Delta = (s', s''] \times (t', t''] \subset D$ the random variable $B(\Delta)$ defined by

$$B(\Delta) = B(s'', t'') - B(s', t'') - B(s'', t') + B(s', t')$$

is destributed by $N(0, (s'' - s')(t'' - t'))$.

3° For every finite collection $\{\Delta_1, \cdots, \Delta_n\}$ of disjoint rectangles of the type above, the system of random variables $\{B(\Delta_1), \cdots, B(\Delta_n)\}$ in an independent system.

4° B is an $\mathfrak{F}_{s,t}$ adapted stochastic process.

We write $\partial B = 0$ for a 2-parameter Brownian motion B when every sample function of B vaninshes on ∂D. □

DEFINITION 2.2. A 2-parameter stochastic process X on a probability space $(\Omega, \mathfrak{F}, P)$ and D is called a measurable process if it is $\sigma(\mathfrak{B}(D) \times \mathfrak{F})/\mathfrak{B}(\boldsymbol{R})$ measurable. A 2-parameter stochastic process X on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and D is said to be progressively measurable if for every $(s, t) \in D$ the restriction of X to $[0, s] \times [0, t] \times \Omega$ is $\sigma(\mathfrak{B}([0, s] \times [0, t]) \times \mathfrak{F}_{s,t})/\mathfrak{B}(\boldsymbol{R})$ measurable. □

PROPOSITION 2.3. *Let X be an $\mathfrak{F}_{s,t}$ adapted 2-parameter stochastic process on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. If every sample function of X is right continuous* i.e., *for every $\omega \in \Omega$*

$$\lim_{\substack{(u, v) \to (s, t) \\ (s, t) < (u, v)}} X(u, v, \omega) = X(s, t, \omega) \quad \textit{for} \quad (s, t) \in D\ ,$$

then X is progressively measurable. The same holds when every sample function of X is left continuous.

DEFINITION 2.4. By $\mathfrak{L}_p(\mathfrak{F}_{s,t})$, $p \geqq 1$, we mean the linear space of equivalence classes of 2-parameter stochastic process Φ on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and D which satisfy the following conditions:

1° Φ is $\mathfrak{F}_{s,t}$ adapted.

2° Φ is a measurable process.

3° For every $T > 0$

$$\|\Phi\|_{p,T} \equiv \boldsymbol{E}\Big[\int_{[0,T]\times[0,T]} |\Phi(s, t, \cdot)|^p m_L(d(s, t))\Big]^{1/p} < \infty\ .$$

For $\Phi \in \mathfrak{L}_p(\mathfrak{F}_{s,t})$ we define

$$\|\Phi\|_p = \sum_{k=1}^{\infty} 2^{-k}\{\|\Phi\|_{p,k} \wedge 1\}\ ,$$

and for $(s, t) \in D$ we define

$$\|\Phi\|_{p,(s,t)} = E\left[\int_{[0,s]\times[0,t]} |\Phi(u, v, \cdot)|^p m_L(d(s, t))\right]^{1/p}. \qquad \square$$

It is easy to verify that $\|\cdot\|_p$ is a norm on $\mathfrak{L}_p(\mathfrak{F}_{s,t})$ and $\mathfrak{L}_p(\mathfrak{F}_{s,t})$ is a Banach space relative to this norm.

PROPOSITION 2.5. *Consider the space* $\mathfrak{L}_p(\mathfrak{F}_{s,t})$, $p \geqq 1$, *on an equipped probability space* $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. *Every member of* $\mathfrak{L}_p(\mathfrak{F}_{s,t})$ *has a progressively measurable version. In fact if* Φ *is a 2-parameter stochastic process on* $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ *which satisfies the conditions* 1°, 2°, 3° *of Definition* 2.4, *then there exists a null set* N *in* $(\Omega, \mathfrak{F}, P)$ *such that if we define*

$$\Psi(s, t, \omega) = \begin{cases} \limsup_{h \downarrow 0} h^{-2} \int_{[s-h,s]\times[t-h,t]} \Phi(u, v, \omega) m_L(d(u, v)) \\ \qquad \text{for} \quad (s, t) \in D \quad \text{and} \quad \omega \in N^c, \\ 0 \qquad \text{for} \quad (s, t) \in D \quad \text{and} \quad \omega \in N, \end{cases}$$

then Ψ *satisfies* 1°, 2°, 3° *of Definition* 2.4, *and is progressively measurable and* $\Psi = \Phi$ i.e., $\Psi(\cdot, \cdot, \omega) = \Phi(\cdot, \cdot, \omega)$ *for* a.e. $\omega \in \Omega$. $\square$

DEFINITION 2.6. By $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ we mean the linear space of equivalence classes of 2-parameter stochastic processes Φ on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and D which satisfies the following conditions:

1° Φ is $\mathfrak{F}_{s,t}$ adapted.

2° Φ is a measurable process.

3° Φ is bounded in the sense that there exists $M > 0$ such that

$$|\Phi(s, t, \omega)| \leqq M \quad \text{for} \quad (s, t) \in D \quad \text{and a.e.} \quad \omega \in \Omega .$$

4° There exist $0 = s_0^* < s_1^* < \cdots$ with $\lim_{i\to\infty} s_i^* = \infty$ and $0 = t_0^* < t_1^* < \cdots$ with $\lim_{j\to\infty} t_j^* = \infty$ such that

$$\Phi(s, t, \omega) = \Phi(s_i^*, t_j^*, \omega) \quad \text{for} \quad (s, t) \in [s_i^*, s_{i+1}^*) \times [t_j^*, t_{j+1}^*) ,$$
$$i, j = 0, 1, 2, \cdots \quad \text{for a.e.} \quad \omega \in \Omega. \qquad \square$$

Clearly $\mathfrak{L}_0(\mathfrak{F}_{s,t}) \subset \mathfrak{L}_p(\mathfrak{F}_{s,t})$ for $p \geqq 1$.

PROPOSITION 2.7. *For every* $p \geqq 1$, $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ *is dense in* $\mathfrak{L}_p(\mathfrak{F}_{s,t})$ *relative to the metric associated with* $\|\cdot\|_p$, i.e., *for every* $\Phi \in \mathfrak{L}_p(\mathfrak{F}_{s,t})$ *there exists a sequence* $\{\Phi_n, n = 1, 2, \cdots\}$ *in* $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ *such that* $\lim_{n\to\infty} \|\Phi_n - \Phi\|_p = 0$. $\square$

DEFINITION 2.8. Let B be an $\mathfrak{F}_{s,t}$ Brownian motion on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and let Φ be a 2-parameter stochastic process of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ represented as

$$(1) \quad \Phi(s, t, \omega) = \Phi(s_i^*, t_j^*, \omega) \quad \text{for} \quad (s, t) \in [s_i^*, s_{i+1}^*) \times [t_i^*, t_{j+1}^*) , \quad i, j = 0, 1, 2, \cdots \quad \text{and} \quad \omega \in N^c ,$$

where N is a null set in $(\Omega, \mathfrak{F}, P)$. The stochastic integral $I(\Phi)$ of Φ with respect to B is a 2-parameter stochastic process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ and D defined as follows:

For $(s, t) \in D$, say, $(s, t) \in [s_{m-1}^*, s_m^*) \times [t_{n-1}^*, t_n^*)$ let

$$(2) \quad \begin{cases} s_i = s_i^* & \text{for} \quad i = 0, 1, \cdots, m-1 \quad \text{and} \quad s_m = s \\ t_j = t_j^* & \text{for} \quad j = 0, 1, \cdots, n-1 \quad \text{and} \quad t_n = t \end{cases}$$

and

$$(3) \quad \begin{cases} \varphi_{i,j}(\omega) = \Phi(s_i, t_j, \omega) & \text{for} \quad i = 0, 1, \cdots, m ; \\ & \qquad j = 1, 2, \cdots, n \quad \text{and} \quad \omega \in N^c , \\ \beta_{i,j}(\omega) = B((s_{i-1}, s_i] \times (t_{j-1}, t_j])(\omega) & \\ & \text{for} \quad i = 1, 2, \cdots, m ; \\ & \qquad j = 1, 2, \cdots, n \quad \text{and} \quad \omega \in \Omega . \end{cases}$$

We define

$$(4) \quad I(\Phi)(s, t, \omega) = \begin{cases} \sum_{i=1}^{m} \sum_{j=1}^{n} \varphi_{i-1,j-1}(\omega)\beta_{i,j}(\omega) & \text{for} \quad \omega \in N^c , \\ 0 & \text{for} \quad \omega \in N . \end{cases} \qquad \square$$

COROLLARY 2.9. *$I(\Phi)$ as defined above for Φ in the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ has the following properties:*

(1) *$I(\Phi)$ does not depend on the representation* (1) *of Φ in Definition* 2.8. *Also, if Φ and Ψ are equivalent so are $I(\Phi)$ and $I(\Psi)$.*

(2) *$I(\Phi)$ is an $\mathfrak{F}_{s,t}$ adapted process.*

(3) *Every sample function of $I(\Phi)$ is continuous on D.*

(4) $E[|(\Phi)(s, t, \cdot)|^2] < \infty$ *from every $(s, t) \in D$ (from the boundedness of Φ).*

(5) $\partial[I(\Phi)] = 0.$ $\square$

PROPOSITION 2.10. *$I(\Phi)$ as defined in Definition* 2.8 *for Φ in the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ is a* 2-*parameter martingale of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$.* $\square$

PROPOSITION 2.11. *Let $I(\Phi)$ be as in Definition* 2.8 *for Φ in the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$. Then*

$$(1) \qquad |I(\Phi)|_{(s,t)} = \|\Phi\|_{2,(s,t)} \quad \textit{for every} \quad (s, t) \in D$$

so that in particular

$$(2) \qquad |I(\Phi)|_T = \|\Phi\|_{2,T} \quad \textit{for every} \quad T > 0$$

and

$$(3) \qquad |I(\Phi)| = \|\Phi\|_2 . \qquad \square$$

Let Φ be a 2-parameter stochastic process of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ and let B be an $\mathfrak{F}_{s,t}$ Brownian motion on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. According to Proposition 2.7 there exists a sequence $\{\Phi_n, n = 1, 2, \cdots\}$ of 2-parameter stochastic processes of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ satisfying the condition $\lim_{n\to\infty} \|\Phi_n - \Phi\|_2 = 0$. According to Proposition 2.10 the stochastic integral $I(\Phi_n)$ of Φ_n with respect to B is a 2-parameter martingale of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ and according to Proposition 2.11 $|I(\Phi_m) - I(\Phi_n)| = \|\Phi_m - \Phi_n\|_2$ so that $\{I(\Phi_n), n = 1, 2, \cdots\}$ is a Cauchy sequence in $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$. Since $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ is a complete metric space by Proposition 1.5, there exists a 2-parameter martingale of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$, which we denote by $I(\Phi)$, such that

$$(2.1) \qquad \lim_{n\to\infty} |I(\Phi_n) - I(\Phi)| = 0 .$$

Observe that if $\{\Phi_n', n = 1, 2, \cdots\}$ is another sequence of 2-parameter stochastic processes of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ such that $\lim_{n\to\infty} \|\Phi_n' - \Phi\|_2 = 0$ and if $I(\Phi')$ is a 2-parameter martingale of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ such that $\lim_{n\to\infty} |I(\Phi_n') - I(\Phi')| = 0$ then by considering the sequence $\{\Phi_1, \Phi_1', \Phi_2, \Phi_2', \cdots\}$ we conclude that $I(\Phi) = I(\Phi')$ in $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ i.e., the two 2-parameter stochastic processes $I(\Phi)$ and $I(\Phi')$ are equivalent.

