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A "MAXIMAL TORUS" TYPE THEOREM FOR COMPLETE RIEMANNIAN MANIFOLDS

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The purpose of this paper is to show the existence of "Maximal tori" in a number of complete Riemannian manifolds. More specifically we are looking for what we call submanifold systems which are defined as follows.

DEFINITION. A submanifold system $\mathscr{S}$, through a point p, in a complete connected Riemannian manifold M, is a collection $\mathscr{S} = \{N_\alpha\}$ of connected submanifolds of M such that:

(a) $N_\alpha \in \mathscr{S}$ is complete, topologically closed and totally geodesic.

(b) $p \in N_\alpha$ for each $N_\alpha \in \mathscr{S}$.

(c) $M = \bigcup_\alpha N_\alpha$.

(d) For N_α, $N_\beta \in \mathscr{S}$ there is an isometry I_β^α from N_α to N_β with $I_\beta^\alpha(p) = p$. (The isometry I_β^α need not be the restriction of an isometry of M.)

Note. If the isometries I_β^α are in fact restrictions of isometries of M then we call $\mathscr{S}$ a conjugate submanifold system.

Clearly such objects are rare with the exception of the trivial case $\mathscr{S} = \{M\}$.

An example of such a submanifold system is given by the collection of maximal tori in a Lie group.

Other standard examples are:

(1) The collection of S^r's passing through a point $p \in S^n$.

(2) The collection of CP^r's passing through a point $p \in CP^n$.

(3) The collection of L_q^r's passing through a point $p \in L_q^n (n, r \geqq 3$, odd). All of the above are conjugate submanifold systems.

An example of a nonconjugate submanifold system is given by all the geodesics through a point $p \in M$, where M is a simply connected manifold of variable negative curvature.

In this paper we show that the existence of sufficiently many totally geodesic submanifolds (with a mild convexity condition) through $p \in M$ is sufficient to guarantee the existence of a nontrivial submanifold system through p.

This condition is somewhat more general than it at first appears. For example, let I_p be the group of isometries of M that leave p fixed. Then I_p will act orthogonally on the unit $n-1$ sphere in T_pM. If this representation has nontrivial principal isotropy subgroup, then the above condition will hold. (We will see this in §1.) Wu-Yi Hsiang has essentially classified the possible orthogonal representations on spheres with nontrivial principal isotropy subgroups [4].

In particular, the above will imply the following. Let M be a compact manifold and G be a compact lie group acting effectively on M. Then either the principal orbit of the induced action on TM is diffeomorphic to G or in any G-averaged metric on M there is a nontrivial submanifold system through each point $p \in M$.

The paper is divided into four sections. In the first section we define the notions we need and state the theorem. In the second section we state results whose proofs follow directly the proofs in a previous paper [2]. In the third section we prove the theorem. The fourth section consists of remarks.

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I. Statement of the theorem. In a previous paper [2] we define convexity conditions ATC (almost totally convex) and CC (completely convex). These give rise to invariants $\mathrm{ATC}_p(M)$ and $\mathrm{CC}_p(M)$ for each point p in a complete Riemannian manifold M^n. $\mathrm{ATC}_p(M)$ and $\mathrm{CC}_p(M)$ are integers satisfying: $0 \leqq \mathrm{ATC}_p(M) \leqq n$, $1 \leqq \mathrm{CC}_p(M) \leqq n$, and $\mathrm{ATC}_p(M) \leqq \mathrm{CC}_p(M)$.

We now proceed in the same fashion.

DEFINITION. Given $\varepsilon > 0$, a submanifold N of a complete Riemannian manifold M is said to be ε-convex if whenever $x, y \in N$ and γ is a geodesic from x to y of length less than ε then $\gamma \subset N$.

REMARK. This clearly implies that N is totally geodesic.

DEFINITION. Given $\varepsilon > 0$ and $R \subset T_pM$ a linear subspace let N_R^ε represent the smallest connected, complete, topologically closed, totally geodesic submanifold through p such that $R \subset T_pN_R^\varepsilon$ and N_R^ε is ε-convex.

