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TENSOR PRODUCTS FOR SL_2 ($\mathfrak{k}$) II, SUPERCUSPIDAL REPRESENTATIONS

C. ASMUTH AND J. REPKA

Certain pairs of quadratic extension Weil representations of $SL_2(\mathfrak{k})$ have as their tensor product the quaternion Weil representations. This fact is used to develop a method for decomposing tensor products of certain pairs of irreducible supercuspidal representations of $SL_2(\mathfrak{k})$.

1. The object of this paper is to give decompositions of tensor products of certain pairs of supercuspidal representations of $SL_2(\mathfrak{k})$ where $\mathfrak{k}$ is a p-adic field of odd residual characteristic. These tensor products are summands of the quaternion Weil representation. The second section includes preliminaries concerning the quaternion Weil representation and its relation to quadratic extension Weil representations.

The third section sets up the basic mechanism by which the tensor product summands in the quaternion Weil representation are analyzed. It ends with what is the central theorem of the paper. This theorem provides information on decompositions of tensor products in terms of characters of certain multiplicative subgroups of the quaternions.

The fourth section is a catalogue of data on characters of multiplicative subgroups of the quaternions. It is based on [3] and to an extent on [5]. Unifortunately, the work in [5] excludes the cases needed here. For that reason I would like to particularly thank L. Corwin for a manuscript version [4] which includes some specific computations for the quaternion case. The computations in [4] and [5] are similar.

The fifth section gives the decompositions of tensor products explicitly. The main result of §3 and the data in §4 combine to produce the end results.

The sixth section is independent of the others. It gives a (brief) description of how these and other results ([6] and [7]) can be used to give partial results for the tensor products of pairs of supercuspidal representations not covered in the above work. Specifically, we can describe which tensor products contain a continuous part in their decompositions, and give the multiplicities explicitly. We can also give the multiplicities for *some* of the discrete components.

2. Let $\mathfrak{k}$ be a p-adic field with odd residual characteristic. Let

$\mathfrak{o}$ be the ring of integers and let $\mathfrak{p}$ denote its prime ideal. Let $K = \mathfrak{o}/\mathfrak{p}$ be the residue class field of order q. Let $Q = \{\varepsilon, \pi, \varepsilon\pi\}$. Then for $\theta \in Q$ and $\lambda \in \mathfrak{k}^\times$, we obtain a Weil representation $T(\theta, \lambda)$ of $SL_2(\mathfrak{k})$ with representation space $C_c^\infty(\mathfrak{k}(\sqrt{\theta}))$. Here $C_c^\infty(k(\sqrt{\theta}))$ denotes the space of C-valved compactly supported locally constant functions on $\mathfrak{k}(\sqrt{\theta})$. ($T(\theta, \lambda)$ corresponds to $D(\Phi_\lambda, \mathfrak{k}(\sqrt{\theta}))$ in [10] where Φ is some fixed character of $\mathfrak{k}^+$ and $\Phi_\lambda(x) = \Phi(x\lambda)$.)

Let ν_θ be the norm map of $\mathfrak{k}(\sqrt{\theta})$ over $\mathfrak{k}$. Let $C^\theta = \{\alpha \in \mathfrak{k}(\sqrt{\theta})$: $\nu_\theta(\alpha) = 1\}$. From this group we get the decomposition

$$T(\theta, \lambda) = \coprod_{\psi \in \hat{C}^\theta} T(\theta, \lambda, \psi) \qquad ([10]) .$$

Each $T(\theta, \lambda, \psi)$ has the representation space $C_c^\infty(\theta, \psi) = \{f \in C_c^\infty(\mathfrak{k}(\sqrt{\theta}))$: $\forall \alpha \in C^\theta, f(z\alpha) = f(z)\psi(\alpha)\}$.

Let D denote the division algebra of quaternions over $\mathfrak{k}$. Let P denote the prime ideal of its ring of integers. For $\lambda \in \mathfrak{k}^\times$ we have a Weil representation $T(D, \lambda)$ of $G = SL_2(\mathfrak{k})$ in $C_c^\infty(D)$. An explicit formula for $T(D, \lambda)$ is found in [9]. While all choices of λ give equivalent representations, we retain this parameter in order to easily express the quaternion Weil representation as a tensor product.

Let ν_D be the reduced norm of D over $\mathfrak{k}$. Set $\Gamma = \{\gamma \in D$: $\nu_D(\gamma) = 1\}$. For $U \in \hat{\Gamma}$, let Ω_U denote its character. Then the space $C_c^\infty(D, U) = \left\{f \in C_c^\infty(D): \int_\Gamma f(z\gamma)\overline{\Omega_U(\gamma)}d\gamma = f(z)\right\}$ is invariant and we have the corresponding decomposition

$$T(D, \lambda) = \coprod_{U \in \hat{\Gamma}} T(D, \lambda, U) .$$

We will need to choose specific imbeddings of the various $\mathfrak{k}(\sqrt{\theta})$ in D for $\theta \in Q$. A basis for D over $\mathfrak{k}$ can be given by the set $\{1, i, j, k\}$ where $i^2 = \varepsilon$, $j^2 = \pi$, and $ij = -ji = k$. We choose imbeddings of $\mathfrak{k}(\sqrt{\varepsilon})$ and $\mathfrak{k}(\sqrt{\pi})$ in D to consist respectively of elements of the form $a + bi$ and $a + bj$. If -1 is a square in $\mathfrak{k}$, we may imbed $\mathfrak{k}(\sqrt{\varepsilon\pi})$ in D as elements of the form $a + bk$. If not, let ζ be a fixed primitive $q^2 - 1$ root of unity in $\mathfrak{k}(\sqrt{\varepsilon}) \subset D$. Then choose $\mathfrak{k}(\sqrt{\varepsilon\pi}) \subset D$ to consist of elements of the form $a + b\zeta j$. We shall refer to these imbeddings as primary imbeddings of the various $\mathfrak{k}(\sqrt{\theta})$ in D.

Let $t(\theta) \in D$ be given by $t(\varepsilon) = j$ and $t(\pi) = t(\varepsilon\pi) = i$. This allows us to identify $\mathfrak{k}(\sqrt{\theta}) \oplus \mathfrak{k}(\sqrt{\theta})$ with D by the map $(u, v) \mapsto u + vt(\theta)$. Using this identification along with formulae from [9] we have the following.