According to Definition 2.8 every sample function of $I(\Phi_n)$ is continuous on D. By Proposition 1.6 there exists a null set N in $(\Omega, \mathfrak{F}, P)$ and a subsequence $\{m\}$ of $\{n\}$ such that

$$(2.2) \qquad \lim_{m\to\infty} I(\Phi_m)(s, t, \omega) = I(\Phi)(s, t, \omega)$$

uniformly on every bounded subset of D when $\omega \in N^c$ so that $I(\Phi)(\cdot, \cdot, \omega)$ is continuous on D for $\omega \in N^c$. Let us define

$$(2.3) \qquad I(\Phi)(s, t, \omega) = 0 \quad \text{for} \quad (s, t) \in D \quad \text{when} \quad \omega \in N .$$

Then every sample function of $I(\Phi)$ is continuous on D.

DEFINITION 2.12. By the stochastic integral of a 2-parameter stochastic process Φ of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ with respect to an $\mathfrak{F}_{s,t}$ Brownian motion B on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ we mean the 2-parameter martingale $I(\Phi)$ of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$ defined by (2.1), (2.2) and (2.3). We also use the notation

$$(2.4) \qquad \int_{[0,s]\times[0,t]} \Phi(u, v)dB(u, v) \quad \text{for} \quad I(\Phi)(s, t, \cdot) \quad \text{for} \quad (s, t) \in D . \qquad \square$$

Note that from $\lim_{n\to\infty}\|\Phi_n\|_2 = \|\Phi\|_2$, $\lim_{n\to\infty}|I(\Phi_n)| = |I(\Phi)|$ and $\|\Phi_n\|_2 = |I(\Phi_n)|$ for $n = 1, 2, \cdots$, we have

$$(2.5) \qquad |I(\Phi)| = \|\Phi\|_2 .$$

This implies

$$(2.6) \qquad |I(\Phi)|_{(s,t)} = \|\Phi\|_{2,(s,t)} \quad \text{for every} \quad (s, t) \in D .$$

Given an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$, let Ψ be a 2-parameter stochastic process of the class $\mathfrak{L}_1(\mathfrak{F}_{s,t})$. By Proposition 2.5 we assume that Ψ is a progressively measurable version. By 3° of Definition 2.4 there exists a null set N in $(\Omega, \mathfrak{F}, P)$ such that

$$\int_{[0,T]\times[0,T]} |\Psi(s, t, \omega)|(m_L(d(s, t))) < \infty \quad \text{for all} \quad T > 0 \quad \text{when} \quad \omega \in N^c .$$

Let us define a real valued function $J(\Psi)$ on $D \times \Omega$ by

$$(2.7) \qquad J(\Psi)(s, t, \omega) = \begin{cases} \displaystyle\int_{[0,s]\times[0,t]} \Psi(u, v, \omega) m_L(d(u, v)) & \\ \qquad \text{for} \quad (s, t) \in D \quad \text{when} \quad \omega \in N^c , & \\ 0 \qquad \text{for} \quad (s, t) \in D \quad \text{when} \quad \omega \in N . & \end{cases}$$

The measurability of Ψ and the fact that $N \in \mathfrak{F}$ imply that $J(\Psi)$ is a stochastic process on $(\Omega, \mathfrak{F}, P)$ and D. The progressive measurability of Ψ and the fact that $N \in \mathfrak{F}_{s,t}$ for every $(s, t) \in D$ imply that $J(\Psi)$ is an $\mathfrak{F}_{s,t}$ adapted process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. Also by (2.7) every sample function of $J(\Psi)$ is continuous.

DEFINITION 2.13. By the stochastic integral of a 2-parameter stochastic process Ψ of the class $\mathfrak{L}_1(\mathfrak{F}_{s,t})$ with respect to the Lebesgue measure we mean the $\mathfrak{F}_{s,t}$ adapted 2-parameter stochastic process $J(\Psi)$ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ defined by (2.7). We also use the notation

$$(2.8) \qquad \int_{[0,s]\times[0,t]} \Psi(u, v) d(u, v) \quad \text{for} \quad J(\Psi)(s, t, \cdot) \quad \text{for} \quad (s, t) \in D . \qquad \square$$

3. Stochastic differential equations in the plane. The following function spaces are needed in the definition of our stochastic differential equation and its solution.

Let W be the collection of all real valued continuous functions w on D. Let

$$\rho_W(w_1, w_2) = \sum_{k=1}^{\infty} 2^{-k} \{ \max_{(s,t)<(k,k)} |w_1(s, t) - w_2(s, t)| \wedge 1\} \quad \text{for} \quad w_1, w_2 \in W .$$

Then ρ_W is a metric on W and W is a separable complete metric space relative to ρ_W. A sequence $\{w_n, n = 1, 2, \cdots\}$ in W converges to $w \in W$ in the metric ρ_W if and only if w_n converges to w uniformly on every bounden subset of D.

Let $\mathfrak{Z}$ be the collection of cylinder sets C in W of the type $C = \{w \in W;\, w(s, t) \in E\}$ for some $(s, t) \in D$ and $E \in \mathfrak{B}(\boldsymbol{R})$. Then $\mathfrak{Z} \subset W$ and in fact for the σ-algebra $\mathfrak{B}(W)$ of Borel sets in the metric space W we have

$$\mathfrak{B}(W) = \sigma(\mathfrak{Z}) .$$

For $(s, t) \in D$ let $\mathfrak{Z}_{s,t}$ be the collection of the cylinder sets C in W of the type $C = \{w \in W;\, w(u, v) \in E\}$ for some $(u, v) \prec (s, t)$ and $E \in \mathfrak{B}(\boldsymbol{R})$. We define

$$\mathfrak{B}_{s,t}(W) = \sigma(\mathfrak{Z}_{s,t}) .$$

Let ∂W and $W^{(1)}$ be the collections of all real valued continuous functions on ∂D and $[0, \infty)$ respectively. We define a metric $\rho_{\partial W}$ on ∂W and metric $\rho_{W^{(1)}}$ on $W^{(1)}$ in the same way as we defined ρ_W on W. Then ∂W and $W^{(1)}$ are separable complete metric spaces with respect to $\rho_{\partial W}$ and $\rho_{W^{(1)}}$ respectively. We write $\mathfrak{B}(\partial W)$ and $\mathfrak{B}(W^{(1)})$ for the σ-algebras of Borel sets in the two metric spaces.

PROPOSITION 3.1. *For every transformation θ of an arbitrary measurable space $(\Omega, \mathfrak{F})$ into the measurable space $(W, \mathfrak{B}(W))$ the following hold:*

(1) *θ is $\mathfrak{F}/\mathfrak{B}(W)$ measurable if and only if for every $(s, t) \in D$, $[\theta(\cdot)](s, t)$ is an $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of Ω into $\boldsymbol{R}$.*

(2) *For each $(s, t) \in D$, θ is $\mathfrak{F}/\mathfrak{B}_{s,t}(W)$ measurable if and only if for every $(u, v) \prec (s, t)$, $[\theta(\cdot)](u, v)$ is an $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of Ω into $\boldsymbol{R}$.* $\square$

Proof. Since (1) and (2) can be proved in the same way let us prove (2) here. Let $(s, t) \in D$ be fixed.

To prove the necessity of the condition, note that since the transformation of W into $\boldsymbol{R}$ defined by $w \to w(u, v)$ is $\mathfrak{B}_{s,t}(W)/\mathfrak{B}(\boldsymbol{R})$ measurable for every $(u, v) \prec (s, t)$ assumption of the $\mathfrak{F}/\mathfrak{B}_{s,t}(W)$ measurability of θ implies the $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurability of $[\theta(\cdot)](u, v)$.

To prove the sufficiency of the condition suppose that for every $(u, v) \prec (s, t)$, $[\theta(\cdot)](u, v)$ is an $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of Ω into $\boldsymbol{R}$. Let $C \in \mathfrak{Z}_{s,t}$ be given by

$$C = \{w \in W;\, w(u, v) \in E\} \quad \text{where} \quad E \in \mathfrak{B}(\boldsymbol{R}) .$$

Then from the $\mathfrak{F}/\mathfrak{B}(\boldsymbol{R})$ measurability of $[\theta(\cdot)](u, v)$,

$$\theta^{-1}C = \{\omega \in \Omega;\, [\theta(\omega)](u, v) \in E\} \in \mathfrak{F} .$$

From the arbitrariness of $C \in \mathfrak{Z}_{s,t}$ we have $\theta^{-1}(\mathfrak{Z}_{s,t}) \subset \mathfrak{F}$ and therefore

$$\theta^{-1}(\mathfrak{B}_{s,t}(W)) = \theta^{-1}(\sigma(\mathfrak{Z}_{s,t})) = \sigma(\theta^{-1}(\mathfrak{Z}_{s,t})) \subset \mathfrak{F}$$

i.e., θ is $\mathfrak{F}/\mathfrak{B}_{s,t}(W)$ measurable. □

DEFINITION 3.2. We write $M(D \times W)$ for the collection of all real valued functions α on $D \times W$ satisfying the following measurability conditions:

1° α is a $\sigma(\mathfrak{B}(D) \times \mathfrak{B}(W))/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of $D \times W$ into $\boldsymbol{R}$.

2° For every $(s, t) \in D$, $\alpha(s, t, \cdot)$ is a $\mathfrak{B}_{s,t}(W)/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of W into $\boldsymbol{R}$. □

DEFINITION 3.3. Let $\alpha, \beta \in M(D \times W)$. By a solution of the stochastic differential equation

$$dX(s, t) = \alpha(s, t, X)dB(s, t) + \beta(s, t, X)d(s, t) \tag{3.1}$$

we mean a system of two 2-parameter stochastic processes (X, B) on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ satisfying the following conditions:

1° B is an $\mathfrak{F}_{s,t}$ Brownian motion with $\partial B = 0$.

