PACIFIC JOURNAL OF MATHEMATICS
Vol. 97, No. 1, 1981

SAMPLE FUNCTIONS OF POLYA PROCESSES

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For a nonnegative measurable function f satisfying

$$\int_{-\infty}^{\infty} f(x)dx = 1 ,$$

define

$$r(t) = \int_{-\infty}^{\infty} \min\{f(x),\, f(x+t)\}dx .$$

Berman proved, extending so-called "Polya characteristic function", that the r is the characteristic function of an absolutely continuous distribution. The positive-definiteness of the r corresponds to a stationary Gaussian process, which is called Polya-Covariance process or simply Polya process.

In this paper, some analytic properties of its sample functions are studied: (1) continuity, (2) differentiability, (3) quadratic variation, and (4) upper and lower class.

1. Introduction.

Berman [2] extended a class of characteristic functions described by Polya [7]: Let $f(x)$ be a nonnegative measurable function satisfying

$$\int_{-\infty}^{\infty} f(x)dx = 1 . \tag{1.1}$$

Put

$$r(t) = \int_{-\infty}^{\infty} \min\{f(x),\, f(x+t)\}dx . \tag{1.2}$$

The r is a characteristic function, corresponding to an absolutely continuous distribution. Since the r is considered as a covariance function, there corresponds to a stationary Gaussian process X with mean zero and with the covariance function r. This process is often called *Polya Covariance Process* [2] or simply *Polya process* (cf: the review for [2], MR 52 (1976), ♯9345).

The Polya process X has a representation by Cabaña and Wschebor, using the plane Wiener process W:

$$X(t) = \int_{(y>0)}\int I\{f(x+t) - y\} W(dx \times dy) ,$$

where $I(u) = 1$ for $u > 0$ and $I(u) = 0$ for $u \leqq 0$ (Berman [2]). This representation is not used in this paper, but in terms of the covariance function (1.2), some analytic properties of sample functions of Polya processes are studied.

In §2, a basic hypothesis (H) is introduced, which reveals a foundation of the properties of their sample functions in terms of the r, that is, of f.

2. A basic hypothesis. Let f be a function of bounded variation in an interval $[a, b]$. Denote the total variation by $V_f([a, b])$. The $V_f([a, b])$ is decomposed into the positive variation $V_f^{\mathrm{pos}}([a, b])$ and the negative variation $V_f^{\mathrm{neg}}([a, b])$:

$$V_f([a, b]) = V_f^{\mathrm{pos}}([a, b]) + V_f^{\mathrm{neg}}([a, b]) ,$$

and moreover

$$V_f([a, b]) = 2V_f^{\mathrm{neg}}([a, b]) + \{f(b) - f(a)\} . \tag{2.1}$$

When f is defined in $(-\infty, \infty)$, being monotone outside of an interval $[a, b]$ and $f(\pm\infty) = 0$, define the total variation in $(-\infty, \infty)$ or denoting by $V(f)$ simply:

$$V(f) = V_f([a, b]) + \{f(b) + f(a)\} .$$

In more general, the total variation in $(-\infty, \infty)$ for a function f being given in $(-\infty, \infty)$ is defined, denoting by $V(f)$ again, as the limit of the total variation in any finite interval $[a, b]$, $V_f([a, b])$ as $a \to -\infty$ and $b \to \infty$. Then we have similarity

$$V(f) = V^{\mathrm{pos}}(f) + V^{\mathrm{neg}}(f) ,$$

and

$$V(f) = 2V^{\mathrm{neg}}(f) . \tag{2.2}$$

Now, define

$$S(\varepsilon) = \frac{1 - r(\varepsilon)}{\varepsilon}$$

and introduce a basic hypothesis:

$$\lim_{\varepsilon \downarrow 0} S(\varepsilon) = \frac{1}{2} V(f) . \tag{H}$$

LEMMA. *If f is continuous in $(-\infty, \infty)$, is monotone outside of a finite interval, and of bounded variation satisfying* (1.1) *and $f(\pm\infty) = 0$, then for the r defined by the f, the hypothesis* (H) *holds.*

Proof. (i) (A motive case). Take a unimodal function f satisfying the assumption of the lemma. Let $q(t)$ be the coordinate where the curves $y = f(x)$ and $y = f(x + t)$ intersect. Then we have

$$\begin{aligned} 1 - r(t) &= \int_{-\infty}^{\infty} f(x)dx - \int_{-\infty}^{\infty} \min\{f(x), f(x+t)\}dx \\ &= \int_{q(t)}^{\infty} \{f(x) - f(x+t)\}dx \\ &= \int_{q(t)}^{\infty} f(x)dx - \int_{q(t)+t}^{\infty} f(x)dx \\ &= tf(q(t) + t\theta),\ \theta \in (0, 1) . \end{aligned}$$

Then $(1 - r(t))/t = f(q(t) + t\theta)$ converges to the maximum of f in $(-\infty, \infty)$, that is, to $V(f)/2$.