PROPOSITION 2.1. *For* $0 \neq \lambda \in \mathfrak{k}$ *and* $\theta \in Q$, *there exists a unique*

$\lambda' \in \mathfrak{k}$ *such that* $T(\theta, \lambda) \otimes T(\theta, \lambda') \simeq T(D, \lambda)$ *where* $(f \otimes g)(u + vt(\theta)) = f(u)g(v)$.

Thus it will be tensor products of the form $T(\theta, \lambda, \psi_1) \otimes T(\theta, \lambda', \psi_2)$ which we analyze in this paper.

Let $U \in \hat{\Gamma}$ be a nontrivial representation. Then $T(D, \lambda, U)$ consists entirely of supercuspidal summands. Let U correspond to $\phi \in \hat{C}^{\delta}$ as given by [3]. Let σ_δ be the nontrivial character on the units in the integers of $\mathfrak{k}(\sqrt{\delta})$ whose kernel is the squares. Then by [1] we have, considering σ_δ restricted to C^δ, $T(D, \lambda, U) \simeq [\deg. U]^2[T(\delta, \lambda, \phi\sigma_\delta) + T(\delta, \lambda^*, \phi\sigma_\delta)]$ where $\lambda^*/\lambda \notin \nu_\delta(\mathfrak{k}(\sqrt{\delta}))$. Here, as in [1], we assume $\phi^2 \not\equiv 1$ if $\delta \neq \varepsilon$.

For $x \in \{1, \varepsilon, \pi, \varepsilon\pi\}$, let $H^\times = \{f \in C_c^\infty(D): f(z) \neq 0 \Rightarrow z \in \sqrt{x}\,\Gamma(\mathfrak{k}^\times)^2\}$. Let $H^x(U) = H^x \cap C_c^\infty(D, U)$. From [1] we see that $H^1(U) \oplus H^\delta(U)$ under $T(D, \lambda, U)$ is isomorphic to $[\deg U]^2 C_c^\infty(\delta, \phi\sigma_\theta)$ under $T(\delta, \lambda, \phi\sigma_\theta)$. Let $\{\delta_1, \delta_2\} = Q - \{\delta\}$. Then the action of $T(D, \lambda, U)$ on $H^{\delta_1}(U) \oplus H^{\delta_2}(U)$ is equivalent to $[\deg U]^2$ copies of $T(\delta, \lambda^*, \phi\sigma_\delta)$. If $U^2 \equiv 1$ and U is nontrivial then each $H^\times(U)$ is G-invariant and distinct. Let $r_z: C_c^\infty(D) \to C_c^\infty(z\Gamma)$ for $z \in D$ be the restriction map. The following lemma is easily derived from the facts in [10] and [1].

LEMMA 2.2. *Let W be a G-invariant subspace of $C_c^\infty(D)$. Let V be an irreducible supercuspidal summand meeting $H^\delta(U)$. If $z \in \sqrt{x}\,\Gamma(\mathfrak{k}^\times)^2$ then the multiplicity of summands of type V in W which also meet $H^x(U)$ is equal to the dimension of $r_z(W \cap H^\times(U))$.*

Let $H(\psi_1, \psi_2) \subseteq C_c^\infty(D)$ be the subspace identified with $C_c^\infty(\theta, \psi_1) \otimes C_c^\infty(\theta, \psi_2)$ as prescribed in Proposition 2.1. Let$H_z(\psi_1, \psi_2) = r_z(H(\psi_1, \psi_2))$. Let $C_c^\infty(z\Gamma, U) = r_z(C_c^\infty(D, U))$. Lemma 2.2 says that our main object should be to compute the dimensions of the spaces $H_z(\psi_1, \psi_2) \cap C_c^\infty(z\Gamma, U)$.

3. Let $B = (\Gamma \times \Gamma)/\{\pm(1, 1)\}$. We define a map $\Phi: B \to z\Gamma$ given by $(\delta, \gamma) \to \delta^{-1}z\gamma$. Φ is clearly a well defined surjection. It follows that the map Φ^* given by $\Phi^* f = f \circ \Phi$ is an injection of $C_c^\infty(z\Gamma)$ into $C_c^\infty(B)$.

In what follows, if M is a group and L is a subgroup with a one-dimensional representation T, we denote by $I_R(M, T)$ the set $\{f \in C_c^\infty(M): \forall x \in L, f(xz) = T(x)f(z)\}$. We also set $I_L(M, T) = \{f \in C_c^\infty(M): \forall x \in L, f(zx) = T(x)^{-1}f(z)\}$. If U is any irreducible representation of M, we set $R(M, U)$ and $L(M, U)$ to be right and left regular representation subspaces of type U respectively.

Let $W(z)$ be the trivial representation of the group $\{(z\gamma z^{-1}, \gamma): \gamma \in \Gamma\}/\{\pm(1, 1)\}$. Then we have

PROPOSITION 3.1. $\Phi^*(C_c^\infty(z\Gamma)) = I_R(B, W(z))$. *We think of* $\hat{B}$ *as* $\{U \times V \in (\Gamma \times \Gamma)\hat{}: U(-1) = V(-1)\}$.

PROPOSITION 3.2. *Let* $U^z(\gamma) = U(z\gamma z^{-1})$.
Then we have the decomposition

$$I_R(B, W(z)) = \coprod_{U \in \hat{\Gamma}} [I_R(B, W(z)) \cap R(B, \bar{U}^z \otimes U)] .$$

Each summand is irreduble under the right regular representation action of B.

Proof. This is a simple consequence of group character computations.

Let ζ be the fixed $q^2 - 1$ root of unity of §2. For $\theta \in \{\pi, \varepsilon\}$ we define

$$\omega(\theta) = \begin{cases} \zeta & \text{if} \quad \theta = \varepsilon \\ j & \text{if} \quad \theta = \pi . \end{cases}$$

Since for our purposes of computation, π and $\varepsilon\pi$ are interchangeable we discuss only $\theta = \pi$ and ε.