2° X is an $\mathfrak{F}_{s,t}$ adapted process whose sample functions are all continuous on D.

3° If we set

$$\Phi(s, t, \omega) = \alpha(s, t, X(\cdot, \cdot, \omega)) \quad \text{for} \quad (s, t, \omega) \in D \times \Omega$$
$$\Psi(s, t, \omega) = \beta(s, t, X(\cdot, \cdot, \omega)) \quad \text{for} \quad (s, t, \omega) \in D \times \Omega$$

then Φ and Ψ are in the classes $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ and $\mathfrak{L}_1(\mathfrak{F}_{s,t})$ respectively.

4° With probability 1

$$\begin{aligned} X(s, t) - X(s, 0) - X(0, t) + X(0, 0) \\ = \int_{[0,s]\times[0,t]} \alpha(u, v, X)\, dB(u, v) + \int_{[0,s]\times[0,t]} \beta(u, v, X)d(u, v) \\ \text{for all} \quad (s, t) \in D\,. \end{aligned} \tag{3.2}$$

□

REMARK 3.4. The condition 2° of Definition 3.3 is equivalent to the condition that X is an $\mathfrak{F}/\mathfrak{B}(W)$ measurable transformation of Ω into W and furthermore for every $(s, t) \in D$, X is an $\mathfrak{F}_{s,t}/\mathfrak{B}_{s,t}(W)$ measurable transformation of Ω into W. □

REMARK 3.5. Φ and Ψ as defined in 3° of Definition 3.3 are measurable and $\mathfrak{F}_{s,t}$ adapted processes on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. The assumption in 3° that Φ and Ψ are in the classes $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ and $\mathfrak{L}_1(\mathfrak{F}_{s,t})$ respectively implies according to Proposition 2.5 that they have equivalent processes on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ which are progressively measurable.

Therefore we shall always assume that Φ and Ψ themselves are progressively measurable. $\square$

DEFINITION 3.6. A solution (X, B) of the stochastic differential equation (3.1) on an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ is called a strong solution if there exists a transformation F of $\partial W \times W$ into W satisfying the following conditions:

1° F is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(W)$ measurable.

2° For each $z \in \partial W$, $F[z, \cdot]$ is a $\mathfrak{B}_{s,t}(W)/\mathfrak{B}_{s,t}(W)$ measurable transformation of W into W for every $(s, t) \in D$.

3° With the boundary stochastic process ∂X of X we have

$$X(\cdot, \cdot, \omega) = F[\partial X(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for a.e.} \quad \omega \in \Omega . \qquad \square$$

DEFINITION 3.7. We say that the solution of the stochastic differential equation (3.1) is pathwise unique if whenever there are two solutions (X, B) and (X', B) with the same B on the same equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ such that $\partial X = \partial X'$, then $X = X'$. $\square$

As sufficient conditions for the pathwise uniqueness of the solution and for the existence of a strong solution of the stochastic differential equation (3.1) we have the following Lipschitz condition and growth condition on the coefficients α, β in (3.1): There exists a Borel measure λ on D which is finite for every compact subset of D and satisfies the condition that for every $T > 0$ there exists $L_T > 0$ such that

$$\begin{aligned} (3.3) \qquad & |\alpha(s, t, w) - \alpha(s, t, w')|^2 + |\beta(s, t, w) - \beta(s, t, w')|^2 \\ & \leqq L_T \left\{ \int_{[0,s]\times[0,t]} |w(u, v) - w'(u, v)|^2 \lambda(d(u, v)) \right. \\ & \qquad \left. + |w(s, t) - w'(s, t)|^2 \right\} , \end{aligned}$$

$$\begin{aligned} (3.4) \qquad & |\alpha(s, t, w)|^2 + |\beta(s, t, w)|^2 \\ & \leqq L_T \left\{ \int_{[0,s]\times[0,t]} |w(u, v)|^2 \lambda(d(u, v)) + |w(s, t)|^2 + 1 \right\} \end{aligned}$$

for all $(s, t) \prec (T, T)$ and $w, w' \in W$. We shall show by the method of successive approximation that (3.3) ensures the pathwise uniqueness of the solution and that (3.3) and (3.4) together ensure the existence of a strong solution for (3.1).

THEOREM 3.8. *If the coefficients α and β in the stochastic differential equation* (3.1) *in Definition* 3.3 *satisfy the Lipschitz condition* (3.3) *then its solution is pathwise unique.* $\square$

Proof. Let (X, B) and (X', B) be two solutions of (3.1) on the same equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ with $\partial X = \partial X'$. We proceed to show that $X = X'$. Since the sample functions of X and X' are all continuous on D, it suffices to show that for each $(s, t) \in D$ we have $X(s, t) = X'(s, t)$, i.e., $X(s, t, \omega) = X'(s, t, \omega)$ for a.e. $\omega \in \Omega$. We shall show this by showing

$$(1) \qquad P\{|X(s, t) - X'(s, t)| > 0\} = 0 \quad \text{for every} \quad (s, t) \in D .$$

Now since both X and X' satisfy (3.2) and since $\partial X = \partial X'$, we have with probability 1

$$(2) \qquad X(s, t) - X'(s, t) = \int_{[0,s]\times[0,t]} \{\alpha(u, v, X) - \alpha(u, v, X')\} dB(u, v) + \int_{[0,s]\times[0,t]} \{\beta(u, v, X) - \beta(u, v, X')\} d(u, v) \quad \text{for all} \quad (s, t) \in D .$$

For each positive integer N and $(s, t) \in D$ let

$$(3) \qquad A^N_{s,t} = \{\omega \in \Omega; \sup_{(u,v)\prec(s,t)} (|X(u, v, \omega)|^2 + |X'(u, v, \omega)|^2) \leqq N\} .$$

Since $|X|$ and $|X'|$ are $\mathfrak{F}_{s,t}$ adapted processes on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$, $|X(u, v, \cdot)|$ and $|X'(u, v, \cdot)|$ are $\mathfrak{F}_{s,t}$ measurable random variables for every $(u, v) \prec (s, t)$. This implies that $A^N_{s,t} \in \mathfrak{F}_{s,t}$ for each $(s, t) \in D$. Let us write $\chi(\cdot, A)$ for the characteristic function of a subset A of Ω and consider the $\{0, 1\}$ valued 2-parameter stochastic process I^N on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ defined by

$$(4) \qquad I^N(s, t, \omega) = \chi(\omega, A^N_{s,t}) \quad \text{for} \quad (s, t, \omega) \in D \times \Omega .$$

Since $A^N_{s,t} \in \mathfrak{F}_{s,t}$ for every $(s, t) \in D$, I^N is an $\mathfrak{F}_{s,t}$ adapted process. Also since the sample functions of $|X|$ and $|X'|$ are all continuous and since for every $\omega \in \Omega$ the function f_ω defined by

$$f_\omega(s, t) = \sup_{(u,v)\prec(s,t)} (|X(u, v, \omega)|^2 + |X'(u, v, \omega)|^2) \quad \text{for} \quad (s, t) \in D$$

is an increasing function on D in the sense of the partial order $\prec$, each sample function of I^N is left continuous in the sense that

$$\lim_{\substack{(u,v)\to(s,t)\\(u,v)\prec(s,t)}} I^N(u, v, \omega) = I^N(s, t, \omega) \quad \text{for} \quad (s, t) \in D .$$

Therefore by Proposition 2.3, I^N is a progressively measurable process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$. Now

$$\{\chi(\omega, A^N_{s\ t})\}^2 = \chi(\omega, A^N_{s,t})$$

and

$$(u, v) \prec (s, t) \Rightarrow A^N_{u,v} \supset A^N_{s,t} \Rightarrow \chi(\omega, A^N_{u,v})\chi(\omega, A^N_{s,t}) = \chi(\omega, A^N_{s,t})$$

and consequently

$$(5) \qquad \{I^N(s, t, \omega)\}^2 = I^N(s, t, \omega) ,$$

$$(6) \qquad (u, v) \prec (s, t) \Rightarrow I^N(u, v, \omega) \geqq I^N(s, t, \omega) ,$$

$$(7) \qquad (u, v) \prec (s, t) \Rightarrow I^N(u, v, \omega)I^N(s, t, \omega) = I^N(s, t, \omega) .$$

By (2), (5) and (6) we have with probability 1

$$I^N(s, t)\{X(s, t) - X'(s, t)\}^2 \\ \leqq 2I^N(s, t)[\{I(\Phi)(s, t)\}^2 + \{J(\Psi)(s, t)\}^2] \quad \text{for all} \quad (s, t) \in D$$

where Φ and Ψ are defined by

$$(8) \qquad \Phi(s, t, \omega) = I^N(s, t, \omega)[\alpha(s, t, X(\cdot, \cdot, \omega)) - \alpha(s, t, X'(\cdot, \cdot, \omega))] ,$$

$$(9) \qquad \Psi(s, t, \omega) = I^N(s, t, \omega)[\beta(s, t, X(\cdot, \cdot, \omega)) - \beta(s, t, X'(\cdot, \cdot, \omega))]$$

for $(s, t, \omega) \in D \times \Omega$ and $I(\Phi)$ and $J(\Psi)$ are stochastic integrals of Φ and Ψ with respect to the $\mathfrak{F}_{s,t}$ Brownian motion and the Lebesgue measure respectively as in (2.4) and (2.8) of §2. Note that by 3° of Definition 3.3, Φ and Ψ are 2-parameter stochastic processes on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ and $\mathfrak{L}_1(\mathfrak{F}_{s,t})$ respectively. From the last inequality we have

$$(10) \qquad \boldsymbol{E}[I^N(s, t)\{X(s, t) - X'(s, t)\}^2] \\ \leqq 2\boldsymbol{E}[|I(\Phi)(s, t)|^2] + 2\boldsymbol{E}[|J(\Psi)(s, t)|^2] \quad \text{for} \quad (s, t) \in D .$$

Let us show that for every $(s, t) \in D$, Φ and Ψ are bounded on $[0, s] \times [0, t] \times \Omega$. Consider Φ for instance. Let $T > 0$ be so large that $(s, t) \prec (T, T)$. By applying (5) and the Lipschitz condition (3.3) to (8) we have for every $\omega \in \Omega$

$$|\Phi(u, v, \omega)|^2 \\ \leqq I^N(u, v, \omega)L_T\left\{\int_{[0,u]\times[0,v]} |X(u', v', \omega) - X'(u', v', \omega)|^2\lambda(d(u', v')) \right. \\ \left. + |X(u, v, \omega) - X'(u, v, \omega)|^2\right\} \quad \text{for} \quad (u, v) \prec (s, t) .$$

Taking $I^N(u, v, \omega)$ under the integral sign for the first term on the right side and then recalling (4) and (3) we have for every $\omega \in \Omega$

$$|\Phi(u, v, \omega)|^2 \leqq L_T\{2N\lambda([0, s] \times [0, t]) + 2N\} \quad \text{for} \quad (u, v) \prec (s, t) .$$

This proves the boundedness of Φ on $[0, s] \times [0, t] \times \Omega$. The same estimate holds for Ψ. Also this implies in particular that Ψ belongs not only in the class $\mathfrak{L}_1(\mathfrak{F}_{s,t})$ but also in the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$.