(ii) (A preparative remark). We take a partition π_n of a finite interval $[a, b]$:

$$\pi_n\ ; \quad a = x_0 < x_1 < \cdots < x_n = b .$$

Let's denote by $\omega(x_k, x_{k+1})$ the oscillation of f in $[x_k, x_{k+1}]$, and define $\omega(\pi_n) = \sum_k \omega(x_k, x_{k+1})$. Then, it is well known that

$$\omega(\pi_n) \longrightarrow V_f([a, b]) \text{ as the mesh } (\pi_n) \longrightarrow 0 .$$

(iii) (General case). For a given positive t, the intersection-points of the curves $y = f(x)$ and $y = f(x + t)$ are at most denumerable. Write their coordinates as follows; $\cdots < b_{k-1}(t) < a_k(t) < b_k(t) < a_{k+1}(t) < \cdots$ for which for any x in the interval $[a_k(t), b_k(t)]$, $f(x) > f(x + t)$ holds. This is considered one of partition in $(-\infty, \infty)$, which is denoted by $\pi(t)$. Denote the cardinality of such intervals by $K(t)$. Write by $a_\infty(t)$ the coordinate of the intersection which has not the corresponding $b(t)$, that is, the largest among the $\{a_k(t)\}$. Then we obtain, writing $\{a_k\}$, $\{b_k\}$ simply for $\{a_k(t)\}$, $\{b_k(t)\}$,

$$\begin{aligned} 1 - r(t) &= \sum_{k=1}^{K(t)} \int_{a_k}^{b_k} \{f(x) - f(x+t)\}dx + \int_{a_\infty}^{\infty} \{f(x) - f(x+t)\}dx \\ &= \sum_{k=1}^{K(t)} \left\{\int_{a_k}^{t+a_k} f(x)dx - \int_{b_k}^{t+b_k} f(x)dx\right\} + \int_{a_\infty}^{t+a_\infty} f(x)dx \qquad (2.3) \\ &= t\sum_{k=1}^{K(t)} \{f(a_k + t\theta_k(t)) - f(b_k + t\theta'_k(t))\} + tf(a_\infty + t\theta_\infty(t)) \\ &= tS(t) , \end{aligned}$$

finding the sequences $\{\theta_k(t)\}$ and $\{\theta'_k(t)\}$ from the interval $[0, 1]$. Then we have

$$S(t) \leqq V^{\text{neg}}(f) ,$$

and applying the remark (ii) to the present case,

$$\omega(\pi(t)) \longrightarrow V(f) \qquad \text{as} \quad t \downarrow 0 .$$

On the other hand, for any $\varepsilon > 0$, it is possible from (2.3) and (2.2) to find a sequence $\{t_n \downarrow 0\}$ such that

$$\frac{1}{2}\omega(\pi(t_n)) \leqq S(t_n) + \varepsilon .$$

These establish the lemma.

The lemma shows only one example of classes satisfying the hypothesis (H).

3. Sample continuity. Since Polya process X is a stationary Gaussian process, it has either sample continuous version or sample unbounded one (Ju. K. Beljaev [1]). This alternative depends on the function f, appearing in (1.2). The variance of the increment of Polya process X has the exact form;

$$E\{(X(s) - X(t))^2\} = 2(1 - r(|s - t|)) . \tag{3.1}$$

Now, it is possible to make use of the Fernique's condition for a separable stationary Gaussian process Y to have sample continuous version ([4]) in a following modification:

Suppose that there exists, for a real-valued, separable stationary Gaussian process Y with mean zero, a function ϕ monotone increasing in the neighborhood of the origin and satisfying, for $(s, t) \in R^2$,

$$k_2\phi^2(|s - t|) \leqq E\{(Y(s) - Y(t))^2\} \leqq k_1\phi^2(|s - t|) , \quad (|s - t| \longrightarrow 0) , \tag{3.2}$$

where k_1 and k_2 are absolute constants.

If

$$\int^{\infty} \phi^2(e^{-x^2})dx < \infty , \quad (= \infty) , \tag{3.3}$$

then the Y has sample continuous version (sample unbounded version).

