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STRONG COMPLETENESS IN PROFINITE GROUPS

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A profinite group is strongly complete if every subgroup of finite index is open. In this paper it is shown that a profinite group with finitely generated p-Sylow subgroups is strongly complete and that if G is a finitely generated strongly complete profinite group and A is a finitely generated pseudocompact G-modulo then any extension of A by G is strongly complete.

The purpose of this paper is to extend some results of Anderson [1] in the theory of strong completeness of profinite groups. A *profinite group* is a topological group whose topology is Hausdorff, compact and has neighborhood base of the identity consisting of certain subgroups of finite index. A profinite group is *strongly complete* if every subgroups of finite index is open. Since all open subgroups are also closed, a strongly complete profinite group has no dense subgroups of finite index except itself.

Our first result is:

THEOREM 1. *Let G be a profinite group, G_p a p-Sylow subgroup, $U \trianglelefteq G$ with $(G\colon U) = n < \infty$. U is open in G if and only if $U \cap G_p$ is open in G_p for every prime p which divides n.*

COROLLARY 1. *Let G be a profinite group all of whose p-Sylow subgroups are finitely generated. Then G is strongly complete.*

Our second result is:

THEOREM 2. *Let $A \rightarrowtail E \twoheadrightarrow G$ be a short exact sequence of profinite groups. If G is a finitely generated strongly complete profinite group and A is a finitely generated pseudocompact $\hat{\boldsymbol{Z}}[[G]]$-module then E is strongly complete.*

COROLLARY 1. *Let $A \rightarrowtail E \twoheadrightarrow G$ be a short exact sequence of profinite groups where G is as in the theorem and A contains a finite sequence of subgroups which are normal in E: $A = A_0 \geqq A_1 \geqq \cdots \geqq A_n = (e)$ such that A_i/A_{i+1} is a finitely generated pseudocompact $\hat{\boldsymbol{Z}}[[G]]$-module for $i = 0, \cdots, n-1$. Then E is strongly complete.*

In this paper all groups are profinite, all subgroups are closed, and all homomorphisms are continuous unless otherwise stated. We will call a proper subgroup of finite index *large*.

1. For any group, G, $x \in G$, the closed subgroup generated by x, $\overline{\langle x\rangle}$, is cyclic and so there is a continuous homomorphism ρ: $\hat{\boldsymbol{Z}} \twoheadrightarrow \overline{\langle x\rangle}$ defined by $\rho(\lambda) = x^{\lambda}$. Writing $\hat{\boldsymbol{Z}}$ as $\prod_p \hat{\boldsymbol{Z}}_p$, the product over all primes p of p-adic integers, and then as $\hat{\boldsymbol{Z}}_p \times \prod_{q\neq p} \hat{\boldsymbol{Z}}_q$ and allowing the generator of $\hat{\boldsymbol{Z}}_p \times (0)$ to be $(1, 0)$ and the generator of $(0) \times \prod_{q\neq p} \hat{\boldsymbol{Z}}_q$ to be $(0, 1)$ one sees that $\overline{\langle x^{(1,0)}\rangle}$ is the p-Sylow subgroup of $\overline{\langle x\rangle}$ and $\overline{\langle x^{(0,1)}\rangle}$ its p-complement. Finitely generated pro-abelian groups are known to be strongly complete. Hence any homomorphism from $\overline{\langle x\rangle}$ to a finite group is continuous. With this we prove:

PROPOSITION 1. *Let U be a large normal subgroup of G, U not necessarily open, $x \in G$ such that $\bar{x} \in (G/U)_p$, p-Sylow subgroup of G/U. Then $\overline{x^{(1,0)}} = \bar{x}$ in G/U.*

Proof. The morphism $\overline{\langle x\rangle}$ to $\langle \bar{x}\rangle \leqq G/U$ is continuous as we have noted. $\langle \bar{x}\rangle$ is a finite cyclic p-group. Since $x = x^{(1,0)} \cdot x^{(0,1)}$ and $x^{(0,1)}$ is an element of G whose order is prime to p, its image $\langle \bar{x}\rangle$ is the identity. Hence

$$\bar{x} = \overline{x^{(1,0)} \cdot x^{(0,1)}} = \overline{x^{(1,0)}} \cdot \overline{x^{(0,1)}} = \overline{x^{(1,0)}}\ . \qquad \square$$

We call an element of G a p-element if it belongs to some p-Sylow subgroup of G. For all x in G, $x^{(1,0)}$ is a p-element and x is a p-element if and only if $x = x^{(1,0)}$ (see [4]).

A net of elements $\{x_\alpha\}$ of a profinite group G converges to an element x if for all open normal subgroups V of G, $x_\alpha V = xV$ for almost all α.