Returning to the stochastic integral $I(\Phi)$ we have by (2.6) in § 2 and the Fubini theorem

$$(11)\qquad E[|I(\Phi)(s,t)|^2]=\int_{[0,s]\times[0,t]}E[|\Phi(u,v)|^2]m_L(d(u,v))\,.$$

For the stochastic integral $J(\Psi)$ we have by the Schwarz Inequality and the Fubini theorem

$$(12)\qquad E[|J(\Psi)(s,t)|^2]\leqq st\int_{[0,s]\times[0,t]}E[|\Psi(u,v,\cdot)|^2]m_L(d(u,v))\,.$$

Using (11), (12) in (10), applying (3.3) and recalling that $I^N(u,v)\leqq I^N(u',v')$ for $(u',v')\prec(u,v)$ by (6) we obtain

$$\begin{aligned}(13)\quad & E[I^N(s,t)\{X(s,t)-X'(s,t)\}^2]\\ &\leqq 2(1+st)\int_{[0,s]\times[0,t]}E[|\Phi(u,v)|^2+|\Psi(u,v)|^2]m_L(d(u,v))\\ &\leqq 2(1+st)L_T\int_{[0,s]\times[0,t]}E\Big[\int_{[0,u]\times[0,v]}I^N(u',v')\\ &\qquad\times\{X(u',v')-X'(u',v')\}\lambda(d(u',v'))\\ &\qquad+I^N(u,v)\{X(u,v)-X'(u,v)\}^2\Big]m_L(d(u,v))\\ &\qquad\qquad\text{for}\quad (s,t)\prec(T,T)\,.\end{aligned}$$

Let

$$(14)\qquad c(s,t)=\sup_{(u,v)\prec(s,t)}E[I^N(u,v)\{X(u,v)-X'(u,v)\}^2]\quad\text{for}\quad(s,t)\in D\,.$$

Using (14) in (13) we obtain

$$\begin{aligned}&E[I^N(s,t)\{X(s,t)-X'(s,t)\}^2]\\ &\quad\leqq 2(1+T^2)L_T\{\lambda([0,T]\times[0,T])+1\}\int_{[0,s]\times[0,t]}c(u,v)m_L(d(u,v))\\ &\qquad\qquad\text{for}\quad(s,t)\prec(T,T)\,.\end{aligned}$$

Since c is nonnegative, the last inequality implies

$$\begin{aligned}&E[I^N(u,v)\{X(u,v)-X'(u,v)\}^2]\\ &\quad\leqq 2(1+T^2)L_T\{\lambda([0,T]\times[0,T])+1\}\int_{[0,s]\times[0,t]}c(u',v')m_L(d(u',v'))\\ &\qquad\qquad\text{for}\quad(u,v)\prec(s,t)\prec(T,T)\,.\end{aligned}$$

Taking the supremum over $(u,v)\prec(s,t)$ on the left side we have by (14)

$$\begin{aligned}(15)\qquad c(s,t)&\leqq 2(1+T^2)L_T\{\lambda([0,T]\times[0,T])+1\}\\ &\quad\times\int_{[0,s]\times[0,t]}c(u,v)m_L(d(u,v))\quad\text{for}\quad(s,t)\prec(T,T)\,.\end{aligned}$$

Now for every $T > 0$, the Borel measurable function c on D is bounded on $[0, T] \times [0, T]$ since

$$c(s, t) \leqq c(T, T) < \infty \quad \text{for} \quad (s, t) \prec (T, T)$$

by (14), (10) and the boundedness of Φ and Ψ on $[0, T] \times [0, T] \times \Omega$. For brevily let us write

$$a = 2(1 + T^2)L_T\{\lambda([0, T] \times [0, T]) + 1\} .$$

Then by iterating (15) n times we have

$$\begin{aligned} c(s, t) &\leqq a^{n+1}\int_{[0,s]\times[0,t]}\int_{[0,u_1]\times[0,v_1]}\cdots\int_{[0,u_n]\times[0,v_n]} c(u_{n+1}, v_{n+1}) \\ &\qquad \times m_L(d(u_{n+1}, v_{n+1}))\cdots m_L(d(u_2, v_2))m_L(d(u_1, v_1)) \\ &\leqq a^{n+1}c(T, T)\frac{(st)^{n+1}}{\{(n+1)!\}^2} \quad \text{for} \quad (s, t) \prec (T, T) . \end{aligned}$$

Since this holds for every positive integer n and since $T > 0$ is arbitrary, we have

$$c(s, t) = 0 \quad \text{for} \quad (s, t) \in D .$$

Using this in (14) and recalling (4) we have

$$E[\chi(\cdot, A^N_{s,t})\{X(s, t) - X'(s, t)\}^2] = 0 \quad \text{for} \quad (s, t) \in D .$$

Since the second factor in the expectation is nonnegative, the last equality implies

$$\{\omega \in \Omega; |X(s, t, \omega) - X'(s, t, \omega)| > 0\} \subset (A^N_{s\,t})^c \cup A^*_{s\,t} \quad \text{for} \quad (s, t) \in D$$

where $A^*_{s\,t}$ is a null set in $(\Omega, \mathfrak{F}, P)$. Thus

$$(16) \qquad P\{|X(s, t) - X'(s, t)| > 0\} \leqq P((A^N_{s,t})^c) \quad \text{for} \quad (s, t) \in D .$$

Since $\{A^N_{s,t},\ N = 1, 2, \cdots\}$ is a monotone increasing sequence of sets, $\{P((A^N_{s,t})^c),\ N = 1, 2, \cdots\}$ is a monotone decreasing sequence of nonnegative numbers. To show that it actually converges to 0, assume the contrary. Then there exists $\varepsilon > 0$ such that

$$P\left(\bigcap_{N=1}^{\infty} (A^N_{s,t})^c\right) = \lim_{N\to\infty} P((A^N_{s,t})^c) \geqq \varepsilon .$$

In other words

$$P\{\omega \in \Omega;\ \sup_{(u,v)\prec(s,t)} (|X(u, v, \omega)|^2 + |X'(u, v, \omega)|^2) > N \text{ for all } N\} \geqq \varepsilon ,$$

i.e.,

$$P\{\omega \in \Omega;\ \sup_{(u,v)\prec(s,t)} (|X(u, v, \omega)|^2 + |X'(u, v, \omega)|^2) = \infty\} \geqq \varepsilon .$$

This contradicts the fact that the sample functions of X and X' are all continuous on D and consequently the supremum in the above expression is finite for every $\omega \in \Omega$. Therefore

$$\lim_{N\to\infty} P((A_{s,t}^N)^c) = 0 . \tag{17}$$

Letting $N \to \infty$ on the right side of (16) and applying (17) we have (1). $\square$

DEFINITION 3.9. Given an equipped probability space $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$, let $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ be the collection of 2-parameter stochastic processes X on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ which satisfy the following conditions:

1° X is $\mathfrak{F}_{s,t}$ adapted.

2° Every sample function of X is continuous on D.

3° For every $T > 0$

$$\gamma(T; X) \equiv \sup_{(s,t) \prec (T,T)} \boldsymbol{E}[|X(s, t)|^2] < \infty .$$

Let $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t} | \partial D)$ be the collection of boundary stochastic processes Z on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ which satisfy the following conditions:

4° Z is $\mathfrak{F}_{s,t}$ adapted.

5° Every sample function of Z is continuous on ∂D.

6° For every $T > 0$

$$\gamma(T; Z) \equiv \sup_{\substack{(s,t) \prec (T,T) \\ (s,t) \in \partial D}} \boldsymbol{E}[|Z(s, t)|^2] < \infty . \qquad \square$$

Note that if X is in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$, then X is of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$.

LEMMA 3.10. *Let $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ be an equipped probability space on which an $\mathfrak{F}_{s,t}$ Brownian motion B with $\partial B = 0$ exists. Suppose that the coefficients α and β in the stochastic differential equation* (3.1) *in Definition* 3.3 *satisfy the growth condition* (3.4). *Let $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t} | \partial D)$ be fixed.*

For every $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ define a 2-parameter stochastic process τX on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ by

$$\begin{aligned}(\tau X)(s, t) = &-Z(0, 0) + Z(s, 0) + Z(0, t) \\ &+ \int_{[0,s]\times[0,t]} \alpha(u, v, X)dB(u, v) \\ &+ \int_{[0,s]\times[0,t]} \beta(u, v, X)d(u, v) \quad \text{for} \quad (s, t) \in D .\end{aligned} \tag{1}$$

Then

$$\tau X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t}) . \tag{2}$$

With the transformation τ of $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ into itself as given above, define a sequence $\{X^{(i)},\ i = 0, 1, 2, \cdots\}$ in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ by

$$(3)\qquad \begin{cases} X^{(0)}(s, t) = -Z(0, 0) + Z(s, 0) + Z(0, t) & \text{for} \quad (s, t) \in D\,, \\ X^{(i)}(s, t) = (\tau X^{(i-1)})(s, t) & \text{for} \quad (s, t) \in D \quad \text{and} \quad i = 1, 2, \cdots\,. \end{cases}$$

Then for every $T > 0$ there exists $M_T > 0$ such that

$$(4)\qquad \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^{(i)}(s, t)|^2] \leqq M_T \quad \text{for} \quad i = 0, 1, 2, \cdots\,. \qquad \square$$

Proof. Let $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$. As we saw in Remark 3.5, the fact that $\alpha, \beta \in \boldsymbol{M}(D \times W)$ and X is an $\mathfrak{F}_{s,t}$ adapted process whose sample functions are all continuous on D implies that the two 2-parameter stochastic process Φ and Ψ on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ defined by

$$(5)\qquad \begin{cases} \Phi(s, t, \omega) = \alpha(s, t, X(\cdot, \cdot, \omega)) \\ \Psi(s, t, \omega) = \beta(s, t, X(\cdot, \cdot, \omega)) \end{cases}$$

are $\mathfrak{F}_{s,t}$ adapted and measurable processes. Furthermore the assumption that α and β satisfy (3.4) implies that Φ and Ψ are both of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$. In fact for every $T > 0$ we have by (3.4)

$$\begin{aligned}(6)\qquad & \boldsymbol{E}\left[\iint_{[0,T]\times[0,T]} \{|\Phi(s, t)|^2 + |\Psi(s, t)|^2\} m_L(d(s, t))\right] \\ & \leqq \boldsymbol{E}\left[\iint_{[0,T]\times[0,T]} L_T\left\{\iint_{[0,s]\times[0,t]} |X(u, v)|^2 \lambda(d(u, v))\right.\right. \\ & \qquad \left.\left. + |X(s, t)|^2 + 1\right\} m_L(d(s, t))\right] \\ & \leqq L_T[\gamma(T; X)\{\lambda([0, T] \times [0, T]) + 1\} + 1]T^2 < \infty\,.\end{aligned}$$

Now that Φ and Ψ are of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ the two stochastic integrals in (1) exist and τX is defined. Clearly τX is an $\mathfrak{F}_{s,t}$ adapted process whose sample functions are all continuous on D.