Indeed, take ψ as the inverse function of ϕ. The change of variable: $\exp(-x^2) = \psi(u)$ implies the integral (3.3) is equivalent to

$$\int_0 (\log(1/\psi(u))^{1/2}du . \tag{3.4}$$

On the other hand, let $N(V, q^{-k})$ be the minimal number of open d-ball of the radius q^{-k}, $(q > 1)$, covering $V = [-1/2, 1/2]$, with pseudo-metric $d(s, t) = E^{1/2}\{(Y(s) - Y(t))^2\}$. Then it leads to

$$N(V, q^{-k}) = O([1/2\psi(q^{-k}/C)]) , \quad (k \longrightarrow \infty) , \tag{3.5}$$

where the bracket [ ] indicates the integral part and C is a constant relying on the k_1 and k_2 in (3.2). Then (3.4) and (3.5) implies the Fernique's condition:

$$\sum q^{-k}(\log N(V, q^{-k}))^{1/2} .$$

Therefore the separable Polya process X has continuous version or unbounded version according to

$$\int^{\infty}(1 - r(e^{-x^2}))^{1/2}dx < \infty , \quad \text{or} = \infty , \tag{3.6}$$

since $1 - r(t)$ satisfies the conditions of ϕ in the Fernique's condition.

THEOREM 1. *If the covariance function r of a separable Polya process satisfies the hypothesis* (H), *then sample functions are continuous almost surely.*

Proof. From (H), we have for a sufficiently small ε

$$1 - r(\varepsilon) \leqq \varepsilon V(f) .$$

Therefore

$$\int^{\infty}\{2(1 - r(e^{-x^2}))\}^{1/2}dx < 2\int^{\infty}\{V(f)e^{-x^2}\}^{1/2}dx .$$

This establishes the theorem.

The following example shows the sample unboundedness of Polya process, being defined by the f which does not satisfy conditions in the Theorem 1, that is, f is not continuous at the two points and not of bounded variation. This Polya process is an example of the class considered by Fernique [3].

EXAMPLE 3.1. Let's define g by

$$g(x) = \frac{d}{dx}\left\{1/|\log x| \prod_{j=2}^{n} |\log_j x|^2\right\} , \quad x \in (0, 1/e^{*n}) ,$$

where $\log_j x$ is the j-times iterated logarithm of x, and e^{*n} is defined recursively as follows: $e^{*k} = e^{e^{*(k-1)}}$, $e^{*2} = e^e$, $e^{*1} = e$, $e^{*0} = 1$, $(k = 3, 4, 5, \cdots)$. Set

$$E = \left\{\left(\prod_{k=0}^{n-2} e^{*k}\right)^2 e^{*(n-1)}\right\}^{-1} .$$

and define

$$f(x) = \begin{cases} g(x)/E , & \text{for } \ x \in (0, 1/e^{*n}) , \\ 0 , & \text{otherwise} . \end{cases}$$

Then, the Polya process X, corresponding to the r defined by the f, is sample unbounded: In fact,

$$\begin{aligned} r(t) &= \int_{-\infty}^{\infty} \min\{f(x), f(x+t)\}dx , \quad (t > 0) \\ &= \int_0^{\infty} f(x+t)dx \\ &= \int_t^{\infty} f(y)dy . \end{aligned}$$

Since

$$\phi^2(t) = E\{(X(s+t) - X(s))^2\} = 2(1 - r(t)) = 2\int_0^t f(x)dx ,$$

we have

$$\int^{\infty} \phi(e^{-t^2})dt = \infty ,$$

which implies the unboundedness of almost all sample functions.

But, even when it assumed that f is not of bounded variation in opposition to the Theorem 1, an example is constructed, which shows the sample continuity of the corresponding Polya process.

EXAMPLE 3.2. Let's define g by

$$g(x) = \begin{cases} x|\sin(1/x)| , & \text{for } \ x \in [0, 1/\pi] , \\ 0 , & \text{otherwise} . \end{cases}$$

Set $K = \int_0^{1/\pi} g(x)dx$. Define f by

$$f(x) = \begin{cases} g(x)/K , & \text{for } \ x \in [0, 1/\pi] , \\ 0 , & \text{otherwise} . \end{cases}$$

The $f(x)$ is nonnegative continuous in $(-\infty, \infty)$ satisfying (1.1), but not of bounded variation. But, the Polya process, corresponding to the covariance $r(t) = \int_{-\infty}^{\infty} \min\{f(x), f(x+t)\}dx$, has sample continuity: In fact, by taking a positive c and a positive M, such that $3M = c$, we have

$$\begin{aligned} &\int_c^{\infty} (1 - r(e^{-x^2}))^{1/2}dx \\ (3.7) \qquad &= \sum_{k=0}^{\infty} \int_{c+kM}^{c+(k+1)M} (1 - r(e^{-x^2}))^{1/2}dx \\ &= M\sum_{k=0}^{\infty} (1 - r(\exp\{-(c + (k+\theta_k)M)^2\}))^{1/2} , \end{aligned}$$

where for each k, $\theta_k \in (0, 1)$.

