PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

CONVEXITY AND THE TABLE THEOREM

MARK D. MEYERSON

In "The Table Theorem" ([1]), Roger Fenn proves that one can "balance" a square "table" on any "hill". One of the hypotheses he suggests relaxing is that the hill has convex support. We show here the necessity of such a hypothesis.

More precisely, Fenn showed that if $D \subset \boldsymbol{R}^2$ is a compact, convex disk and $f: \boldsymbol{R}^2 \to \boldsymbol{R}$ is a map which is zero outside D and nonnegative inside D, then, given $s > 0$, there is a square of side s in $\boldsymbol{R}^2$ with center in D such that f takes the same values at all the vertices of the square. Below, we shall give an example where D is not convex and the conclusion fails to hold.

The two following lemmas will be useful:

LEMMA 1. *Suppose $K \subset \boldsymbol{R}^2$ is compact and $\varepsilon > 0$ is given. Then there is a $\delta > 0$ such that if all the points of K' are within δ of K, then any square with vertices in K' has each of its vertices within ε of a point of K such that these 4 points of K form the vertices of a square. Four copies of a single point are viewed as the vertices of a (degenerate) square.*

Proof. Each square in $\boldsymbol{R}^2$ corresponds to a point in $\boldsymbol{R}^8$. The square with vertices (x_1, y_1), (x_2, y_2), (x_3, y_3), and (x_4, y_4), traveling counterclockwise around the square, is given by equations: $x_3 = x_2 + y_1 - y_2$, $y_3 = y_2 + x_2 - x_1$, $x_4 = x_1 + y_1 - y_2$, and $y_4 = y_1 + x_2 - x_1$. Thus all squares are represented by the 4-dimensional flat, F, in $\boldsymbol{R}^8$ which is the intersection of these 4 hyperplanes. Also represented are degenerate squares.

Let $N \subset \boldsymbol{R}^8$ be the set of points within $\varepsilon/2$ of $K^4 \cap F$ ($K^4 = K \times K \times K \times K$). Let $\delta > 0$ be smaller than $\varepsilon/4$ and smaller than half the distance from F to $K^4 \backslash N$. Let V' be a square with vertices within δ of K. Then in $\boldsymbol{R}^8$, V' is represented by a point $v \in F$ within $2\delta < \varepsilon/2$ of a point k of K^4. Now v is more than 2δ from $K^4 \backslash N$ so $k \in K^4 \cap N$. Thus k is within $\varepsilon/2$ of $K^4 \cap F$. Hence v is within ε of $K^4 \cap F$. $\square$

The next lemma follows from the fact that an angle greater than 90° with vertex at the center of a square must contain a vertex of the square.

LEMMA 2. *Suppose at every point of region D there is the vertex of an angle greater than 90° with an ε-radius sector in the interior of D (except perhaps for the vertex of the angle). Then for $\delta < \sqrt{2}\varepsilon$, if a square of side δ has center in D, at least one vertex of the square is in the interior of D.* □

We now construct the promised example.

EXAMPLE. A compact disk $D \subset \boldsymbol{R}^2$, a map $f: \boldsymbol{R}^2 \to \boldsymbol{R}$ zero outside D and nonnegative inside D, and $s > 0$, such that if f takes the same values on the vertices of a square of side s then the square's center is outside D.

Construction. Start with the unit circle in the plane. For $\varepsilon > 0$ small add the two segments from $(-\varepsilon, 0)$ to the circle which make angles of $\pm\varepsilon$ with the positive x-axis. Throw out the small arc of the circle subtended by the angle the two segments determine. "Round off" the two corners on the unit circle with arcs of small circles so that the resulting curve, C:

(i) is strongly starlike from any point $(-\gamma, 0)$ with $\varepsilon < \gamma < 1$ and

(ii) the points from but not including $(-\varepsilon, 0)$ to the origin lie outside C.

Note that the above can be achieved so that the only squares with all 4 vertices on C have center at the origin and vertices on the unit circle. One way to check this is to start with the unit circle and a single radius. On this compact set the claim is true. Then, using Lemma 1, we only need check for new squares which are very close to the old ones or very small.

Next, using Lemma 2, choose a small $s > 0$ so that no square of side s can have its center on or inside C and all its vertices on or outside C.

The boundary of $D(\partial D)$ is C translated to the right by γ, $\varepsilon < \gamma < 1$ (to be determined later). For $0 \leqq r \leqq 1$, the level curve on which f takes value r is given by multiplying the points of ∂D (treating them as vectors) by $1 - r$. So the level curves are all similar to C. By condition (i) above, this will indeed give us a continuous function. Now the only nonzero level on which our "table can balance" has $1 - r$ (the radius of the circle about the level curve) equal to $s/\sqrt{2}$, i.e., $r = 1 - s/\sqrt{2}$. For this r we choose γ so that $\varepsilon < \gamma < \varepsilon/r$ and hence $(1 - r)\gamma > \gamma - \varepsilon$. So the only center of a square "balancing" at a nonzero level lies over a point outside D (at $((1 - r)\gamma, 0)$).

Thus the only "tables which balance" on the graph of f are at the zero level or $r = 1 - s/\sqrt{2}$ level, and in both cases the centers lie outside D. $\square$

By putting the level curves at heights $g(r)$ for appropriate g, rather than r, we could have permitted a smoothing out of C at $(-\varepsilon, 0)$. In fact we could have made ∂D and f as smooth as desired.

Reference

1. Roger Fenn, *The Table Theorem*, Bull. London Math. Soc., **2** (1970), 73–76.

Received September 3, 1980.

Department of Mathematics
U. S. Naval Academy
Annapolis, MD 21402