To show that τX is in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ it remains to verify that for every $T > 0$

$$(7)\qquad \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|(\tau X)(s, t)|^2] < \infty\,.$$

This can be shown by the same kind of estimates used in (6). However we shall obtain a more precise estimate than (7) which is needed in proving (4) and at the same time implies (7). First of all, for any $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ there exists a nonnegative Borel measurable function $A(s, t; X)$ for $(s, t) \prec (T, T)$ which is increasing in the sense of "$\prec$" in D and satisfies the condition

$$(8)\qquad \sup_{(u,v)\prec(s,t)} \boldsymbol{E}[|X(u, v)|^2] + 1 \leqq A(s, t; X) \quad \text{for} \quad (s, t) \prec (T, T)\,.$$

Indeed the fact that X is in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ implies that

$$\sup_{(u,v)\prec(s,t)} \boldsymbol{E}[|X(u, v)|^2] \leqq \gamma(s \vee t; X)$$

so that if we define

$$A(s, t; X) = \gamma(s \vee t; X) + 1 \quad \text{for} \quad (s, t) \prec (T, T) ,$$

then $A(\cdot, \cdot; X)$ is increasing in the sense of "$\prec$" in D and (8) holds. We show next that for any increasing function $A(\cdot, \cdot; X)$ on $[0, T] \times [0, T]$ which satisfies (8) we have

$$\text{(9)} \quad \boldsymbol{E}[|(\tau X)(s, t)|^2] \leqq 5\left\{3\gamma(T; Z) + B_T \int_{[0,s]\times[0,t]} A(u, v; X) m_L(d(u, v))\right\} \quad \text{for} \quad (s, t) \prec (T, T)$$

where

$$\text{(10)} \qquad B_T = (1 + T^2)L_T\{\lambda([0, T] \times [0, T]) + 1\} .$$

Clearly (9) and (10) imply (7). To prove (9) note that from (1) and (5)

$$|(\tau X)(s, t)|^2 \leqq 5\{|Z(0, 0)|^2 + |Z(s, 0)|^2 + |Z(0, t)|^2 + |I(\Phi)(s, t)|^2 + |J(\Psi)(s, t)|^2\} \quad \text{for} \quad (s, t) \prec (T, T)$$

and therefore by (11) and (12) in the proof of Theorem 3.8 and then by (3.4) and (8) we have

$$\begin{aligned}
&\boldsymbol{E}[|(\tau X)(s, t)|^2] \\
&\quad \leqq 5\left\{3\gamma(T; Z) + \int_{[0,s]\times[0,t]} \boldsymbol{E}[|\Phi(u, v)|^2] m_L(d(u, v)) \right. \\
&\qquad \left. + st \int_{[0,s]\times[0,t]} \boldsymbol{E}[|\Psi(u, v)|^2] m_L(d(u, v))\right\} \\
&\quad \leqq 5\left\{3\gamma(T; Z) + (1 + st) \int_{[0,s]\times[0,t]} \boldsymbol{E}[|\alpha(u, v, X)|^2 \right. \\
&\qquad \left. + |\beta(u, v, X)|^2] m_L(d(u, v))\right\} \\
&\quad \leqq 5\left\{3\gamma(T; Z) + (1 + T^2)L_T \int_{[0,s]\times[0,t]} \boldsymbol{E}\left[\int_{[0,u]\times[0,v]} |X(u', v')|^2 \right.\right. \\
&\qquad \left.\left. \times \lambda(d(u', v')) + |X(u, v)|^2 + 1\right] m_L(d(u, v))\right\} \\
&\quad \leqq 5\left\{3\gamma(T; Z) + (1 + T^2)L_T[\lambda([0, T] \times [0, T]) + 1] \right. \\
&\qquad \left. \times \int_{[0,s]\times[0,t]} A(u, v; X) m_L(d(u, v))\right\} ,
\end{aligned}$$

which proves (9) in view of (10).

We now turn to the proof of (4) under the definition of

$\{X^{(i)},\ i = 0, 1, 2, \cdots\}$ by (3). Clearly $X^{(0)}$ is an $\mathfrak{F}_{s,t}$ adapted process on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ whose sample functions are all continuous on D and

$$(11)\qquad \begin{aligned}\gamma(T; X^{(0)}) &= \sup_{(s,t)\prec(T,T)} E[|X^{(0)}(s, t)|^2] \\ &\leqq 3 \sup_{0\leqq s,t\leqq T} \{|Z(0, 0)|^2 + |Z(s, 0)|^2 + |Z(0, t)|^2\} \\ &\leqq 3\gamma(T; Z) < \infty .\end{aligned}$$

Thus $X^{(0)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$. Then by (2) we have $X^{(i)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ for $i = 1, 2, \cdots$ also.

To prove (4) let us write for brevity

$$c = 3\gamma(T: Z)$$

and show by induction on i that

$$(12)\qquad \begin{aligned}\boldsymbol{E}[|X^{(i)}(s, t)|^2] \leqq 5\Big\{c \sum_{k=0}^{i-1} (5B_T st)^k (k!)^{-2} + 5^{i-1}(c + 1)(B_T st)^i (i!)^{-2} \\ + \sum_{k=1}^{i-1} 5^{k-1}(B_T st)^k (k!)^{-2}\Big\} \\ \text{for}\quad (s, t) \prec (T, T) \quad\text{and}\quad i = 1, 2, \cdots .\end{aligned}$$

To show that (12) holds for $i = 1$ let

$$A(s, t; X^{(0)}) = 3c + 1 \quad\text{for}\quad (s, t) \prec (T, T) .$$

Then A is trivially a nonnegative Borel measurable function which is increasing in the sense of "$\prec$" in D. Also by (11), A satisfies (8) with $X^{(0)}$ in the place of X so that by (9)

$$\boldsymbol{E}[|X^{(1)}(s, t)|^2] \leqq 5\{c + (c + 1)B_T st\} ,$$

which is (12) for $i = 1$. Next assume that (12) holds for some positive integer i. Define

$$\begin{aligned}A(s, t; X^{(i)}) = 5\Big\{c \sum_{k=0}^{i-1} (5B_T st)^k (k!)^{-2} + 5^{i-1}(c + 1)(B_T st)^i (i!)^{-2} \\ + \sum_{k=1}^{i-1} 5^{k-1}(B_T st)^k (k!)^{-2}\Big\} + 1 \quad\text{for}\quad (s, t) \prec (T, T) .\end{aligned}$$

Then A is a nonnegative Borel measurable function which is increasing in the sense of "$\prec$" in D and satisfies (8) with $X^{(i)}$ in the place of X. Thus by (9)

$$\begin{aligned}\boldsymbol{E}[|X^{(i+1)}(s, t)|^2] &\leqq 5\Big\{c + B_T \int_{[0,s]\times[0,t]} A(u, v; X^{(i)}) m_L(d(u, v))\Big\} \\ &= 5\Big\{c \sum_{k=0}^{i} (5B_T st)^k (k!)^{-2} + 5^i (c + 1)(B_T st)^{i+1}[(i + 1)!]^{-2} \\ &\qquad + \sum_{k=1}^{i} 5^{k-1}(B_T st)^k (k!)^{-2}\Big\} ,\end{aligned}$$

which is (12) for $i+1$. Thus by induction (12) holds for $i = 1, 2, \cdots$. From (12) we obtain the simpler estimate

$$(13) \qquad E[|X^{(i)}(s, t)|^2 \leqq 5[3\gamma(T; Z) + 1] \sum_{k=0}^{i} (5B_T st)^k (k!)^{-2} \quad \text{for} \quad (s, t) \prec (T, T)$$

which holds not only for $i = 1, 2, \cdots$ but also for $i = 0$ on account of (11). From (13) we have (4) by setting

$$M_T \equiv 5[3\gamma(T; Z) + 1] \sum_{k=0}^{\infty} (5B_T T^2)^k (k!)^{-2} < \infty .$$

This completes the proof of the lemma. □

LEMMA 3.11. *Under the same hypothesis as in Lemma* 3.10, *assume further that* α *and* β *satisfy the Lipschitz condition* (3.3) *also. Then for every* $T > 0$ *there exists* $N_T > 0$ *such that*

$$E[\sup_{(u,v) \prec (s\ t)} |(\tau X)(u, v) - (\tau X')(u, v)|^2] \leqq N_T \int_{[0,s]\times[0,t]} \sup_{(u',v') \prec (u,v)} E[|X(u', v') - X'(u', v')|^2] m_L(d(u, v))$$

for $(s, t) \prec (T, T)$ *for any* $X, X' \in L_2^c(\mathfrak{F}_{s,t})$. □

Proof. Let $X, X' \in L_2^c(\mathfrak{F}_{s,t})$. According to Lemma 3.10, τX and $\tau X'$ exist in $L_2^c(\mathfrak{F}_{s,t})$. Now from the defining equation (1) for τ in Lemma 3.10 we have

$$(1) \qquad E[\sup_{(u,v) \prec (s,t)} |(\tau X)(u, v) - (\tau X')(u, v)|^2] \leqq 2E[\sup_{(u,v) \prec (s,t)} |I(\Phi)(u, v)|^2] + 2E[\sup_{(u,v) \prec (s,t)} |J(\Psi)(u, v)|^2]$$

for $(s, t) \in D$ where

$$(2) \qquad \begin{cases} \Phi(s, t, w) = \alpha(s, t, X(\cdot, \cdot, \omega)) - \alpha(s, t, X'(\cdot, \cdot, \omega)) \\ \Psi(s, t, w) = \beta(s, t, X(\cdot, \cdot, \omega)) - \beta(s, t, X'(\cdot, \cdot, \omega)) \end{cases}$$

and $I(\Phi)$ and $J(\Psi)$ are the stochastic integrals of Φ and Ψ with respect to the $\mathfrak{F}_{s,t}$ Brownian motion B with $\partial B = 0$ and the Lebesgue measure respectively. Note that by using (3.3) we have the same kind of estimate as (6) in the proof of Lemma 3.10 for our Φ and Ψ to show that they are of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ and this ensures the existence of the stochastic integrals $I(\Phi)$ and $J(\Psi)$.