The square of a summand, $(1 - r(\exp\{-(c + (k + \theta_k)M)^2\}))$ is, as stated in the the proof of lemma, equal to the total area enclosed by the curves $y_1 = f(x)$ and $y_2 = f(x + \mathrm{ext}\{-(c+(k + \theta_k)M)^2\})$, where $y_1 > y_2$ in $[0, 1/\pi)$. The coordinates of zeros of $f(x)$ are $1/2n\pi$ and $1/(2n + 1)\pi$, $(n = 1, 2, 3, \cdots)$. The distance between a zero and a consecutive local maximal extreme; $(1/2n\pi) - (1/(2n\pi + \pi/2))$ or $(1/(2n + 1)\pi) - (1/((2n + 1)\pi + \pi/2)$ is smaller than $1/8n^2$. Determine an integer n such that the shift-width does not exceed over $1/8n^2$, that is, $1/8n^2 > \exp\{-(c + (k + \theta_k)M)^2\}$, namely,

$$n < (e^{\{c+(k+1)M\}^2}/8)^{1/2} . \tag{3.8}$$

For n's satisfy (3.8), the area, which is generated by a shift $\exp\{-(c + (k + \theta_k)M)^2\}$, being enclosed by $y_1 = f(x)$ and $y_2 = f(x + \exp\{-(c + (k + \theta_k)M)^2\})$, where $y_1 > y_2$, is estimated by an absolute constant multiple of

$$e^{-\{c+(k-1)M\}^2}\Big\{\sum_{n \text{ satisfying}(3.8)} \Big\{1/\Big(2n\pi + \frac{\pi}{2}\Big) + 1\Big/\Big((2n + 1)\pi + \frac{\pi}{2}\Big)\Big\} . \tag{3.9}$$

Using (3.8) for estimate of the cardinality of the summands in (3.9), it is moreover majorated by a constant multiple of

$$\exp\{-(c + (k - 1)M)^2\}(\exp\{(c + (k + 1)M)^2\}/8)^{1/2} ,$$

that is, by a constant multiple of

$$\exp\{-(Mk - (3M - c))^2/2\} . \tag{3.10}$$

Next, for the area of the part in which n does not satisfy (3.8), set $n_0 = (\exp\{c + (k + 1)M\}^2/8)^{1/2}$ which appeared in (3.8), and take a rectangular triangle of which vertices are $(0, 0)$, $(0, 1, 2n_0\pi)$ and $(1/2n_0\pi, 1/K(2n_0\pi))$. Then the rectangular triangle covers the part enclosed by the curve $y_1 = f(x)$ and the line $y = 0$. The area of the rectangular triangle, $1/2K(2n_0\pi)^2$ is estimated by

$$\exp\{-(c + (k + 1)M)^2\}/\pi^2K . \tag{3.11}$$

By (3.10) and (3.11), we have

$$1 - r(\exp\{-(c + (k + \theta_k)M)^2\}) \leqq \text{Const. } e^{-(Mk-(3M-c))^2}/2 + \text{Const. } e^{-(c+(k+1)M)^2} .$$

As we set $3M = c$,

$$\{1 - r(\exp\{-(c + (k + \theta_k)M)^2\})\}^{1/2} \leqq \text{Const. } e^{-(Mk)^2/4} .$$

This assures the convergence of the series in (3.7). It implies the assertion by the Fernique's condition.

EXAMPLE 3.3. Let's define

$$f(x) = \begin{cases} \exp(-x/2)/2\,, & \text{for} \quad x \in [0, \infty)\,, \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Then we see that $r(t) = \exp(-t/2)$, for a nonnegative t. The Polya process corresponding to the r is an Ornstein — Uhlenbeck's Brownian motion X:

$$X(t) = \exp(-t/2)\int_{-\infty}^{t} \exp(u/2)dB(u)\,,$$

where B stands for the ordinary Brownian motion.