Let $B^\theta = \langle \Gamma \times \Gamma, (\omega(\theta), \omega(\theta))\rangle / \langle (x, x): x \in \mathfrak{f}\rangle$. Let $\widetilde{\Phi}$ be the obvious well defined extension of Φ to B^θ. Notice that $[B^\theta: B] = 2$. As before, we construct the dual map $\widetilde{\Phi}^*: C_c^\infty(z\Gamma) \to C_c^\infty(B^\theta)$. Let $\widetilde{W}(z)$ be the trivial representation of the subgroup of B^θ given by the elements of the form $(z\gamma z^{-1}, \gamma)$ for $\gamma \in D$.

Let $s: C_c^\infty(B^\theta) \to C_c^\infty(B)$ be the restriction map.

PROPOSITION 3.3. *The following is a commuting diagram of bijections.*

$$\begin{tikzcd} & I_R(B^\theta, \widetilde{W}(z)) \arrow[d, "s"] \\ C_c^\infty(z\Gamma) \arrow[ur, "\widetilde{\Phi}^*"] \arrow[r, "\Phi^*"] & I_R(B, W(z)) \end{tikzcd}$$

Proof. $\widetilde{\Phi}^*$ is an injection since $\widetilde{\Phi}$ is a surjection. $\widetilde{\Phi}^*$ and $\widetilde{\Phi}$ are both clearly surjections by the definition of induced representation. If $p: B \to B^\theta$ is the inclusion, then $\widetilde{\Phi} = \Phi \circ p$, hence the diagram commutes. Therefore s is also a bijection.

Now set $Z^\theta \subset B^\theta$ to be the subgroup of B^θ generated by images of $C^\theta \times C^\theta$ (considered as a subgroup of $\Gamma \times \Gamma$ given by the primary imbedding of $\mathfrak{f}(\sqrt{\theta})$) and $(\omega(\theta), \omega(\theta))$. Since $\omega(\theta)$ centralizes C^θ, we may express irreducible characters χ of Z^θ as triples (ρ_1, ρ_2, w) where $\rho_1(-1)\rho_2(-1) = 1$, χ restricted to $C^\theta \times C^\theta$ is $\rho_1 \otimes \rho_2$, and $\chi(\omega(\theta), \omega(\theta)) = w \in \boldsymbol{C}$.

PROPOSITION 3.4. *Let* $\chi = (\psi_1\psi_2,\ \bar{\psi}_1\psi_2,\ \bar{\psi}_2(\omega(\theta)^{-1}\omega(\bar{\theta})))$. *Then* $\tilde{\Phi}^*(H_z(\psi_1, \psi_2)) = I_L(B^\theta, \chi)$.

Proof. We need to pull right translation by elements in Z^θ back to $C_c^\infty(z\Gamma)$ and see what happens. A simple calculation gives $(f \otimes g)(\alpha^{-1}(u + vt(\theta))\beta) = (f \otimes g)(u + vt(\theta))\bar{\psi}_1\bar{\psi}_2(\alpha)\psi_1\bar{\psi}_2(\beta)$ for $(\alpha, \beta) \in C^\theta \times C^\theta$ and $f \otimes g \in C_c^\infty(\theta, \psi_1) \times C_c^\infty(\theta, \psi_2)$. Similarly $(f \otimes g)(\omega(\theta)^{-1}(u + vt(\theta))\omega(\theta)) = (f \otimes g)(u + vt(\theta))\psi_2(\omega(\theta)^{-1}\bar{\omega}(\theta))$. The definition of induced representation gives containment. Equality follows from the fact that $\psi_1 \otimes \psi_2 \to (\psi_1\psi_2,\ \bar{\psi}_1\psi_2,\ \psi_2(\omega(\theta)^{-1}\bar{\omega}(\theta)))$ is one to one.

PROPOSITION 3.5. $\tilde{\Phi}^*(H_z(\psi_1, \psi_2)) = \coprod_{M \in \hat{B}^\theta} [R(B^\theta, M) \cap I_R(B^\theta, \tilde{W}(z)) \cap I_L(B^\theta, \chi)]$ *with* χ *as in Proposition* 3.4. *The dimension of each summand is the multiplicity of* χ *in* $\bar{M}|_{Z^\theta}$ *where* $R(B^\theta, M) \cap I_R(B^\theta, \tilde{W}(z)) \neq \{0\}$.

Proof. Clearly, by Propositions 3.3 and 3.4,

$$\tilde{\Phi}^*(H_z(\psi_1, \psi_2)) = [I_R(B^\theta, \tilde{W}(z)) \cap I_L(B^\theta, \chi)] \cap \coprod_{M \in \hat{B}^\theta} R(B^\theta, M) .$$

We need to see that intersection commutes with direct sum in this case. We may write $I_R(B^\theta, \tilde{W}(z)) = \coprod_{M \in \hat{B}^\theta} I_R(B^\theta, \tilde{W}(z)) \cap R(B^\theta, M)$. Similarly $I_L(B^\theta, \chi) = \coprod_{M \in \hat{B}^\theta} I_L(B^\theta, \chi) \cap L(B^\theta, \bar{M}) = \coprod_{M \in \hat{B}^\theta} I_L(B^\theta, \chi) \cap R(B^\theta, M)$. This is what was needed.

By Proposition 3.3 we see that the multiplicity of summands of type M in $I_R(B^\theta, \tilde{W}(z))$ is either one or zero since the same is true for $I_R(B, W(z))$. (Proposition 3.2) We now look at the somewhat elaborated expression for $\tilde{\Phi}^*(H_z(\psi_1, \psi_2))$,

$$\coprod_{M \in \hat{B}^\theta} [I_R(B^\theta, \tilde{W}(z)) \cap R(B^\theta, M)] \cap [I_L(B^\theta, \chi) \cap L(B^\theta, \bar{M})] .$$

For M occuring in $I_R(B^\theta, \tilde{W}(z))$, the left hand expression in square brackets is an irreducible right regular subspace of type M of $C_c^\infty(B^\theta)$. The right hand half is a sum of left regular subspaces of type $\bar{M}$. This fact gives the rest of the proposition immediately, since irreducible spaces of these two sorts have intersections of dimension one.

4. Two make use of Proposition 3.5, we need to determine the characters on Z^θ of representations of B^θ. Since $[B^\theta : B] = 2$, we know that a representation of B^θ is either an extension or is induced from a representation of B.