The existence and uniqueness of N_R^ε follows from the fact that the properties are preserved under intersections and taking connected components.

DEFINITION. For fixed $p \in M$ and $\varepsilon > 0$ let $C_p^\varepsilon(M) = \min\{\dim R \mid R \in T_pM \text{ and } N_R^\varepsilon = M\}$.

We will assume for the remainder of this paper, that M is a connected complete Riemannian manifold with nonvanishing injectivity radius $i(M)$.

Now for $i(M) \geqq \varepsilon > 0$ we have directly from the definitions that

$$\text{ATC} \Longrightarrow \text{CC} \Longrightarrow \varepsilon\text{-convex} .$$

Hence $N_R^{\text{ATC}} \supseteqq N_R^{\text{CC}} \supseteqq N_R^{\varepsilon}$, hence $\text{ATC}_p(M) \leqq \text{CC}_p(M) \leqq \text{C}_p^{\varepsilon}(M) \leqq n$.

Now let I_p be the group of isometries of M leaving p fixed. Results in [2] show that if the induced orthogonal representation on S^{n-1} has nontrivial isotropy subgroup then $1 < \text{CC}_p(M)$. Hence $C_p^{\varepsilon}(M) \geqq 2$. We also have the formula:

$$\dim I_p \leqq \frac{\text{CC}_p(M)}{2}(2n - \text{CC}_p - 1) \leqq \frac{C_p^{\varepsilon}(M)}{2}(2n - C_p^{\varepsilon}(M) - 1) .$$

Hence if $\dim I_p$ is very large so is $C_p^{\varepsilon}(M)$.

The theorem will give us a nontrivial submanifold system $\mathscr{S}^r$ for every $1 \leqq r < C_p^{\varepsilon}(M)$. Thus there exists a nontrivial submanifold system whenever $C_p^{\varepsilon}(M) \geqq 2$.

In order to define precisely what $\mathscr{S}^r$ is we need some further notation.

Let $G^r(M) \xrightarrow{\pi^r} M$ be the Grassman manifold of r-dimensional linear subspaces. Let $G(M) \xrightarrow{\pi} M$ be the bundle where $G(M) = G^0(M) + G^1(M) + \cdots + G^n(M)$ (+denoting disjoint union).

Now define the following functions:

$$d\colon G(M) \longrightarrow \boldsymbol{Z} \text{ by } d(S) = \text{dimension of } S .$$
$$C^{\varepsilon}\colon M \longrightarrow \boldsymbol{Z} \text{ by } C^{\varepsilon}(p) = C_p^{\varepsilon}(M) .$$
$$f_{\varepsilon}\colon G(M) \longrightarrow G(M) \text{ by } f_{\varepsilon}(R) = T_{\pi(R)}N_R^{\varepsilon} .$$

Clearly

$$\begin{array}{ccc} G(M) & \xrightarrow{f_{\varepsilon}} & G(M) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ M & \xrightarrow{\text{id}} & M \end{array}$$

commutes.

We will let $G^r(T_pM) = (\pi^r)^{-1}(p)$ (i.e., the Grassman manifold of r planes in T_pM).

For $1 \leqq r \leqq n$ let $\mathscr{A}^r \subset G^r(T_pM)$ be the set of r-planes R such that $d \circ f_{\varepsilon}(R)$ is maximum. Let $\mathfrak{A}^r = f_{\varepsilon}(\mathscr{A}^r)$.

THEOREM. *Let M^n be a complete Riemannian manifold with a positive injectivity radius $i(M)$. Fix $i(M) \geqq \varepsilon > 0$. Then for each point $p \in M$ and each r, $1 \leqq r \leqq n$, the collection $\mathscr{S}^r = \{\exp_p S \mid S \in \mathfrak{A}^r\}$ is a submanifold system through p.*

REMARK. If $r < C_p^{\varepsilon}$ then $d \circ f_{\varepsilon}(R) < n$ for every $R \in G^r(T_pM)$ hence $\mathfrak{A}^r \neq \{T_pM\}$ and the submanifold system $\mathscr{S}^r$ is not trivial.