PROPOSITION 2. *Let $\{x_\alpha\}$ be a net in G converging to a p-element x. Then $\{x_\alpha^{(1,0)}\}$ is a net of p-elements which also converges to x.*

Proof. If x is a p-element then for any open normal subgroup V of G, xV is a p-element in G/V. By Proposition 1, $x_\alpha V = x_\alpha^{(1,0)} V$ if $x_\alpha V = xV$. The set $\{x_\alpha^{(1,0)}\}$ is clearly a net and hence the result.

Before proving Theorem 1 we need the following lemma.

LEMMA 1. *Let $U \trianglelefteq G$, U not necessarily closed, such that for some p-Sylow subgroup G_p of G, $U \cap G_p$ is closed in G_p. The set of all p-elements in U is closed in G.*

Proof. *Let* $U_p = U \cap G_p$. The set of all p-elements in U is

$$\bigcup_{x \in G} U \cap G_p^x = \bigcup_{x \in G_p} U_p^x$$

since U is normal. Consider the function $U_p \times G \to G$ defined by $(u, g) \to g^{-1}ug$. Since U_p is closed in G_p it is compact and hence the function, which is easily continuous, is a closed function. Its image, which is precisely the set of p-elements of U, is therefore closed in G. $\square$

Proof of Theorem 1. Let $U \trianglelefteq G$ of finite index. If U is open then $U \cap G_p$ is open in G_p for all G_p. Conversely suppose there exists large U not open, the quotient group $\bar{U}/U$ has a nontrivial p-Sylow subgroup for some prime p. Hence there exists $x \notin U$ such that $\bar{e} \neq \bar{x} \in \bar{U}/U$ is a nontrivial p-element. By Proposition 1 we may assume x is a p-element of G. Since $x \in \bar{U}$ there is a net $\{x_\alpha\}$ of elements of U which converges to x. By Proposition 2, the net $\{x_\alpha^{(1,0)}\}$ also converges to x. Clearly, $x_\alpha \in U$ then $x_\alpha^{(1,0)} \in U$ by the strong completeness of $\overline{\langle x_\alpha \rangle}$. Hence the net $\{x_\alpha^{(1,0)}\}$ is a net of p-elements in U which converge to a p-element x not in U. By hypothesis and Lemma 1, the set of p-element of U is closed in G. Hence x must be a p-element of U, contradiction. $\square$

Proof of Corollary 1 *to Theorem* 1. Finitely generated pro-p-groups are strongly complete, [1], [6]. Hence if $U \trianglelefteq G$, U large then $U \cap G_p$ is large in G_p and so open. Therefore the theorem applies.

The above corollary is another proof of the result due to Oltikar and Ribes, [5], that finitely generated prosupersolvable groups are strongly complete since in the same paper they prove that such groups have finitely generated p-Sylow subgroups.

2. In this section we first show that the completed group algebra $\hat{\boldsymbol{Z}}[[G]]$ (which we denote by Δ) for a finitely generated profinite group, G, is in some sense strongly complete. Let Mod (G) be the category of G-modules, G considered as an abstract group.

PROPOSITION 3. *Let G be a finitely generated profinite group, $A \leqq \Delta$ such that Δ/A is finite and $A \in$ Mod (G). Then A is open in the topology of Δ.*

Before proving Proposition 3 we first review the topological structure of Δ.

$$\Delta \simeq \varprojlim_{n,\overline{U} \text{ open}} \boldsymbol{Z}/n\boldsymbol{Z}(G/U) .$$

A neighborhood base of (0) consists of the kernals, $\pi_{n,U}$ of the continuous morphisms $\Delta \twoheadrightarrow \boldsymbol{Z}/n\boldsymbol{Z}(G/U)$. In [2], Brummer notes that $\pi_{n,U}$ is the closed ideal generated by $\{(u-1) \mid u \in U\}$. In fact, as a pseudocompact Δ-module, $\pi_{n,U}$ is precisely $\overline{n\Delta + \sum \Delta(u_i - 1)}$ where $\{u_i\}$ is a set of topological generators of U. Therefore if G and hence U is finitely generated $\boldsymbol{I}_{n,U}$ is a finitely generated pseudocompact Δ-module.