4. Nowhere differentiability of sample functions. In [5], a condition of nowhere differentiability of sample functions of a Gaussian process X is given: *If there can be found a function ϕ such that*

$$E[\{X(s+h) - X(s)\}^2] \geqq \phi^2(h)\,, \tag{4.1}$$

and if there exists a positive integer q such that

$$\lim_{h \downarrow 0} \left\{\frac{h}{\phi(h)}\right\}^q /h = 0 \tag{4.2}$$

and finding a positive integer p

$$\limsup_{h \downarrow 0} \left\{ \sup_{|t-s| \geqq ph} |\psi(h;\, t,\, s)| \right\} \leqq 1/2q\,, \tag{4.3}$$

where $\psi(h;\, t,\, s) = \text{Correlation } \{X(t+h) - X(t),\, X(s+h) - X(s)\}$, *then*

$$\limsup_{h \downarrow 0} h^{-1}|X(t+h) - X(t)| = \infty\,,$$

for all t in $[0, 1]$, *almost surely.*

Since the present process is stationary, it is sufficient to consider only in $[0, 1]$. Using this proposition, we establish the following theorem:

THEOREM 2. *If the covariance function r of of Polya process X satisfies* (H), *then the sample function are nowhere differentiable in* $[0, 1]$ *almost surely.*

Proof. Since it can be taken $2(1 - r(t))$ as $\phi^2(t)$,

$$h^{-1}(h/\phi(h))^q \leqq \left(\frac{h}{2}\right)^{(q/2)-1} \{V^{\text{neg}}(f)\}^{-q/2}\,,$$

for a sufficiently small h by (H). This ensures (4.2) for each $q > 2$.

Next, it must be found a positive integer p satisfying (4.3). Since

$$\sup_{|t-s|\geqq ph} |\psi(h;\, t,\, s)| = \sup_{p\in N}\ \sup_{ph\leqq|t-s|<(p+1)h} |\psi(h;\, t,\, s)|\ ,$$

it is sufficient to check for each p

$$\lim_{h\downarrow 0}\ \sup_{ph\leqq|t-s|<(p+1)h} |\psi(h;\, t,\, s)|\ .$$

Then we have for t and s in $[ph, (p+1)h)$, $(s<t)$,

$$\begin{aligned}\psi(h;\, t,\, s) = \Big\{&S\Big(h\Big(\frac{t-s}{h}-1\Big)\Big)h\Big(\frac{t-s}{h}-1\Big)+S\Big(h\Big(\frac{t-s}{h}+1\Big)\Big)h\Big(\frac{t-s}{h}+1\Big)\\ &-2S\Big(h\Big(\frac{t-s}{h}\Big)\Big)h\Big(\frac{t-s}{h}\Big)\Big\}\Big/ 2S(h)h\ .\end{aligned}$$

Since $S(\varepsilon)\to V^{\mathrm{neg}}(f)$ under the hypothesis (H) as $\varepsilon\to 0$, we see that $\psi(h;\, t,\, s)$ is vanishing as $h\downarrow 0$. This ensures (4.3) for taking any positive integer p. Thus the proof is completed.

5. **Quadratic variations.** For the process, the method of Klein and Giné [6] yields the following:

THEOREM 3. *Let* $\{X(t);\, t\in[0,1]\}$ *be a Polya process satisfying the hypothesis* (H). *Let* $\{\pi_n\}_{n=1}^{\infty}$ *be a sequence of partitions of the interval* $[0, 1]$; $\pi_n = \{0 = t_0^{(n)} < t_1^{(n)} < \cdots < t_{c(\pi_n)}^{(n)} = 1\}$, *such that* $\max\{(t_i^{(n)} - t_{i-1}^{(n)}):\, t_i^{(n)}\in\pi_n\} = o(1/\log n)$, $(n\to\infty)$. *Then, denoting*

$$B_n = \sum_{i=1}^{c(\pi_n)} \{X(t_i^{(n)}) - X(t_{i-1}^{(n)})\}^2\ ,$$

we have almost surely

$$\lim_{n\to\infty} B_n = V(f)\ .$$

The proof is quite same as in [6] which contributes to find a sequence $\{\varepsilon_n\downarrow 0\}$ satisfying

$$\sum_{n=1}^{\infty} P[\,|B_n - E(B_n)|\geqq \varepsilon_n] < \infty\ ,$$

being based on the bound of Hanson and Wright (1971). Hence, only some different points from the proof in [6] will be noted here. In the proof, it necessitates to evaluate $E[(X(t_i^{(n)}) - X(t_{i-1}^{(n)}))(X(t_j^{(n)}) - X(t_{j-1}^{(n)}))]$. This requires generally to consider the singularity of the second derivatives of the covariance function. But for the present process, which is stationary, it is easy to carry out it under the (H).