From [3], we obtain a parametrization of $\hat{\Gamma}$. When $-1 \notin (\mathfrak{k}^\times)^2$, we choose secondary imbeddings of $\mathfrak{k}(\sqrt{\pi})$ and $\mathfrak{k}(\sqrt{\varepsilon\pi})$ in D. Let

$\xi = \zeta^{q-1}$. Then these imbeddings are given by elements of the form $a + b\xi j$ and $a + b\xi\zeta j$ respectively. The norm one subgroups in these imbeddings are denoted $C^{\pi'}$ and $C^{\varepsilon\pi'}$ respectively. C^{π} and $C^{\pi'}$ are conjugate in D but not in Γ. Thus when -1 is not a square, we have a set of norm one subgroups C^{δ} where $\delta \in Q' = Q \cup \{\pi', \varepsilon\pi'\}$. Elements of $\hat{\Gamma}$ are expressed as $U = U(\delta, \phi)$ where $\delta \in Q$ or Q' as appropriate, $\phi \in \hat{C}^{\delta}$ and we require that $\phi^2 \not\equiv 1$ if $\delta \neq \varepsilon$. The only equivalences occur when $-1 \notin (\mathfrak{k}^{\pi})^2$ and $\delta \neq \varepsilon$ where $U(\delta, \phi) \cong U(\delta, \bar{\phi})$.

The following information on characters is derived directly from [4]. The methods in [5] can be extended to produce the same results. Let $U = U(\delta, \psi)$ as above. We denote by $\mathscr{E}(U, \theta)$ the character of U restricted to C^{θ}. Let D^{θ} be $\langle \Gamma, \omega(\theta)\rangle$ if $\theta = \varepsilon$ and $\langle \Gamma, \omega(\theta)\rangle / \langle \omega(\theta)^2\rangle$ if $\theta \neq \varepsilon$. Let E^{θ} be the subgroup of D^{θ} generated by $\omega(\theta)$ and C^{θ}. If $V \in \hat{D}^{\theta}$, we let $\tilde{\mathscr{E}}(V, \theta)$ denote the character of its restriction to E^{θ}.

Since π may be chosen to be any generator of $\mathfrak{p}$, we lose nothing by considering only the cases $\theta = \varepsilon$ and $\theta = \pi$. Also, in what follows, if X is any multiplicative subgroup of D, we let X_N denote $X \cap (1 + P^N)$ where P is the prime ideal of the integers in D.

LEMMA 4.1. *Let $\theta = \varepsilon$. Let $U = U(\delta, \phi)$ have conductor Γ_N.*
(a) *If $\delta = \varepsilon$ we write $\mathscr{E}(U, \varepsilon) = \sum_{\rho \in \hat{C}^{\varepsilon}} a_{\rho}\rho$ where*

$$a_{\rho} = \begin{cases} 1 & \text{if } \rho = \phi \\ 2 & \text{if conductor } \rho\bar{\phi} = C^{\varepsilon}_{s} \text{ and } 4 \mid (N - 1 - s), \text{ and } s < N . \\ 0 & \text{otherwise} \end{cases}$$

Furthermore if we set $\mathscr{C}(\phi) = \{\rho \in \hat{C}^{\varepsilon} : a_{\rho} = 2\}$, we have the property $\rho \in \mathscr{C}(\phi)$ if and only if $\phi \in \mathscr{C}(\rho)$.

(b) *If $\delta \neq \varepsilon$ then $\mathscr{E}(U, \varepsilon)$ is the sum of all characters of C^{ε} which are trivial on $C^{\varepsilon}_{N} = C^{\varepsilon} \cap (1 + P^N)$ and agree with ϕ on -1.*

We now consider the case $\theta = \pi$.

LEMMA 4.2. *Let $U = U(\varepsilon, \phi)$ have conductor Γ_N. Then $\mathscr{E}(U, \pi)$ is the sum of all characters trivial on C^{π}_{N} which agree with ϕ on -1.*

When $U = U(\delta, \phi)$ and $\delta \neq \varepsilon$, the situation is more complicated. Let Φ be a fixed character of $\mathfrak{k}^{+}$ with conductor $\mathfrak{p}$. Let $S = \{x \in \mathfrak{k}\sqrt{\varepsilon} \subset D : x = 0 \text{ or } x^{q^2-1} = 1\}$. If N is even, we may express elements of Γ_{N-1}/Γ_N in the form $1 + \alpha j^{N-1}$ where $\alpha \in S$. Any $\tau \in (\Gamma_{N-1}/\Gamma_N)^{\wedge}$ can be expressed as $\tau(1 + \alpha j^{N-1}) = \Phi(\text{trace}(\bar{\mu}\alpha))$ for some $\mu \in S$. Here "trace" is taken to be of $\mathfrak{k}(\sqrt{\varepsilon})$ over $\mathfrak{k}$. If $\delta \neq \varepsilon$ and

$\phi \in \hat{C}^\delta$ has conductor C_N, there exists a unique $\tau \in (\Gamma_{N-1}/\Gamma_N)^\wedge$ such that τ and ϕ agree on Γ_{N-1} and τ is centralized by the quadratic extension in D containing C^δ. Let $\mu(\phi)$ denote the corresponding element of S which gives τ. If ϕ is trivial on C^δ_{N-1}, set $\mu(\phi) = 0$.