Proof of Proposition 3. Since Δ/A is finite, there exists n such that $n\Delta \leqq A$. As well, Δ/A is trivial U-action for some large but not necessarily open subgroup U of G. However U contains the topological generators $\{u_1, \cdots, u_s\}$ of $\bar{U}$, its closure in G. In this case $\boldsymbol{I}_{n,\bar{U}} = \overline{n\Delta + \sum_{i=1}^{s} \Delta(u_i - 1)} = n\Delta + \sum_{i=1}^{s} \Delta(u_i - 1)$ and since clearly $B = \sum_{i=1}^{s} \Delta(u_i - 1) \leqq A$ one has $\boldsymbol{I}_{n,\bar{U}} \leqq A$ which implies A is open as well. $\square$

The category of pseudocompact Δ-modules, PC_G^p, is studied by Brummer, [2], and in the thesis of Gabriel. These modules are inverse limits of finite discrete G-modules with the corresponding profinite topology. If $M \in PC_G^p$ and M is (topologically) finitely generated then M is the continuous homomorphic image of $\bigoplus^m \Delta$, for some finite m.

COROLLARY 1. *Let G be a finitely generated profinite group, $M \in PC_G^p$, M finitely generated. If $A \leqq M$ such that M/A is finite and $A \in \mathrm{Mod}(G)$, then A is open in M.*

Proof. If $\pi\colon \bigoplus^m \Delta \twoheadrightarrow M$ is defined, which is the case for M finitely generated by at most m elements, then one easily shows $\pi^{-1}(A)$ open in $\bigoplus^m \Delta$ and hence A is open in M. $\square$

We now prove Theorem 2.

Proof of Theorem 2. If U is a large normal subgroup of E but not necessarily open, its image in G is open since G is strongly complete and $U \cap A$ is open in A by Corollary 1 to Proposition 3 since $U \cap A$ is preserved under the action of G and hence belongs to $\mathrm{Mod}(G)$.

Consider the following commutative diagram of profinite groups

$$\begin{tikzcd}
A \arrow[r, tail] \arrow[d, two heads] & E \arrow[r, "\pi"] \arrow[d, two heads, "\rho"'] & G \arrow[d, no head] \\
A/U \cap A \arrow[r, two heads] & E/U \cap A \arrow[r, two heads, "\pi_1"] & G .
\end{tikzcd}$$

Clearly $\rho^{-1}(\rho(U)) = U$ and ρ is continuous so it suffices to show $\rho(U)$ is open or closed in $E/U \cap A$.

However, π_1 is a monomorphism when restricted to $\rho(U)$ and $\pi_1 \circ \rho(U)$ is open in G. Therefore, restricted to $\pi_1 \circ \rho(U)$, π_1 has an inverse π_1^{-1}, such that $\pi_1 \circ \pi_1^{-1} = 1_{\pi_1 \circ \rho(U)}$ and $\pi_1^{-1} \circ \pi_1 = 1_{\rho(U)}$. Hence there is a topology which we can place on $\rho(U)$ to make it a profinite group. Namely, $V \leqq \rho(U)$ is open iff $\pi_1(V)$ is open in G. But this is clearly the original relative topology on $\rho(U)$. We argue as follows: Let $V \leqq E/A \cap U$ be open in $E/A \cap U$. Then $\pi_1(V \cap \rho(U))$ is open in G. Hence $V \cap \rho(U)$ is open in $\rho(U)$ equipped with its profinite topology.

Hence the profinite topology of $\rho(U)$ is finer than its relative topology. Conversely, if the profinite topology is properly finer then we extend this topology to a profinite topology on $E/A \cap U$. Hence $E/A \cap U$ can be equipped with two profinite topologies, one coarser than the other and this is impossible. Hence the two topologies on $\rho(U)$ are identical so that $\rho(U)$ is closed in $E/A \cap U$ since it is compact. Hence the result. $\square$

Proof of Corollary 1 *to Theorem* 2. The profinite group, E, of Theorem 2 is finitely generated. By the Theorem, E/A_1 is strongly complete. By induction, if E/A_i is strongly complete then the short exact sequence $A_i/A_{i+1} \rightarrowtail E/A_{i+1} \twoheadrightarrow E/A_i$ shows E/A_{i+1} to be strongly complete. Hence, by induction, the corollary holds. $\square$

Finally we notice that in the case $A \rightarrowtail E \twoheadrightarrow G$ verifies the hypothesis of Corollary 1 to Theorem 2 then E is finitely generated.

PROPOSITION 4. *Let $A \rightarrowtail E \twoheadrightarrow G$ be a split short exact sequence of profinite groups where E is generated by n elements and A is abelian. Then A is a pseudocompact Δ-module generated by n elements.*

Proof. A similar results is proved by Hartley, [3, Lemma 5] for finite groups and easily carries over to profinite groups. $\square$

COROLLARY 2 TO THEOREM 2. *If $A \rightarrowtail E \twoheadrightarrow G$ is a split short exact sequence of profinite groups such that E is finitely generated, A is abelian and G is strongly complete, then E is strongly complete.*

Proof. Proposition 4 allows us to say A is a finitely generated pseudocompact G-module and so we may apply the theorem. $\square$

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