REMARK. By definition, for $S \in \mathfrak{A}^r$ there is an $R \in \mathscr{A}^r$ such that $S = f_\varepsilon R$ hence $\exp_p S = N^\varepsilon_R$. Thus the first two conditions for a submanifold system are satisfied by $\mathscr{S}^r$.

II. Some facts. In this section we state some results whose proofs follow the same lines as the proofs in [2] for the corresponding results about ATC and CC.

Fact 1. *The image of f_ε is closed in $G(M)$.*

Fact 2. *The function $d\circ f_\varepsilon$ is lower semi-continuous. In particular $\mathscr{A}^r$ is open in $G^r(T_pM)$.*

Fact 3. *The function C^ε is upper semi-continuous and its image consists of at most two consecutive integers.*

REMARK. This says that if for some $p \in M$, $C^\varepsilon_p(M) \geqq 3$ then for all $q \in M$, $C^\varepsilon_q(M) \geqq 2$. Hence every q will have a nontrivial submanifold system.

Fact 4. *Let $\tau\colon [0, 1] \to G^s(T_pM)$ be a piecewise C^∞ curve such that $\tau(t)$ is in the image of f_ε for every $t \in [0, 1]$. Then* $\exp_p(\tau(0))$ *is isometric to* $\mathrm{Exp}_p(\tau(\varepsilon))$. *Further the isometry is of the form* $\mathrm{Exp}_p\circ I_t\circ \mathrm{Exp}_p^{-1}$, *where I_t is an isometry from $\tau(0)$ to $\tau(t)$.*

REMARK. To define isometries I_t in the above it is sufficient to choose an appropriate orthonormal basis for each $\tau(t)$. The choice that gives the above result is gotten by lifting τ to the Stiefle manifold horizontally with respect to the canonical connection (see [2]).

Many of the results in [2] and [3] hold for ε-convex as well. We will only use those stated above.

III. Proof of the Theorem. To prove the theorem we will show that $\mathfrak{A}^r$ is a smooth, closed, connected submanifold of $G^s(T_pM)$, where $s = d\circ f_\varepsilon(\mathscr{A}^r)$. To prove that $\mathfrak{A}^r$ is a submanifold we will first show that f_ε is smooth when restricted to $\mathscr{A}^r$ (f_ε in general is not even continuous). Lemmas 3.1 to 3.3 are technical lemmas used to show that f_ε is smooth.

For $S \subset T_pM$ a linear subspace, let N^F_S be the set of points $x \in M$ such that there exists a finite chain $p = x_0, x_1, \cdots, x_r = x$ of points in M and geodesics $\gamma_1, \cdots, \gamma_r$ with γ_i going from x_{i-1} to x_i satisfying the following property. Let $S_0 = S \subset T_pM$. Let $S_i \subset T_{x_i}M$ be the parallel translate of S_{i-1} along γ_i. We require that $\gamma'_i(x_{i-1}) \in S_{i-1}$ (hence $\gamma'_i(x_i) \in S_i$). We will call such a chain $(\{x_i\}^r_0, \{\gamma_i\}^r_1, \{S_i\}^r_1)$ an appropriate chain. Since N_S^ε is totally geodesic $N_S^F \subset N_S^\varepsilon$.

LEMMA 3.1. *For $S \subset T_pM$, S is in the image of f_ε if and only if N_S^F has the following property: For every $q_1, q_2 \in N_S^F$ and every appropriate chain $(\{x_i\}_0^r, \{\gamma_i\}_1^r, \{S_i\}_0^r)$ from p to q_1 and every geodesic γ from q_1 to q_2 of length less than ε we have $\gamma'(q_1) \in S_r$. Further if this is the case then $N_S^F = N_S^\varepsilon$.*

REMARK. A similar lemma is true for ATC and CC.