LEMMA 4.3. *Assume that $\theta = \pi$ and $\delta = \varepsilon\pi$ or $\varepsilon\pi'$. Let $U = U(\delta, \phi)$ have conductor Γ_N. If $\rho \in \hat{C}^\pi$ then $\mu(\rho) \in S \cap \mathfrak{k}$. If we write $\mathscr{E}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} a_\rho \rho$ then*

$$a_\rho = \begin{cases} 1 & \textit{if } \mu(\rho) = 1/2 \textit{ trace } (\alpha^2\mu(\phi)), \qquad \alpha \in C^\varepsilon \cap S \\ 0 & \textit{otherwise}. \end{cases}$$

LEMMA 4.4. *Assume $\theta = \pi$ and $\delta = \pi'$. (Of course $-1 \notin (\mathfrak{k}^\times)^2$ in this case.) Let $U = U(\delta, \phi)$. Then $\mathscr{E}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} b_\rho \rho$ where*

$$b_\rho = \begin{cases} 2 & \textit{if for some } \alpha \in C^\varepsilon \cap S, \qquad \mu(\rho) = 1/2 \textit{ trace } (\alpha^2\mu(\phi)) \\ 0 & \textit{otherwise}. \end{cases}$$

When $\delta = \theta = \pi$, the situation is more complicated. For $\phi \in \hat{C}^\pi$ we will construct a set $\mathscr{C}(\phi)$ as in the case $\theta = \varepsilon$. If s is odd then C^π_s/C^π_{s+1} can be identified with $S \cap \mathfrak{k}$ by expressing elements in the form $1 + \alpha j^s$ (modulo C^π_{s+1}) where $\alpha \in S \cap \mathfrak{k}$. Let $\lambda(s, u)$ be the character on C^π_s/C^π_{s+1} whose value at $1 + \alpha j^s$ is $\Phi(u\alpha)$. Let $\Lambda(s, u, \phi) = \{\rho \in \hat{C}^\pi : \bar{\rho}\phi\lambda(s, u) \equiv 1 \text{ on } C^\pi_s\}$. Let

$$\mathscr{C}(\phi) = \bigcup_{n=1}^{N/2} \bigcup_{0 \neq x \in S \cap \mathfrak{k}} \Lambda(2n-1, -2\varepsilon\mu(\phi)x^2(-1)^{N/2-n}, \phi).$$

LEMMA 4.5. *Let $U = U(\pi, \phi)$. We write $\mathscr{E}(U, \pi)$ as $\sum_{\rho \in \hat{C}^\pi} a_\rho \rho$ where a_ρ has the following values*

(a) *If $-1 \in (\mathfrak{k}^\times)^2$ we have*

$$a_\rho = \begin{cases} 1 & \textit{for } \rho = \phi \\ 2 & \textit{for } \rho \in \mathscr{C}(\phi) \\ 2 & \textit{for } \mu(\rho) = 1/2(\textit{trace } \alpha^2)\mu(\phi), \qquad \alpha \in S \cap C^\varepsilon \\ 0 & \textit{otherwise}. \end{cases}$$

(b) *If $-1 \notin (\mathfrak{k})^{2\times}$ we have*

$$a_\rho = \begin{cases} 1 & \textit{for } \rho = \phi \textit{ or } \rho = \bar{\phi} \\ 2 & \textit{for } \rho \in \mathscr{C}(\phi) \textit{ or } \rho \in \mathscr{C}(\bar{\phi}) \\ 2 & \textit{for } \mu(\rho) = 1/2(\textit{trace } \alpha^2)\mu(\phi), \qquad \alpha^2 \neq -1, \qquad \alpha \in S \cap C^\varepsilon \\ 0 & \textit{otherwise}. \end{cases}$$

As before, we write irreducible representations of B in the form

$U \otimes V$ where U and V are irreducible representations of Γ which agree on -1.

LEMMA 4.6. *An irreducible component of $W(z)$ extends to B^θ if and only if the corresponding representation of $\Gamma \times \Gamma$ extends to $D^\theta \times D^\theta$.*

Proof. This follows from definitions of B^θ and D^θ and Proposition 3.2.

LEMMA 4.7. *If $U \otimes V$ is a component of $W(z)$, there is a unique component of $\widetilde{W}(z)$ whose restriction to B includes $U \otimes V$.*

Proof. This follows from Proposition 3.3.

From [3] we get the following:

LEMMA 4.8. *Let $U = U(\delta, \phi) \in \hat{\Gamma}$. Then U extends to D^θ if and only if $\delta = \theta$ or $\delta = \theta'$. Also U extends to D^θ if and only if $\bar{U}^z$ does as well.*

Now let U be an arbitrary representation of Γ. If U does not extend to D^θ, then $\bar{U}^z \otimes U$ induces irreducibly to B^θ. On the other hand, if U extends to D^θ, then the unique extension of $\bar{U}^z \otimes U$ to B^θ referred to in Lemma 4.7 is a restriction of certain extensions of $\bar{U}^z \times U$ to $D^\theta \times D^\theta$. Let $F^\theta = \langle \omega(\theta)^2, \Gamma \rangle$ so that F^θ is an extension by central elements. $[D^\theta : F^\theta] = 2$. Let $\{U_x\}$ be the set of extensions of U to F^θ where x ranges over some appropriate index set. For any such U_x, let U_{x1} and U_{x2} denote the two extensions to D^θ. We may similarly identify extensions of characters of C^θ to E^θ; since in fact two distinct extensions of U to D^θ differ by a character of E^θ / C^θ, we may denote by $\{\phi_{xn}\}$ the set of extensions of ϕ to E^θ where x ranges over the same index set and $n = 1$ or 2. Under this arrangement, ϕ_{x1} and ϕ_{x2} both agree with the central character of U_x on the center of D^θ.

LEMMA 4.9. *Let $\theta = \varepsilon$ and $U = U(\varepsilon, \phi)$ for any $\phi \in \hat{C}^\varepsilon$. Then we may further order things so that for $m = 1$ or 2 we have*

$$\widetilde{\mathscr{E}}(U_{xm}, \varepsilon) = \phi_{xm} + \sum_{\rho \in \mathscr{E}(\phi)} \rho_{x1} + \rho_{x2} .$$

LEMMA 4.10. *Let $\theta = \pi$ and assume that $-1 \in (\mathfrak{k}^\times)^2$. Let $U = U(\pi, \phi)$ for some $\phi \in \hat{C}^\pi$ with $\phi^2 \not\equiv 1$. If $\mathscr{E}(U, \pi)$ is written $\sum_{\rho \in \hat{C}^\pi} a_\rho \rho$, then*

$$\tilde{\mathscr{E}}(U_{xm}, \varepsilon) = \phi_{xm} + \sum_{a_\rho=2} \rho_{x1} + \rho_{x2} .$$

LEMMA 4.11. *Let $\theta = \pi$ and assume $-1 \notin (\mathfrak{k}^\times)^2$.*
(a) *If $U = U(\pi, \phi)$ and $\mathscr{E}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} a_\rho \rho$ then*

$$\tilde{\mathscr{E}}(U_{xm}, \pi) = \phi_{xm} + (\bar{\phi})_{xm} + \sum_{a_\rho=2} \rho_{x1} + \rho_{x2} .$$

(b) *If $U = U(\pi', \phi)$ and $\tilde{\mathscr{E}}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} b_\rho \rho$ then*

$$\tilde{\mathscr{E}}(U_{xm}, \pi) = \sum_{b_\rho=2} \rho_{x1} + \rho_{x2} .$$

We now have listed sufficient information to give characters on Z^θ for $\theta = \varepsilon$ and π. Let $\mathscr{E}^z(U, \theta)$ be the character of Z^θ corresponding to $\bar{U}^z \otimes U$. We consider the case $\varepsilon = \theta$ first.