In fact, for any pair (i, j), by Schwarz's inequality, $E[(X(t_i^{(n)}) - X(t_{i-1}^{(n)}))(X(t_j^{(n)}) - X(t_{j-1}^{(n)}))] \leqq E^{1/2}\{(X(t_i^{(n)}) - X(t_{i-1}^{(n)}))^2\}E^{1/2}\{(X(t_j^{(n)}) - X(t_{j-1}^{(n)}))^2\} = 2(1 - r(t_i^{(n)} - t_{i-1}^{(n)}))^{1/2}(1 - r(t_j^{(n)} - t_{j-1}^{(n)}))^{1/2} \leqq K(t_i^{(n)} - t_{i-1}^{(n)})^{1/2}(t_j^{(n)} - t_{j-1}^{(n)})^{1/2}$, where the constant K is relevant to the total variation $V(f)$. It remains to find the limit of $E(B_n)$:

$$E(B_n) = \sum_{k=1}^{c(\pi_n)} E\{(X(t_k^{(n)}) - X(t_{k-1}^{(n)}))^2\}$$
$$= \sum_{k=1}^{c(\pi_n)} \frac{2(1 - r(t_k^{(n)} - t_{k-1}^{(n)}))}{t_k^{(n)} - t_{k-1}^{(n)}}(t_K^{(n)} - t_{k-1}^{(n)}) .$$

Hence,

$$\min_k \frac{2(1 - r(t_k^{(n)} - t_{k-1}^{(n)}))}{t_k^{(n)} - t_{k-1}^{(n)}} \sum_{k=1}^{c(\pi_n)} (t_k^{(n)} - t_{k-1}^{(n)}) \leqq E(B_n) ,$$
$$\max_k \frac{2(1 - r(t_k^{(n)} - t_{k-1}^{(n)}))}{t_k^{(n)} - t_{k-1}^{(n)}} \sum_{k=1}^{c(\pi_n)} (t_k^{(n)} - t_{k-1}^{(n)}) \geqq E(B_n) .$$

Since $\sum_{k=1}^{c(\pi_n)}(t_k^{(n)} - t_{k-1}^{(n)}) = 1$, the hypothesis (H) implies

$$\lim_{n\to\infty} E(B_n) = V(f) .$$

This completes the proof.

6. Upper and lower class. This section aims only to restate, in the case of Polya process X, the theorem in [8] which describes the asymptotic behavior of the process.

Let M be the class of monotone, nondecreasing, and continuous functions. A function ϕ in M is called *a function of the upper class with respect to the uniform continuity* of X, if for almost all ω of the random parameter in a probability space, there exists a $\delta(\omega) > 0$ such that $0 < |t - s| < \delta(\omega)$ implies

$$|X(t, \omega) - X(s, \omega)| \leqq \phi(1/t - s|)E^{1/2}\{(X(t) - X(s))^2\} .$$

This class is denoted by unif-U.

On the contrary, a function ϕ in M is called *a function of the lower class with respect to the uniform continuity* of X, if for almost all ω there exists a sequence $\{t_n(\omega); n = 1, 2, 3, \cdots\} \in [0, 1]$ such that

$$|X(t_n) - X(t_{n-1})| > \phi(1/|t_n - t_{n-1}|)E^{1/2}\{(X(t_n) - X(t_{n-1}))^2\}$$

as $|t_n - t_{n-1}| \to 0$, $(n \to \infty)$. This class is denoted by unit-L. Then we have

THEOREM 4. *Let X be a Polya process satisfying the hypothesis*

(H). *Assume that the covariance function $r(t)$ is convex or concave in $(0, \delta)$ for a sufficiently small δ. If, for ϕ in M, the integral*

$$\int^{\infty} \phi^3(t) \exp\{-\phi^2(t)/2\} dt$$

converges, then the ϕ belongs to unif-U.

On the contrary, if the above integral diverges under the sssumption that $r(t)$ is convex in $(0, \delta)$ for a sufficient small δ, then the ϕ belongs to unif-L.

The adaptation of [8] to the other properties of Polya process; the local continuity, the "iterated logarithm"-typed properties etc., are all omitted.

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Received July 29, 1980.

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