Proof. Assume S is in the image of f_ε. Then $\mathrm{Exp}_p(S) = N_S^\varepsilon$ is a complete topologically closed totally geodesic ε-convex submanifold. It is easy to see that $N_S^F = N_S^\varepsilon$ and that $S_r = T_{q_1}N_S^\varepsilon$. Thus by the ε-convexity of N_S^ε we have $\gamma'(q_1) \in T_{q_1}N_S^\varepsilon = S_r$.

Now assume that N_S^F has the stated property. Let $q \in N_S^F$. Let $S_q \subset T_qM$ be a subspace comming from some appropriate chain. Then, by taking a chain with one more link, we have $\mathrm{Exp}_q S_q \subset N_S^F$. Further the property shows that $B(q, \varepsilon) \cap N_S^F = B(q, \varepsilon) \cap \mathrm{Exp}_q S_q$. This shows that S_q is independent of the choice of chain and that N_S^F is a complete, topologically closed, totally geodesic, ε-convex submanifold of M. Further N_S^F is connected and has the same dimension as $S(S = T_pN_S^F)$. Thus $\mathrm{Exp}_p(S) = N_S^\varepsilon = N_S^F$ and so $S = f_\varepsilon(S)$. $\square$

LEMMA 3.2. *Let U be an open subset of $G^r(T_pM)$ such that $d\circ f_\varepsilon(R) > s \geqq r$ for each $R \in U$. Let $f\colon U \to G^s(T_pM)$ be a smooth map such that $R \subset f(R) \subset f_\varepsilon(R)$ for each $R \in U$. Then for each $R \in U$ there is an open set U_R and a smooth map $\tilde{f}\colon U_R \to G^{s+1}(T_pM)$ such that for each $R' \in U_R$ we have $f(R') \subset \tilde{f}(R') \subset f_\varepsilon(R')$.*

Proof. Let $ST^s(T_pM) \xrightarrow{\pi} G^s(T_pM)$ be the bundle of the Stiefle manifold over the Grassman manifold. Let 0 be an open neighborhood of $f(R)$ such that the bundle is trivial over 0. Let $s\colon 0 \to ST^s$ be a smooth section. Then for each $R' \in f^{-1}(0)$ s defines (in a smooth fashion) an orthonormal basis $V_1(R'), \cdots, V_s(R')$ of $f(R')$.

There is thus a $1-1$ correspondence between appropriate chains $(\{x_i\}_0^\alpha, \{\gamma_i\}_1^\alpha, \{S_i\}_0^\alpha)$ in $N_{f(R')}^F$ and finite sequences $Z_1, Z_2, \cdots, Z_\alpha$ in $\boldsymbol{R}^s$. The correspondence is given as follows: $z_1 = \gamma_1'(0)$ with respect to $V_1(R'), \cdots, V_s(R')$ and z_i is the parallel translate of $\gamma_i'(0)$ along $\gamma_1 \cup \cdots \cup \gamma_{i-1}$ with respect to $V_1(R'), \cdots, V_s(R')$, where each γ_i is parameterized on $[0, 1]$ proportional to arc length.

Since $f(R) \neq f_\varepsilon(f(R))$ Lemma 3.1 tells us that there exists points $q_1, q_2 \in N_{f(R)}^F$, an appropriate chain $(\{x_i\}_0^\alpha, \{\gamma_i\}_1^\alpha, \{S_i\}_0^\alpha)$ from p to q_1, and a geodesic γ from q_1 to q_2, of length less than ε such that $\gamma'(0) \notin S_\alpha$. Choose an appropriate chain $(\{y_i\}_0^\beta, \{\tau_i\}_1^\beta, \{T_i\}_0^\beta)$ from p to