PROPOSITION 4.12. *Let $U = U(\varepsilon, \phi)$.*
(a) *If $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup \varepsilon(\mathfrak{k}^\times)^2$ then*

$$\mathscr{E}^z(U, \varepsilon) = (\bar{\phi}, \phi, 1) + \sum_{\rho \in \mathscr{C}(\phi)} \sum_{t=\pm 1} (\bar{\phi}, \rho, t\sqrt{\bar{\phi}\rho(\xi)}) + (\bar{\rho}, \phi, t\sqrt{\bar{\rho}\phi(\xi)}) + 2 \sum_{\rho,\sigma \in \mathscr{C}(\phi)} \sum_{t=\pm 1} (\bar{\rho}, \sigma, t\sqrt{\bar{\rho}\sigma(\xi)}) .$$

(b) *If $\nu_D(z) \notin (\mathfrak{k}^\times)^2 \cup \varepsilon(\mathfrak{k}^\times)^2$ then*

$$\mathscr{E}^z(U, \varepsilon) = (\phi, \phi, \phi(\xi)) + \sum_{\rho \in \mathscr{C}(\phi)} \sum_{t=\pm 1} (\phi, \rho, t\sqrt{\phi\rho(\xi)}) + (\rho, \phi, t\sqrt{\rho\phi(\xi)}) + 2 \sum_{\rho,\sigma \in \mathscr{C}(\phi)} \sum_{t=\pm 1} (\rho, \sigma, t\sqrt{\rho\sigma(\xi)}) .$$

Proof. In case (a), $U^z = U$; in (b) $U^z = \bar{U}$. In either case, $\bar{U}^z \otimes U$ extends to B^θ. Then $\mathscr{E}^z(U, \varepsilon)$ is just the restriction of $\tilde{\mathscr{E}}((\bar{U}_{x1})^z, \varepsilon) \otimes \tilde{\mathscr{E}}(U_{x1}, \varepsilon)$ to Z^θ. In case (a) we must have

$$\tilde{\mathscr{E}}((\bar{U}_{x1})^z, \varepsilon) = \bar{\phi}_{y1} + \sum_{\rho \in \mathscr{C}(\phi)} \rho_{y1} + \rho_{y2}$$

where the product of any ρ_{xn} and ρ_{ym} is trivial on the center of D^θ. Thus if $\bar{\rho}_{yn} \otimes \sigma_{xm}$ is a component of $\tilde{\mathscr{E}}((\bar{U}_{x1})^z, \varepsilon) \otimes \tilde{\mathscr{E}}(U_{x1}, \varepsilon)$, its restriction to Z^θ is given by $(\bar{\rho}, \sigma, \bar{\rho}_{yn}\sigma_{xm}(\zeta))$. But $(\bar{\rho}_{yn}\sigma_{xm}(\zeta))^2 = \bar{\rho}_{yn}\sigma_{xm}(\varepsilon^{-1}\xi) = \bar{\rho}\sigma(\xi)$. Hence the result for (a) follows using Lemma 4.9. The computation for (b) is analogous.

PROPOSITION 4.13. *Assume $\delta \neq \varepsilon$ and that $U = U(\delta, \phi)$ has conductor Γ_N. Then $\mathscr{E}^z(U, \varepsilon)$ is the sum of all characters (ρ, σ, w) which are trivial on $Z^\theta \cap (\Gamma_N \times \Gamma_N)$ and such that $\rho(-1) = \sigma(-1) = \phi(-1)$.*

Proof. Use Lemma 4.1 (b) and the fact that $\bar{U}^z \otimes U$ induces irreducibly to B^θ.

We now consider the case $\theta = \pi$.

PROPOSITION 4.14. *Let $U = U(\pi, \phi)$ and assume*

$$\mathscr{E}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} a_\rho \rho \,.$$

(a) *If $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ and $-1 \in (\mathfrak{k}^\times)^2$, then*

$$\mathscr{E}^z(U, \pi) = (\bar{\phi}, \phi, 1) + \sum_{a_\rho = 2} \sum_{t = \pm 1} (\bar{\phi}, \rho, t) + (\bar{\rho}, \phi, t) + 2 \sum_{a_\rho = 2} \sum_{a_\sigma = 2} \sum_{t = \pm 1} (\bar{\rho}, \sigma, t) \,.$$

(b) *If $\nu_D(z) \notin (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ and $-1 \in (\mathfrak{k}^\times)^2$ then*

$$\mathscr{E}^z(U, \pi) = (\phi, \phi, 1) + \sum_{a_\rho = 2} \sum_{t = \pm 1} (\phi, \rho, t) + (\rho, \phi, t) + 2 \sum_{a_\rho = 2} \sum_{a_\sigma = 2} \sum_{t = 1} (\rho, \sigma, t) \,.$$

(c) *If $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ and $-1 \notin (\mathfrak{k}^\times)^2$ then*

$$\mathscr{E}^z(U, \pi) = (\bar{\phi}, \bar{\phi}, 1) + (\phi, \phi, 1) + (\bar{\phi}, \phi, 1) + (\phi, \bar{\phi}, 1) + \sum_{a_\rho = 2} \sum_{t = \pm 1} [(\bar{\phi}, \rho, t) + (\bar{\rho}, \phi, t)] + 2 \sum_{a_\rho = 2} \sum_{a_\sigma = 2} \sum_{t = \pm 1} (\bar{\rho}, \sigma, t) \,.$$

Proof. In each case $\bar{U}^z \times U$ extends to B^θ. We then use the methods for Proposition 4.12 using data from Lemmas 4.10 and 4.11 as needed.