Since $I(\Phi)$ is of the class $\mathfrak{M}_2^c(\mathfrak{F}_{s,t})$, we have by the martingale inequality (b) in Theorem 1.2 of [3] with $p = 2$ and then by (11) in the proof of Theorem 3.8

$$(3) \qquad E[\sup_{(u,v)\prec(s,t)} |I(\Phi)(u, v)|^2] \leqq 16E[|I(\Phi)(s, t)|^2]$$
$$= 16E\left[\int_{[0,s]\times[0,t]} |\Phi(u, v)|^2 m_L(d(u, v))\right].$$

For $J(\Psi)$ we have by the Schwarz Inequality

$$\begin{aligned}|J(\Psi)(u, v)|^2 &= \left|\int_{[0,u]\times[0,v]} \Psi(u', v') m_L(d(u', v'))\right|^2 \\ &\leqq uv \int_{[0,u]\times[0,v]} |\Psi(u', v')|^2 m_L(d(u', v')) \\ &\leqq st \int_{[0,s]\times[0,t]} |\Psi(u', v')|^2 m_L(d(u', v'))\end{aligned}$$

for $(u, v) \prec (s, t)$ so that

$$(4) \qquad E[\sup_{(u,v)\prec(s,t)} |J(\Psi)(u, v)|^2] \leqq stE\left[\int_{[0,s]\times[0,t]} |\Psi(u, v)|^2 m_L(d(u, v))\right].$$

Using (3) and (4) in (1) and applying (3.3), we have

$$\begin{aligned}&E[\sup_{(u,v)\prec(s,t)} |(\tau X)(u, v) - (\tau X')(u, v)|^2] \\ &\leqq (32 + 2T^2)E\left[\int_{[0,s]\times[0,t]} \{|\Phi(u, v)|^2 + |\Psi(u, v)|^2\} m_L(d(u, v))\right] \\ &\leqq (32 + 2T^2)L_T E\left[\int_{[0,s]\times[0,t]} \left\{\int_{[0,u]\times[0,v]} |X(u', v') - X'(u', v')|^2 \right.\right. \\ &\qquad \left.\left. \times \lambda(d(u', v')) + |X(u, v) - X'(u, v)|^2\right\} m_L(d(u, v))\right] \\ &\leqq (32 + 2T^2)L_T\{\lambda([0, T] \times [0, T]) + 1\} \\ &\qquad \times \int_{[0,s]\times[0,t]} \sup_{(u',v')\prec(u,v)} E[|X(u', v') - X'(u', v')|^2] m_L(d(u, v)).\end{aligned}$$

If we let

$$N_T = (32 + T^2)L_T\{\lambda([0, T] \times [0, T]) + 1\},$$

then the lemma is proved. □

THEOREM 3.12. *Suppose the coefficients α and β in the stochastic differential equation* (3.1) *in Definition* 3.3 *satisfy the conditions* (3.3) *and* (3.4). *Let $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ be an equipped probability space on which an $\mathfrak{F}_{s,t}$ Brownian motion B with $\partial B = 0$ exists. Then for every $Z \in L_2^c(\mathfrak{F}_{s,t}|\partial D)$ there exists a strong solution (X, B) of* (3.1) *in which $X \in L_2^c(\mathfrak{F}_{s,t})$ and $\partial X = Z$. In fact there exists a transformation F of $\partial W \times W$ into W satisfying* 1° *and* 2° *of Definition* 3.6 *such that*

$$X(\cdot, \cdot, \omega) = F[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \textit{for a.e.} \quad \omega \in \Omega$$

for all $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t} | \partial D)$ *and* $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ *in the corresponding solutions* (X, B) *of* (3.1) *with* $\partial X = Z$. □

(Recall that according to Theorem 3.8 the condition $\partial X = Z$ determines X in the solution (X, B) uniquely up to a null set in $(\Omega, \mathfrak{F}, P)$.)

Proof (1). Let $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t} | \partial D)$. Define a sequence $\{X^{(i)}, i = 0, 1, 2, \cdots\}$ in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ as in Lemma 3.10 by

$$(1)\quad \begin{cases} X^{(0)}(s, t) = -Z(0, 0) + Z(s, 0) + Z(0, t) & \text{for } (s, t) \in D\,, \\ X^{(i)}(s, t) = (\tau X^{(i-1)})(s, t) & \text{for } (s, t) \in D \text{ and } i = 1, 2, \cdots\,. \end{cases}$$

By an iterated application of Lemma 3.11 and by the fact that for any $\mathfrak{F}_{s,t}$ adapted process Y on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ the inequality

$$(2)\quad \sup_{(u,v) \prec (s,t)} \boldsymbol{E}[|Y(u, v)|] \leqq \boldsymbol{E}[\sup_{(u,v) \prec (s,t)} |Y(u, v)|] \quad \text{for} \quad (s, t) \in D$$

holds, we have for every $T > 0$ some $N_T > 0$ such that

$$(3)\quad \begin{aligned} &\boldsymbol{E}[\sup_{(s,t) \prec (T,T)} |X^{(i+1)}(s, t) - X^{(i)}(s, t)|^2] \\ &\leqq N_T^i \int_{[0,T]\times[0,T]} \int_{[0,s_1]\times[0,t_1]} \cdots \int_{[0,s_{i-1}]\times[0,t_{i-1}]} \\ &\quad \times \sup_{(u,v) \prec (s_i,t_i)} \boldsymbol{E}[|X^{(1)}(u, v) - X^{(0)}(u, v)|^2] m_L(d(s_i, t_i)) \cdots \\ &\quad m_L(d(s_2, t_2))\, m_L(d(s_1, t_1))\,. \end{aligned}$$

Since $(s_i, t_i) \prec (s_{i-1}, t_{i-1}) \prec \cdots \prec (s_1, t_1) \prec (T, T)$ in the integral above, we have by (4) of Lemma 3.10

$$(4)\quad \sup_{(u,v) \prec (s_i,t_i)} \boldsymbol{E}[|X^{(1)}(u, v) - X^{(0)}(u, v)|^2] \leqq 4M_T\,.$$

Using (4) and (3) and integrating we have

$$(5)\quad \boldsymbol{E}[\sup_{(s,t) \prec (T,T)} |X^{(i+1)}(s, t) - X^{(i)}(s, t)|^2] \leqq 4M_T N_T^i T^{2i} (i!)^{-2}\,.$$

Let

$$A_i = \{\omega \in \Omega;\ \sup_{(s,t) \prec (T,T)} |X^{(i+1)}(s, t) - X^{(i)}(s, t)| > 2^{-i}\} \qquad \text{for} \quad i = 0, 1, 2, \cdots\,.$$

Then by the Chebyshev Inequality and (5)

$$P(A_i) \leqq 16 M_T N_T^i T^{2i} (i!)^{-2} \quad \text{for} \quad i = 0, 1, 2, \cdots\,.$$

Since $\sum_{i=0}^{\infty} N_T^i T^{2i} (i!)^{-2} < \infty$, the Borel-Cantelli lemma applies and consequently $\{X^{(i)}, i = 0, 1, 2, \cdots\}$ converges uniformly for $(s, t) \prec (T, T)$ for a.e. $\omega \in \Omega$. By letting $T = 1, 2, \cdots$, we conclude that there exists a null set N in $(\Omega, \mathfrak{F}, P)$ such that $\{X^{(i)}, i = 0, 1, 2, \cdots\}$ converges on D for each $\omega \in N^c$ where the convergence is uniform on every bounded subset of D for each $\omega \in N^c$.

Let us define a stochastic process X on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ by

$$(6) \qquad X(s, t, \omega) = \begin{cases} \lim_{i\to\infty} X^{(i)}(s, t, \omega) & \text{for } (s, t) \in D \text{ and } \omega \in N^c, \\ 0 & \text{for } (s, t) \in D \text{ and } \omega \in N. \end{cases}$$

Then every sample function of X is continuous on D. Also, since $X^{(i)}$ is $\mathfrak{F}_{s,t}$ adapted and $N \in \mathfrak{F}_{s,t}$ for every $(s, t) \in D$, X is $\mathfrak{F}_{s,t}$ adapted. To show that $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ it remains to show that for every $T > 0$

$$(7) \qquad \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X(s, t)|^2] < \infty .$$

Now for the L_2 norm $\boldsymbol{E}[|\cdot|^2]^{1/2}$ on the Hilbert space $L_2(\Omega, \mathfrak{F}, P)$ we have by the triangle inequality and by (2) and (5)

$$(8) \qquad \begin{aligned} &\sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^{(m)}(s, t) - X^{(n)}(s, t)|^2]^{1/2} \\ &\qquad \leqq \sum_{i=n}^{m-1} \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^{(i+1)}(s, t) - X^{(i)}(s, t)|^2]^{1/2} \\ &\qquad \leqq 2\sqrt{M_T} \sum_{i=n}^{m-1} \sqrt{N_T^i}\, T^i(i!)^{-1} \quad \text{for} \quad (s, t) \in D . \end{aligned}$$

Since $\sum_{i=0}^{\infty} \sqrt{N_T^i}\, T^i(i!)^{-1} < \infty$, the Cauchy Criterion for Uniform Convergence implies the existence of an $\mathfrak{F}_{s,t}$ adapted process X^* on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ such that

$$(9) \qquad \boldsymbol{E}[|X^*(s, t)]|^2 < \infty \quad \text{for} \quad (s, t) \in D$$

and

$$(10) \qquad \lim_{i\to\infty} \{ \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^{(i)}(s, t) - X^*(s, t)|^2]\} = 0 .$$

From (10) and the fact that $X^{(i)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ we have

$$(11) \qquad \begin{aligned} \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^*(s, t)|^2] &\leqq 2 \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^*(s, t) - X^{(i)}(s, t)|^2 \\ &\quad + 2 \sup_{(s,t)\prec(T,T)} \boldsymbol{E}[|X^{(i)}(s, t)|^2] < \infty . \end{aligned}$$

Since convergence in L_2 norm implies the existence of an a.e. convergent subsequence, (6) and (10) imply

$$(12) \qquad X(s, t, \omega) = X^*(s, t, \omega) \quad \text{for a.e.} \quad \omega \in \Omega \quad \text{for each} \quad (s, t) \in D .$$

By (11) and (12) we have (7). This shows that $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$.