q_2. Let $z_1, \cdots, z_\alpha$ and $w_1, \cdots, w_\beta$ be the corresponding finite sequences in $\boldsymbol{R}^s$. These in turn give rise to chains $(\{x_i(R')\}_0^\alpha, \{\gamma_i(R')\}_1^\alpha, \{S_i(R')\}_0^\alpha)$ and $(\{y_i(R')\}_0^\beta, \{\tau_i(R')\}_1^\beta, \{T_i(R')\}_0^\beta)$ in $N^F_{f(R')}$. Thus defining points $q_1(R')$ and $q_2(R')$ in $N^F_{f(R')}$. Everything above varies smoothly in R'. Thus for some open neighborhood $\bar{U}$ of R in $G^r(T_pM)$ we will have the distance from $q_1(R')$ to $q_2(R')$ less than ε. Let $\gamma_{R'}$ be the unique geodesic (since $i(M) \geqq \varepsilon$) from $q_1(R')$ to $q_2(R')$. In some, possibly smaller, neighborhood U_R of R we will have $\gamma'_{R'}(0) \notin S_\alpha(R')$, for any R' in U_R. Let $V(R')$ be the vector in T_pM gotten by parallel translating $\gamma'_{R'}(0)$ back to p along $\gamma_1(R') \cup \cdots \cup \gamma_\alpha(R')$. Now $V(R')$ varies smoothly with R' in $U_{R'}$ $V(R') \notin f(R')$ and $V(R') \in f_\varepsilon(R')$. Hence we need only define $\tilde{f}(R')$ to be the span of $\{V_1(R'), \cdots, V_s(R'), V(R')\}$. □

LEMMA 3.3. *f_ε is a smooth map when restricted to $\mathscr{A}^r$.*

Proof. By *Fact* 2 $\mathscr{A}^r$ is an open subset of $G^r(T_pM)$. By the definition of $\mathscr{A}^r$, $f_\varepsilon(\mathscr{A}^r) = \mathfrak{A}^r \subset G^s(T_pM)$ for some s. Let $R \in \mathscr{A}^r$. Let $f_0\colon \mathscr{A}^r \to G^r(T_p(M)$ be the inclusion. By repeated application of Lemma 3.2 and *Fact* 2 we have open sets U_R^i about R, and smooth maps $f_i\colon U_R^i \to G^{r+i}(T_pM)$ for $0 \leqq i \leqq s-r$, with $\mathscr{A}^r \supseteqq U_R^0 \supseteqq U_R^1 \supseteqq \cdots \supseteqq U_R^{s-r}$ and $f_0(R') \subset f_1(R') \subset \cdots \subset f_{s-r}(R') \subset f_\varepsilon(R;)$ for all $R' \in U^{s-r}$. But for dimension reasons $f_{s-r} = f_\varepsilon$ on U_R^{s-r}. Hence f_ε is smooth in a neighborhood of R. Thus the lemma follows.

LEMMA 3.4. *$\mathfrak{A}^r$ is a submanifold of $G^s(T_pM)$ where $s = d(S)$ for all $S \in \mathfrak{A}^r$.*

Proof. We will show that for each $S \in \mathfrak{A}^r$ there is a coordinate chart U_S about S such that $\mathfrak{A}^r \cap U_S$ is a slice.

Lemma 3.3 tells us that $f_\varepsilon|_{\mathscr{A}^r}$ is smooth. It is not hard to see that on $\mathscr{A}^r$, f_ε has constant rank equal to $r\cdot(n-s)$. Let $S = f_\varepsilon(R)$ for some $R \in \mathscr{A}^r$. A standard result in multivariable calculus tells us that there is an open set 0 around R in $\mathscr{A}^r$ and a coordinate chart U around S in $G^s(T_pM)$ such that $f_\varepsilon(0) \subset U$ is a slice. Let $U_S \subset U$ be so small that for each $S' \in U_S$ there is an $R' \in 0$ such that $R' \subset S'$ (which can be done since 0 is open in $G^r(T_pM)$). Now if $S' \in \mathfrak{A}^r \cap U_S$ then there is an $R' \in 0 \subset \mathscr{A}^r$ such that $R' \subset S'$. Now $f_\varepsilon(R') \subset f_\varepsilon(S') = S'$, so for dimension reasons $f_\varepsilon(R') = S'$. Thus $\mathfrak{A}^r \cap U_S = f_\varepsilon(0) \cap U_S$ is a slice.