PROPOSITION 4.15. *Assume that $-1 \notin (\mathfrak{k}^\times)^2$.*

(a) *Let $U = U(\pi, \phi)$ and assume $\nu_D(z) \notin (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$. Then $\bar{U}^z$ is of the form $U(\pi', \phi')$ with $\mu(\phi') = \xi\mu(\phi)$. Let $(U, \pi) = \sum_{\rho \in \hat{C}^\pi} a_\rho \rho$ and $\xi(\bar{U}^z, \pi) = \sum_{\sigma \in \hat{C}^\pi} b_\sigma \sigma$ as in Lemmas 4.5 and 4.4 respectively. In particular*

$$b_\sigma = \begin{cases} 2 & \text{if } \exists \alpha \in C^\varepsilon \cap S \colon \mu(\sigma) = 1/2 \text{ trace } (\alpha^2 \xi \mu(\phi)) \\ 0 & \text{otherwise} \,. \end{cases}$$

Then

$$\mathscr{E}^z(U, \pi) = \sum_{b_\sigma = 2} \sum_{t = \pm 1} (\sigma, \phi, t) + (\sigma, \bar{\phi}, t) + 2 \sum_{b_\sigma = 2} \sum_{a_\sigma = 2} \sum_{t = \pm 1} (\sigma, \rho, t) \,.$$

(b) *Let $U = U(\pi', \phi')$ and assume $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$. Then $\bar{U}^z$ is of the form $U(\pi, \phi)$ where $\mu(\phi) = \xi\mu(\phi')$. Let $\mathscr{E}(U, \pi) = \sum_{\rho \in \hat{C}^\pi} b_\rho \rho$ and $\mathscr{E}(\bar{U}^z, \pi) = \sum_{\sigma \in \hat{C}^\pi} a_\sigma \sigma$. Then*

$$b_\rho = \begin{cases} 2 & \textit{if } \exists \alpha \in C^\varepsilon \cap S\colon \mu(\rho) = 1/2 \ \textit{trace}\ (\alpha^2 \xi \mu(\phi')) \\ 0 & \textit{otherwise}\ , \end{cases}$$

and

$$\begin{aligned} \mathscr{E}^z(U, \pi) = \sum_{b_\rho = 2} \sum_{t=\pm 1} (\phi, \rho, t) + (\bar{\phi}, \rho, t) \\ + 2 \sum_{b_\rho=2} \sum_{a_\sigma=2} \sum_{t=\pm 1} (\sigma, \rho, t)\ . \end{aligned}$$

Proof. In case (a) we may assume $z^{q-1} = z^{-1}\bar{z} = \xi$, the generator of $C^\varepsilon/C_1^\varepsilon$. It is easy to see that $z^{-1}C^\pi z = C^{\pi'}$ so that $\bar{U}^z$ is of the form $U(\pi', \phi')$. We may write $\phi'(\alpha) = \phi(z\alpha z^{-1})$ for $\alpha \in C^{\pi'}$. (Note that here $\bar{U} \cong U$.) Our choice of z forces $\mu(\phi') = \xi\mu(\phi)$. The result in (a) now follows from Lemma 4.11. Case (b) is more or less the reverse of case (a).

PROPOSITION 4.16. *Assume that* $-1 \notin (\mathfrak{k}^\times)^2$, $U = U(\pi', \phi)$, *and* $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ *so that* $\bar{U}^z = U$. *Let*

$$\mathscr{E}^z(U, \pi) = \sum_{t=\pm 1} \sum_{\sigma, \rho \in \hat{C}^\pi} A_{\sigma\rho}(\sigma, \rho, t)\ .$$

Then

$$A_{\sigma\rho} = \begin{cases} 2 & \textit{if for some } \alpha, \beta \in C^\pi \cap S \textit{ we have } \mu(\sigma) = 1/2(\textit{trace } \alpha^2\mu(\phi)) \\ & \textit{and } \mu(\rho) = 1/2(\textit{trace } \beta^2\mu(\phi))\ . \\ 0 & \textit{otherwise} \end{cases}$$

Proof. Here $\bar{U}^z \otimes U$ extends to B^π. The extension of $\mathscr{E}(\bar{U}^z, \pi) \otimes \mathscr{E}(U, \pi)$ to Z^θ is obtained by using Lemmas 4.4 and 4.11(b).

PROPOSITION 4.17. *Let* $U = U(\delta, \phi)$ *where* $\delta \in \{\varepsilon\pi, \varepsilon\pi'\}$. *Then write*

$$\mathscr{E}^z(U, \pi) = \sum_{t=\pm 1} \sum_{\sigma, \rho \in \hat{C}^\pi} A_{\sigma\rho}(\sigma, \rho, t)\ .$$

(a) *If* $-1 \notin (\mathfrak{k}^\times)^2$ *or* $\nu_D(z) \notin (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ *then*

$$A_{\sigma\rho} = \begin{cases} 1 & \textit{if } \mu(\sigma) \textit{ and } \mu(\rho) \in \{1/2 \ \textit{trace}\ (\alpha^2\mu(\phi))\colon \alpha \in C^\varepsilon \cap S\} \\ 0 & \textit{otherwise}\ . \end{cases}$$

(b) *If* $-1 \in (\mathfrak{k}^\times)^2$ *and* $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ *then*

$$A_{\sigma\rho} = \begin{cases} 1 & \textit{if } -\mu(\sigma) \textit{ and } \mu(\rho) \in \{1/2 \ \textit{trace}\ (\alpha^2\mu(\phi))\colon \alpha \in C^\varepsilon \cap S\} \\ 0 & \textit{otherwise}\ . \end{cases}$$

Proof. Clearly $\bar{U}^z \otimes U$ induces irreducibly to B^π in this case.