Now that $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$, τX exists in $\boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ by Lemma 3.10. We proceed to show that in fact $\tau X = X$. Now for every $T > 0$

$$\sup_{(s,t)\prec(T,T)} |(\tau X)(s, t) - X(s, t)|^2 \leqq 2 \sup_{(s,t)\prec(T,T)} |(\tau X)(s, t) - X^{(i+1)}(s, t)|^2 + 2 \sup_{(s,t)\prec(T,T)} |X^{(i+1)}(s, t) - X(s, t)|^2$$

for each i. Since $\{X^{(i)}, i = 0, 1, 2, \cdots\}$ converges uniformly on every bounded subset of D for each $\omega \in N^c$ to X, we have

$$\lim_{i\to\infty} \{ \sup_{(s,t)\prec(T,T)} |X^{(i+1)}(s, t, \omega) - X(s, t, \omega)|^2\} = 0 \quad \text{for} \quad \omega \in N^c .$$

This implies

$$\sup_{(s,t)\prec(T,T)} |(\tau X)(s, t, \omega) - X(s, t, \omega)|^2 \leqq 2 \liminf_{i\to\infty} \{ \sup_{(s,t)\prec(T,T)} |(\tau X)(s, t, \omega) - X^{(i+1)}(s, t, \omega)|^2\} \quad \text{for} \quad \omega \in N^c .$$

Taking the expectation on both sides of inequality above, then applying Fatou's Lemma and Lemma 3.11 to the right side we have

$$\text{(13)} \qquad \boldsymbol{E}[\sup_{(s,t)\prec(T,T)} |(\tau X)(s, t) - X(s, t)|^2] \leqq 2N_T \liminf_{i\to\infty} \int_{[0,T]\times[0,T]} \sup_{(u,v)\prec(s,t)} \boldsymbol{E}[|X(u, v) - X^{(i)}(u, v)|^2] \times m_L(d(s, t)) .$$

Now by Definition 3.9 and (4) of Lemma 3.10

$$\sup_{(u,v)\prec(s,t)} \boldsymbol{E}[|X(u, v) - X^{(i)}(u, v)|^2] \leqq 2\gamma(T; X) + 2M_T .$$

Let us replace $\liminf_{i\to\infty}$ on the right side of (13) by $\limsup_{i\to\infty}$. The inequality remains valid. Since the integrand on the right side of (13) is bounded by a constant as we have just shown, Fatou's lemma for lim sup applies and we have

$$\boldsymbol{E}[\sup_{(s,t)\prec(T,T)} |(\tau X)(s, t) - X(s, t)|^2] \leqq 2N_T \int_{[0,T]\times[0,T]} \limsup_{i\to\infty} \{ \sup_{(u,v)\prec(s,t)} \boldsymbol{E}[|X(u, v) - X^{(i)}(u, v)|^2]\} m_L(d(s, t))$$

which is equal to 0 by (12) and (10). Therefore

$$\sup_{(s,t)\prec(T,T)} |(\tau X)(s, t, \omega) - X(s, t, \omega)| = 0 \quad \text{for a.e.} \quad \omega \in \Omega .$$

By considering $T = 1, 2, \cdots$, we conclude that

$$(\tau X)(s, t, \omega) = X(s, t, \omega) \quad \text{for all} \quad (s, t) \in D \quad \text{for a.e.} \quad \omega \in \Omega .$$

Thus we have shown that $\tau X = X$.

Recalling the definition of τ by (1) of Lemma 3.10, we have

$$X(s, t) = -Z(0, 0) + Z(s, 0) + Z(0, t) + \int_{[0,s]\times[0,t]} \alpha(u, v, X)dB(u, v) + \int_{[0,s]\times[0,t]} \beta(u, v, X)d(u, v) \,, \tag{14}$$

i.e., (X, B) is a solution of the stochastic differential equation (3.1) on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ with $\partial X = Z$. Bh Theorem 3.8, (X, B) is the unique solution on $(\Omega, \mathfrak{F}, P; \mathfrak{F}_{s,t})$ with $\partial X = Z$.

(2) We proceed to show that there exists a transformation F of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 such that for every $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t}|\partial D)$ the corresponding stochastic process $X \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ defined by (1) and (6) satisfies

$$X(\cdot, \cdot, \omega) = F[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for a.e.} \quad \omega \in \Omega \tag{15}$$

where the exceptional null set in $(\Omega, \mathfrak{F}, P)$ may depend on Z.

We shall show first that for every $i = 0, 1, 2, \cdots$, there exists a transformation $F^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 such that for every $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t}|\partial D)$ the corresponding stochastic process $X^{(i)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ defined by (1) satisfies

$$X^{(i)}(\cdot, \cdot, \omega) = F^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for a.e.} \quad \omega \in \Omega \tag{16}$$

where the exceptional null set in $(\Omega, \mathfrak{F}, P)$ may depend on Z. We shall show this below by induction on i.

(2.1) Let us show the existence of $F^{(0)}$. Recall that by (1),

$$X^{(0)}(s, t, \omega) = -Z(0, 0, \omega) + Z(s, 0, \omega) + Z(0, t, \omega) \quad \text{for} \quad (s, t, \omega) \in D \times \Omega \,.$$

Consider the following transformations:

$$\begin{aligned} &\pi_0; \ \partial W \to \boldsymbol{R} && \text{defined by} \ \ \pi_0 z = z(0, 0) \in \boldsymbol{R} \ \ \text{for} \ \ z \in \partial W \,, \\ &\pi_1; \ \partial W \to W^{(1)} && \text{defined by} \ \ \pi_1 z = z(\cdot, 0) \in W^{(1)} \ \ \text{for} \ \ z \in \partial W \,, \\ &\pi_2; \ \partial W \to W^{(1)} && \text{defined by} \ \ \pi_2 z = z(0, \cdot) \in W^{(1)} \ \ \text{for} \ \ z \in W \,. \end{aligned}$$

Thus defined, π_0 is $\mathfrak{B}(\partial W)/\mathfrak{B}(\boldsymbol{R})$ measurable and π_1 and π_2 are both $\mathfrak{B}(\partial W)/\mathfrak{B}(W^{(1)})$ measurable. Consider the following transformations:

$$\begin{aligned} &e_0; \quad \boldsymbol{R} \to W && \text{defined by} \ \ e_0\zeta = w_\zeta \\ &&& \text{where} \ \ w_\zeta(s, t) = \zeta \ \ \text{for} \ \ (s, t) \in D \,, \\ &e_1; \ W^{(1)} \to W && \text{defined by} \ \ e_1 x = w_x \\ &&& \text{where} \ \ w_x(s, t) = x(s) \ \ \text{for} \ \ (s, t) \in D \,, \\ &e_2; \ W^{(1)} \to W && \text{defined by} \ \ e_1 y = w_y \\ &&& \text{where} \ \ w_y(s, t) = y(t) \ \ \text{for} \ \ (s, t) \in D \,. \end{aligned}$$

Clearly e_0 is $\mathfrak{B}(\boldsymbol{R})/\mathfrak{B}(W)$ measurable and e_1 and e_2 are both $\mathfrak{B}(W^{(1)})/\mathfrak{B}(W)$ measurable. Thus $e_0 \circ \pi_0$, $e_1 \circ \pi_1$ and $e_2 \circ \pi_2$ are all $\mathfrak{B}(\partial W)/\mathfrak{B}(W)$ measurable transformations of ∂W into W. Define a transformation $F^{(0)}$ of $\partial W \times W$ into W by

$$F^{(0)}(z, w) = -(e_0 \circ \pi_0)z + (e_1 \circ \pi_1)z + (e_2 \circ \pi_2)z \quad \text{for} \quad (z, w) \in \partial W \times W .$$

Then $F^{(0)}$ is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(W)$ measurable and 1° and 2° of Definition 3.6 are satisfied trivially. Also

$$\begin{aligned} (17) \qquad & F^{(0)}[Z(\cdot, \cdot, w), B(\cdot, \cdot, w)] \\ & \quad = -(e_0 \circ \pi_0)Z(\cdot, \cdot, \omega) + (e_1 \circ \pi_1)Z(\cdot, \cdot, \omega) + (e_2 \circ \pi_2)Z(\cdot, \cdot, \omega) \\ & \quad = X^{(0)}(\cdot, \cdot, \omega) \quad \text{for} \quad \omega \in \Omega . \end{aligned}$$

This proves the existence of $F^{(0)}$.

(2.2) Next suppose that for some i there exists a transformation $F^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 and (16). Now by (1) and the definition of τ in Lemma 3.10,

$$(18) \qquad X^{(i+1)}(s, t) = X^{(0)}(s, t) + I(\Phi^{(i)})(s, t) + J(\Psi^{(i)})(s, t)$$

where

$$(19) \qquad \begin{cases} \Phi^{(i)}(s, t, \omega) = \alpha[s, t, X^{(i)}(\cdot, \cdot, \omega)] & \text{for} \quad (s, t, \omega) \in D \times \Omega , \\ \Psi^{(i)}(s, t, \omega) = \beta[s, t, X^{(i)}(\cdot, \cdot, \omega)] & \text{for} \quad (s, t, \omega) \in D \times \Omega , \end{cases}$$

and $I(\Phi^{(i)})$ and $J(\Psi^{(i)})$ are the stochastic integrals of $\Phi^{(i)}$ and $\Psi^{(i)}$ with respect to the $\mathfrak{F}_{s,t}$ Brownian motion B with $\partial B = 0$ and the Lebesgue measure respectively. Since we have (17) already, if we show that there exist transformations $G^{(i)}$ and $H^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 such that

$$(20) \qquad \begin{cases} I(\Phi^{(i)})(\cdot, \cdot) = G^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] & \text{for a.e.} \quad \omega \in \Omega , \\ J(\Psi^{(i)})(\cdot, \cdot) = H^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] & \text{for a.e.} \quad \omega \in \Omega , \end{cases}$$

for $X^{(i)}$ defined by (1) with an arbitrary $Z \in L_2^c(\mathfrak{F}_{s,t} | \partial D)$, then the transformation $F^{(i+1)}$ of $\partial W \times W$ into W defined by

$$(21) \qquad F^{(i+1)} = F^{(0)} + G^{(i)} + H^{(i)}$$

satisfies 1° and 2° of Definition 3.6 and also

$$(22) \qquad X^{(i+1)}(\cdot, \cdot, \omega) = F^{(i+1)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for a.e.} \quad \omega \in \Omega .$$

It remains to prove the existence of $G^{(i)}$ and $H^{(i)}$. We do this for $G^{(i)}$ below.

