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COMMON FIXED POINTS OF NONEXPANSIVE MAPPINGS BY ITERATION

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The purpose of this paper is to present an iteration scheme which converges strongly in one setting and weakly in another to a common fixed point of a finite family of nonexpansive mappings.

Let X be a Banach space and C a convex subset of X. Suppose $\{T_i: i = 1, 2, \cdots, k\}$ is a family of nonexpansive self-mappings of C. Define the following mappings: set $U_0 = I$, the identity mapping; then for $0 < \alpha < 1$ let

$$\begin{aligned} U_1 &= (1 - \alpha)I + \alpha T_1 U_0 \, , \\ U_2 &= (1 - \alpha)I + \alpha T_2 U_1 \, , \\ &\cdots \\ U_k &= (1 - \alpha)I + \alpha T_k U_{k-1} \, . \end{aligned}$$

THEOREM 1. *Let C be a convex compact subset of a strictly convex Banach space X and $\{T_i: i = 1, 2, \cdots, k\}$ a family of nonexpansive self-mappings of C with a nonempty set of common fixed points. Then for an arbitrary starting point $x \in C$, the sequence $\{U_k^n x\}$ converges strongly to a common fixed point of $\{T_i: i = 1, 2, \cdots, k\}$.*

REMARK 1. The sequence $\{U_k^n x\}$ can be expressed in the following form: let x_0 be an arbitrary element in C and let

$$\begin{aligned} x_1 &= (1 - \alpha)x_0 + \alpha T_k U_{k-1} x_0 \, , \\ x_2 &= (1 - \alpha)x_1 + \alpha T_k U_{k-1} x_1 \, , \end{aligned}$$

and, in general,

$$(*) \qquad x_{n+1} = (1 - \alpha)x_n + \alpha T_k U_{k-1} x_n \, , \quad n = 0, 1, 2, \cdots \, .$$

Observe that for $k = 1$, the sequence (*) becomes

$$(1) \qquad x_{n+1} = (1 - \alpha)x_n + \alpha T_1 x_n \, ,$$

which converges to a fixed point of T_1 by Edelstein's theorem [3]. The sequence (*) is clearly a generalization of this result.

Proof of Theorem 1. We first note that the mappings U_j and $T_j U_{j-1}$, $j = 1, 2, \cdots, k$, are nonexpansive and map C into itself. It is also easy to check that the families

$$\{U_1, U_2, \cdots, U_k\} \quad \text{and} \quad \{T_1, T_2, \cdots, T_k\}$$

have the same set of common fixed points.

Since the sequence (*) has the same form as (1), $\{U_k^n x\}$ converges to a fixed point y of $T_k U_{k-1}$ by Edelstein's theorem. We wish to show next that y is a common fixed point of T_k and $U_{k-1} (k \geqq 2)$. To this end we first show that $T_{k-1}U_{k-2}y = y (k \geqq 2)$. Suppose not; then the closed line segment $[y, T_{k-1}U_{k-2}y]$ has positive length. Now let

$$z = U_{k-1}y = (1 - \alpha)y + \alpha T_{k-1}U_{k-2}y .$$

By hypothesis there exists a point w such that $T_1 w = T_2 w = \cdots = T_k w = w$. Since $\{T_i\}$ and $\{U_i\}$ have the same common fixed points, it follows that $T_{k-1}U_{k-2}w = w$. By nonexpansiveness

$$\| T_{k-1}U_{k-2}y - w \| \leqq \| y - w \| \tag{2}$$

and

$$\| T_k z - w \| \leqq \| z - w \| .$$

So w is at least as close to $T_k z$ as to z. But $T_k z = T_k U_{k-1} y = y$, so that w is a least as close to y as to $z = (1 - \alpha)y + \alpha T_{k-1}U_{k-2}y$. Since X is strictly convex, we conclude that

$$\| y - w \| < \| T_{k-1}U_{k-2}y - w \| .$$

This contradicts (2), so that $T_{k-1}U_{k-2}y = y$. It now follows from

$$U_{k-1} = (1 - \alpha)I + \alpha T_{k-1}U_{k-2}$$

that $U_{k-1}y = (1 - \alpha)y + \alpha y = y$ and $y = T_k U_{k-1} y = T_k y$. Consequently, y is a common fixed point of T_k and U_{k-1}.

Since $T_{k-1}U_{k-2}y = y$, we may repeat the argument to show that $T_{k-2}U_{k-3}y = y$ and that y must therefore be a common fixed point of T_{k-1} and U_{k-2}. Continuing in this manner, we conclude that $T_1 U_0 y = y$ and that y is a common fixed point of T_2 and U_1. Thus y is a common fixed point of $\{T_i: i = 1, 2, \cdots, k\}$.

REMARK 2. If the family $\{T_i: i = 1, 2, \cdots, k\}$ is commutative, then the assumption that the set of common fixed points is nonempty may be omitted (DeMarr [2]).

THEOREM 2. *If X is a uniformly convex Banach space satisfying Opial's condition (in particular, if X is a Hilbert space) and C a closed convex subset of X, and if the family of mappings $\{T_i: i = 1, 2, \cdots, k\}$ satisfies the conditions in Theorem* 1, *then for any $x \in C$*

the sequence $\{U_k^n x\}$ *converges weakly to a common fixed point.*

Proof. Since $T_k U_{k-1}$ is a nonexpansive self-mapping of C, the sequence $\{U_k^n x\}$ converges weakly to a fixed point y of $T_k U_{k-1}$ (Opial [4]). By the argument in the proof of Theorem 1, y is a common fixed point of $\{T_i\}$.

Suppose, in addition, that C is bounded and the family $\{T_i\}$ commutative. Then, since X is strictly convex and reflexive, the assumption that the set of common fixed points is nonempty may again be omitted (Browder [1]).

Since Theorem 2 remains valid for $C = X$, the iteration scheme can be applied to the solution of systems of equations of the type

$$x - S_i x = f_i\ , \quad i = 1, 2, \cdots, k\ , \tag{3}$$

where each S_i is a nonexpansive self-mapping of X and each f_i a given element of X. To do so, it is sufficient to consider the family

$$T_i x = f_i + S_i x\ , \quad i = 1, 2, \cdots, k\ ,$$

each member of which is also a nonexpansive self-mapping of X, since x is a solution of the system (3) iff x is a common fixed point of $\{T_i\}$.

If C is a proper subset of X (as in Theorem 1) and each S_i a self-mapping of C, then the above procedure applies provided that each T_i maps C into itself.

References

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