LEMMA 3.5. *Let $S^1 \subset S^2 \subset M$ be complete totally geodesic topologically closed submanifolds of M. Assume that S^2 is ε-convex in M. Then S^1 is ε-convex in M if and only if it is ε-convex in S^2.*

REMARK. The lemma is obvious. It is stated here because the corresponding lemma is false for ATC and CC. This is essentially the only point in the proof that cannot be extended to ATC and CC.

PROPOSITION 3.6. *Each connected component of* $\mathfrak{A}^r$ *is a closed submanifold of* $G^s(T_pM)$.

Proof. By Lemma 3.4 we need only show that if $\{S_i\}_{i=0}^{\infty}$ is a sequence in a connected component of $\mathfrak{A}^r$ converging to $S \in G^s(T_pM)$ then $S \in \mathfrak{A}^r$.

By Fact 4 there exists isometries $I_i\colon S_0 \to S_i$ such that $\mathrm{Exp}_p \circ I_i \circ \mathrm{Exp}_p^{-1}|_{\mathrm{Exp}_p(S_0)}$ is an isometry from $\mathrm{Exp}_p(S_0)$ to $\mathrm{Exp}_p(S_i)$. Fix an orthonormal basis for S_0. I_i determines orthonormal bases for S_i. Some subsequence converges to an orthonormal basis for S, defining an isometry I from S_0 to S. Then functions $\mathrm{Exp}_p \circ I_i \circ \mathrm{Exp}_p^{-1}$ converge to $f = \mathrm{Exp}_p \circ I \circ \mathrm{Exp}_p^{-1}$ hence $\mathrm{Exp}_p(S)$ is isometric to $\mathrm{Exp}_p(S_0)$. Let $R \subset S_0$ be an r-dimensional subspace such that $S_0 = f_\varepsilon(R)$. $S \in \mathfrak{A}^r$ will follow if we show that $f_\varepsilon(I(R)) = S$.

We first note that since each S_i is in the image of f_ε so is S by *Fact* 1, hence $S^* \equiv f_\varepsilon(I(R)) \subset S$. Now $\mathrm{Exp}\, S^*$ is ε-convex in M hence (by Lemma 3.5) in $\mathrm{Exp}\, S$. Thus f^{-1} $(\mathrm{Exp}\, S^*)$ is ε-convex in $\mathrm{Exp}\, S_0$ hence in M (Lemma 3.5) so $f_\varepsilon(I^{-1}(S^*)) = I^{-1}(S^*)$. Now, since $R \subset I^{-1}(S^*)$ we have $S_0 = f_\varepsilon(R) \subset f_\varepsilon(I^{-1}(S^*)) = I^{-1}(S^*)$, hence $I^{-1}(S^*) = S_0$ and $S^* = S$.

The theorem now follows from Fact 4 and the following proposition.

PROPOSITION 3.7. $\mathfrak{A}^r$ *is a compact, connected, submanifold of* $G^s(T_pM)$ *such that for every* $R \in G^r(T_pM)$ *there is an* $S \in \mathfrak{A}^r$ *such that* $R \subset S$.

Proof. Proposition 3.6 tells us that each component of $\mathfrak{A}^r$ is a compact submanifold of $G^s(T_pM)$. Consider the bundle $E \xrightarrow{\pi} G^s(T_pM)$ whose fiber at $S \in G^s(T_pM)$ is $\{R \in G^r(T_pM) \mid R \subset S\}$. Let $f\colon E \to G^r(T_pM)$ be the obvious map. Let $\mathscr{A}_0^r$ be a connected component of $\mathscr{A}^r$, and let $\mathfrak{A}_0^r$ be the connected component of $\mathfrak{A}^r$ containing $f_\varepsilon(\mathscr{A}_0^r)$. Let $\tilde{\mathfrak{A}}_0^r \subset E$ be $\pi^{-1}\mathfrak{A}_0^r$. Then $\tilde{\mathfrak{A}}_0^r$ is a compact connected manifold. Let $F = f\,|\mathfrak{A}_0^r$, so $F\colon \tilde{\mathfrak{A}}_0^r \to G^r(T_pM)$. Since for every $R \in \mathscr{A}_0^r$ there is a unique $S \in \mathfrak{A}^r$ such that $R \subset S$, we see that $F^{-1}(R)$ is a single point. Thus since $\mathscr{A}_0^r$ is open in $G^r(T_pM)$ the index modulo 2 of F must be 1. Hence F is onto. Since this must be true for every component of $\mathfrak{A}^r$ and since $R \in \mathscr{A}^r$ implies that there is a unique $S \in \mathfrak{A}^r$