Now according to our induction hypothesis, (16) holds for our i. Then by (19)

$$\begin{aligned} (23) \qquad \Phi^{(i)}(s, t, \omega) &= \alpha(s, t, F^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)]) \\ & \qquad \text{for} \quad (s, t) \in D \quad \text{and a.e.} \quad \omega \in \Omega . \end{aligned}$$

Since $X^{(i)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$, our $\Phi^{(i)}$ defined by (19) is of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$ as we saw in the proof of Lemma 3.10. Let us consider first the particular case where α is such that $\Phi^{(i)}$ is of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$. Then by Definition 2.8 for some null set N in $(\Omega, \mathfrak{F}, P)$

$$(24) \qquad I(\Phi^{(i)})(s, t, \omega) = \begin{cases} \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha(s_{j-1}, t_{k-1}, F^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)]) \\ \quad \times \{B(s_j, t_k, \omega) - B(s_{j-1}, t_k, \omega) - B(s_j, t_{k-1}\omega) \\ \quad + B(s_{j-1}, t_{k-1}, \omega)\} \quad \text{for} \quad (s, t, \omega) \in D \times N^c, \\ 0 \qquad\qquad\qquad\qquad\qquad \text{for} \quad (s, t, \omega) \in D \times N, \end{cases}$$

with the understanding that $s_m = s$ and $t_n = t$ as in Definition 2.8. Define transformations φ and $\iota(\varphi)$ of $D \times \partial W \times W$ into $\boldsymbol{R}$ by

$$(25) \quad \varphi(s, t, z, w) = \alpha(s, t, F^{(i)}[z, w]) \quad \text{for} \quad (s, t, z, w) \in D \times \partial W \times W,$$

$$(26) \quad \iota(\varphi)(s, t, z, w) = \sum_{j=1}^{m} \sum_{k=1}^{n} \alpha(s_{j-1}, t_{k-1}, F^{(i)}[z, w])\{W(s_j, t_k) - w(s_{j-1}, t_k) - W(s_j, t_{k-1}) + w(s_{j-1}, t_{k-1})\} \quad \text{for} \quad (s, t, z, w) \in D \times \partial W \times W,$$

again with the understanding that $s_m = s$ and $t_n = t$. Let us show that the transformation $\iota(\varphi)(\cdot, \cdot, z, w)$ of $(z, w) \in \partial W \times W$ into W is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(W)$ measurable. According to part (1) of Proposition 3.1 it suffices to show that for every $(s, t) \in D$ the transformation $\iota(\varphi)(s, t, z, w)$ of $(z, w) \in \partial W \times W$ into $\boldsymbol{R}$ is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(\boldsymbol{R})$ measurable. Now 1° of Definition 3.6 and 2° of Definition 3.2 imply that the transformation $\alpha(s_{j-1}, t_{k-1}, F^{(i)}[z, w])$ of $(z, w) \in \partial W \times W$ into $\boldsymbol{R}$ is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(\boldsymbol{R})$ measurable. Also the second factor in each summand on the right side of (26) is a $\mathfrak{B}(W)/\mathfrak{B}(\boldsymbol{R})$ measurable transformation of W into $\boldsymbol{R}$. Therefore the transformation $\iota(\varphi)(s, t, z, w)$ of $(z, w) \in \partial W \times W$ into $\boldsymbol{R}$ is $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(\boldsymbol{R})$ measurable. Similarly the fact that for each $z \in \partial W$ the transformation $\iota(\varphi)(\cdot, \cdot, z, w)$ of $w \in W$ into W is $\mathfrak{B}_{s,t}(W)/\mathfrak{B}_{s,t}(W)$ measurable for every $(s, t) \in D$ can be shown by part (2) of Proposition 3.1, 2° of Definition 3.6 and 2° of Definition 3.2. With these two measurability conditions satisfied by $\iota(\varphi)$ we conclude that there exists a transformation $G^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 such that

$$(27) \qquad \iota(\varphi)(\cdot, \cdot, z, w) = G^{(i)}(z, w) \quad \text{for} \quad (z, w) \in \partial W \times W.$$

From (24), (26) and (27) we have (20) for $\Phi^{(i)}$ for the particular case where α is such that $\Phi^{(i)}$ is of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$.

Consider now the general case where $\Phi^{(i)}$ is of the class $\mathfrak{L}_2(\mathfrak{F}_{s,t})$. According to (2.1) in § 2 and Proposition 1.6 there exist a sequence

$\{\Phi_l,\ l=1,2,\cdots\}$ in the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$ and a null set N_0 in $(\Omega, \mathfrak{F}, P)$ such that

$$\lim_{l\to\infty} I(\Phi_l)(s,t,\omega) = I(\Phi^{(i)})(s,t,\omega) \quad \text{uniformly on every bounded subset of } D \quad \text{for} \quad \omega \in N_0^c\,,$$

i.e.,

$$\lim_{l\to\infty} I(\Phi_l)(\cdot,\cdot,\omega) = I(\Phi^{(i)})(\cdot,\cdot,\omega) \quad \text{in the metric topology of } W \quad \text{for} \quad \omega\in N_0^c\,. \tag{28}$$

Since Φ_l is of the class $\mathfrak{L}_0(\mathfrak{F}_{s,t})$, there exists a transformation $G_l^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 and a null set N_l in $(\Omega, \mathfrak{F}, P)$ such that

$$I(\Phi_l)(\cdot,\cdot,\omega) = G_l^{(i)}[Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)] \quad \text{for} \quad \omega\in N_l^c\,. \tag{29}$$

Let

$$N = \bigcup_{l=0}^{\infty} N_l \tag{30}$$

and

$$\Lambda = \{(z,w)\in\partial W\times W;\ \lim_{l\to\infty} G_l^{(i)}[z,w] \text{ exists in } W\}\,. \tag{31}$$

From the $\sigma(\mathfrak{B}(\partial W)\times\mathfrak{B}(W))/\mathfrak{B}(W)$ measurability of $G_l^{(i)}$ for $l=1,2,\cdots$, $\Lambda\in\sigma(\mathfrak{B}(\partial W)\times\mathfrak{B}(W))$. Define a transformation $G^{(i)}$ of $\partial W\times W$ into W by

$$G^{(i)}(z,w) = \begin{cases} \lim_{l\to\infty} G_l^{(i)}(z,w) & \text{for} \quad (z,w)\in\Lambda\,, \\ \text{a fixed element in } W & \text{for} \quad (z,w)\in\Lambda^c\,. \end{cases} \tag{32}$$

Then $G^{(i)}$ satisfies 1° and 2° of Definition 3.6 since $G_l^{(i)}$ does for $l=1,2,\cdots$. Let K be the subset of $\partial W\times W$ covered by the transformation of N^c defined by $[Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)]$ for $\omega\in N^c$. Then for every $(z,w)\in K$ there exists $\omega\in N^c$ such that

$$(z,w) = [Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)]$$

so that by (29), (28) and (30)

$$\begin{aligned}\lim_{l\to\infty} G_l^{(i)}(z,w) &= \lim_{l\to\infty} G_l^{(i)}[Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)] \\ &= \lim_{l\to\infty} I(\Phi_l)(\cdot,\cdot,\omega) = I(\Phi^{(i)})(\cdot,\cdot,\omega) \quad \text{for} \quad (z,w)\in K\,.\end{aligned}$$

Then by (31) $K\subset\Lambda$ so that by (32)

$$\begin{aligned}&\lim_{l\to\infty} G_l^{(i)}[Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)] \\ &\qquad = G^{(i)}[Z(\cdot,\cdot,\omega),\ B(\cdot,\cdot,\omega)] \quad \text{for} \quad \omega\in N^c\,.\end{aligned} \tag{33}$$

Combining (28), (29) and (33) we have

$$I(\Phi^{(i)})(\cdot, \cdot, \omega) = G^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for} \quad \omega \in N^c , \tag{34}$$

proving (20) for $I(\Phi^{(i)})$. We can prove (20) for $J(\Psi^{(i)})$ in the same way. This completes the proof that if $F^{(i)}$ exists so does $F^{(i+1)}$.

(2.3) By induction on i we conclude that for each $i = 0, 1, 2, \cdots$, there exists a transformation $F^{(i)}$ of $\partial W \times W$ into W satisfying 1° and 2° of Definition 3.6 such that for any $Z \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t} | \partial D)$ and the corresponding $X^{(i)} \in \boldsymbol{L}_2^c(\mathfrak{F}_{s,t})$ defined by (1) we have

$$X^{(i)}(\cdot, \cdot, \omega) = F^{(i)}[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for} \quad \omega \in \Omega - N_i(Z) \tag{35}$$

where $N_i(Z)$ is a null set in $(\Omega, \mathfrak{F}, P)$. Let

$$N(Z) = \bigcup_{i=0}^{\infty} N_i(Z) \tag{36}$$

and

$$\Lambda = \{(z, w) \in \partial W \times W; \lim_{i \to \infty} F^{(i)}(z, w) \text{ exists in } W\} . \tag{37}$$

From the $\sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))/\mathfrak{B}(W)$ measurability of $F^{(i)}$, $i = 0, 1, 2, \cdots$, $\Lambda \in \sigma(\mathfrak{B}(\partial W) \times \mathfrak{B}(W))$. Define a transformation F of $\partial W \times W$ into W by

$$F(z, w) = \begin{cases} \lim_{i \to \infty} F^{(i)}(z, w) & \text{for} \quad (z, w) \in \Lambda , \\ \text{a fixed element in } W & \text{for} \quad (z, w) \in \Lambda^c . \end{cases} \tag{38}$$

Then F satisfies 1° and 2° of Definition 3.6 since $F^{(i)}$ does for $i = 0, 1, 2, \cdots$. We saw in (1) that

$$\lim_{i \to \infty} X^{(i)}(\cdot, \cdot, \omega) = X(\cdot, \cdot, \omega) \quad \text{in the metric topology of } W \text{ for } w \in N^c \tag{39}$$

where N is a null set in $(\Omega, \mathfrak{F}, P)$. By the same argument as in proving (34) by means of (28), (29), (31) and (32) we conclude from (39), (35), (36), (37), and (38)

$$X(\cdot, \cdot, \omega) = F[Z(\cdot, \cdot, \omega), B(\cdot, \cdot, \omega)] \quad \text{for} \quad \omega \in (N(Z) \cup N)^c .$$

This completes the proof of the theorem. □

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Received September 2, 1980.

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