with $R \subset S$, we see that there is only one component of $\mathfrak{A}^r$. The proposition follows. $\square$

IV. Remarks. We will assume that M^n is a compact Riemannian manifold and set ε equal to the injectivity radius.

We define, for $p \in M$ and $1 \leqq r \leqq n$, rank (p, r) to be the dimension of $N_\alpha \in \mathscr{S}^r$, where $\mathscr{S}^r$ is the submanifold system through p defined in § I. We also define the "maximal (p, r) submanifold" to be the elements of $\mathscr{S}^r$.

It is easy to see, that if M is a compact Lie group with bi-invariant metric, rank $(p, 1)$ is the rank of M as a Lie group and the maximal $(p, 1)$ submanifolds are the maximal tori.

The (p, r) ranks and submanifolds of the other examples listed in the introduction are easy to see. However, it is interesting to note that for $\boldsymbol{CP}^n$, rank $(p, r) = 2r$ if $r \neq 1$ but rank $(p, 1) = 1$. In particular the submanifold system of $\boldsymbol{Cp}^1$'s in $\boldsymbol{CP}^n$ does not show up as an $\mathscr{S}^r$.

Let M be a manifold such that for some $p \in M$ rank $(p, 1) = 1$. Then it is easy to see that every geodesic through p is a simply closed curve of the same length. A theorem of Bott thus says that M has the integral cohomology ring of a symmetric space of rank one. If every p in M has rank $(p, 1) = 1$ then M is *a* so called simple C_L manifold (see [1]).

Many of the results in [2] and [3] give rise to results about the (p, r) ranks and maximal submanifolds. For example:

For p and q in M and $1 \leqq r \leqq n-1$ rank $(p, r) \leqq$ rank $(q, r+1)$ (see [2]).

If for some $p \in M$ rank $(p, 2) = 2$ then rank $(p, r) = r$ for all $1 \leqq r \leqq n$. Further if M is simply connected then M is diffeomorphic to S^n (see [3]). If this is true at each point of M then M has constant curvature.

Finally we consider the isometry group of the maximal (p, r) submanifold. Consider $\mathfrak{A}^r \subset G^s(T_pM)$, and the principal bundle $ST^s(T_pM) \xrightarrow{\pi} G^s(T_pM)$ with the canonical connection. Let $\tilde{\mathfrak{A}}^r \xrightarrow{\pi} \mathfrak{A}^r$ be the restricted bundle with the restricted connection. Fact 4, and the remark following it, show that elements of the holonomy group at $S \in \mathfrak{A}^r$ induce nontrivial isometries on $\mathrm{Exp}_p(S)$ leaving p fixed. Thus in particular if the maximal (p, r) submanifold does not admit a one-parameter group of isometries leaving p fixed (for example maximal tori) we see that the connection on $\tilde{\mathfrak{A}}^r \to \mathfrak{A}^r$ must be flat.

References

1. A Bessa, *Manifolds All of Whose Geodesics are Closed*, Ergebnisse der Mathema-

tik, Springer, Berlin-Heidelberg-New York, 1978.
2. C. Croke, *Some new Riemannian invariants*, J. Differential Geometry, **15** (1980), 443–466.
3. ———, *Riemannian manifolds with large invariants*, to appear in J. Differential Geometry.
4. W. Y. Hsiang, *On the classification of compact linear groups with nontrivial principal isotropy subgroups*, (preprint).

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