Therefore $\mathscr{E}^z(U, \pi)$ is induced from $\mathscr{E}(\bar{U}^z, \pi) \otimes \mathscr{E}(U, \pi)$ on $C^\pi \times C^\pi$. Thus $\mathscr{E}^z(U, \pi)$ consists of all characters $(\sigma, \rho, \pm 1)$ such that σ is a component of $\mathscr{E}(\bar{U}^z, \pi)$ and ρ is a component of $\mathscr{E}(U, \pi)$. We lack only the information on $\mathscr{E}(\bar{U}^z, \pi)$. Assume that $\bar{U}^z = U(\delta_z, \phi_z)$. If $-1 \in (\mathfrak{k}^\times)^2$ and $\nu(z) \notin (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ then $\bar{U}^z = U$ and $\phi_z = \phi$. Then the formula in case (a) follows from Lemma 4.3. If $-1 \notin (\mathfrak{k}^x)^2$ and $\nu_D(z) \in (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$, we have $\bar{U}^z = U$ and the result in (a) holds. If $-1 \notin (\mathfrak{k}^\times)^2$ and $\nu_D(z) \notin (\mathfrak{k}^\times)^2 \cup (-\pi)(\mathfrak{k}^\times)^2$ we may take $\delta = \varepsilon\pi$ and $\delta_z = \varepsilon\pi'$. We can also take $\mu(\phi_z) = \xi\mu(\phi)$. Also $\mu(\phi) = \zeta x$ for some $x \in \mathfrak{k}$. One can check that when $-1 \notin (\mathfrak{k}^\times)^2$, $\{\mathrm{tr}\,(\alpha^2\zeta x)\colon \alpha \in C^\varepsilon \cap S\} = \{\mathrm{tr}\,(\alpha^2\xi\zeta x)\colon \alpha \in C^\varepsilon \cap S\}$. This also results in the formula in (a). In case (b) we see that $\bar{U}^z = \bar{U}$. We have $\delta_z = \delta = \varepsilon\pi$ and $\phi_z = \bar{\phi}$ so that $\mu(\phi_z) = -\mu(\phi)$. Therefore (b) holds.

PROPOSITION 4.18. *Assume $U = U(\varepsilon, \phi)$ has conductor Γ_N. Then $\mathscr{E}^z(U, \pi)$ is the sum of all characters (ρ, σ, w) of Z^π which are trivial on $Z^\pi \cap (\Gamma_N \times \Gamma_N)$ and such that $\rho(-1) = \sigma(-1) = \phi(-1)$.*

Proof. This result is exactly analogous to Proposition 4.13.

5. Let $\theta = \varepsilon$ or π. We may write $T(\theta, \lambda, \psi_1) \otimes T(\theta, \lambda', \psi_2)$ in the form

$$\begin{aligned} m_0 T(D, 1, 1) &\oplus \coprod_{\phi \in \hat{C}^\varepsilon} m(\varepsilon, 1, \phi) T(\varepsilon, 1, \phi) \oplus m(\varepsilon, \pi, \phi) T(\varepsilon, \pi, \phi) \\ &\oplus \coprod_{\phi \in \hat{C}^\pi, \phi^2 \not\equiv 1} m(\pi, 1, \phi) T(\pi, 1, \phi) \oplus m(\pi, \varepsilon, \phi) T(\pi, \varepsilon, \phi) \\ &\oplus \coprod_{\phi \in \hat{C}^{\varepsilon\pi}, \phi^2 \not\equiv 1} m(\varepsilon\pi, 1, \phi) T(\varepsilon\pi, 1, \phi) \oplus m(\varepsilon\pi, \varepsilon, \phi) T(\varepsilon\pi, \varepsilon, \phi)\ . \end{aligned}$$

One might well ask if $m(\varepsilon, \lambda, \phi)$ is well defined when ϕ is the unique character of order 2 since $T(\varepsilon, \lambda, \phi)$ is reducible in this case. In fact it is well defined since the corresponding representation $U(\varepsilon, \phi)$ is centralized by all of D. Thus the dimension of $H_z(\psi_1, \psi_2) \cap C_c^\infty(D, U(\varepsilon, \phi))$ is independent of z. More details will emerge in proofs in this section.

THEOREM 5.1. *Let $\theta = \varepsilon$. Set $\lambda = 1$ and $\lambda' = \pi$. Let ψ_1 and $\psi_2 \in \hat{C}^\varepsilon$ have conductors $C^\varepsilon \cap \Gamma_{M_n}$ for $n = 1, 2$. Let $\phi \in \hat{C}^\delta$ have conductor $C^\delta \cap \Gamma_N$.*

(a) *If $\delta \neq \varepsilon$, $\lambda_0 \in \mathfrak{k}^\times$, and $\phi \in \hat{C}^\delta$ with $\phi^2 \not\equiv 1$ then*

$$m(\delta, \lambda_0, \phi) = \begin{cases} 1 & \text{if } N > \max\{M_1, M_2\} \\ 0 & \text{otherwise}\ . \end{cases}$$

(b) *For $\phi \in \hat{C}^\varepsilon$ and $\phi \not\equiv 1$ we have*

$$m(\varepsilon, 1, \phi) = \begin{cases} 2 & \text{if } \phi \in \mathscr{C}(\psi_1\psi_2) \cap \mathscr{C}(\bar{\psi}_1\psi_2) \\ 1 & \text{if } \phi = \psi_1\psi_2 \in \mathscr{C}(\psi_1\bar{\psi}_2) \\ 1 & \text{if } \phi = \psi_1\bar{\psi}_2 \in \mathscr{C}(\psi_1\psi_2) \\ 1 & \text{if } \phi = \psi_1 \text{ and } \psi_2 = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$m(\varepsilon, \pi, \phi) = \begin{cases} 2 & \text{if } \phi \in \mathscr{C}(\bar{\psi}_1\bar{\psi}_2) \cap \mathscr{C}(\psi_1\bar{\psi}_2) \\ 1 & \text{if } \phi = \bar{\psi}_1\bar{\psi}_2 \\ 1 & \text{if } \phi = \psi_1\bar{\psi}_2 \\ 1 & \text{if } \phi = \bar{\psi}_1 \\ 0 & \text{otherwise .} \end{cases}$$

(c) $m_0 = \begin{cases} 1 & \text{if } \psi_1 \equiv \psi_2 \equiv 1 \\ 0 & \text{otherwise.} \end{cases}$

Now let $\theta = \pi$. For $\phi \in \hat{C}^\delta$, $\delta \neq \varepsilon$, set $\mathscr{D}(\phi, x) = \{\rho \in \hat{C}^\pi \colon \exists \alpha \in C^\varepsilon \cap S$ such that $\alpha^2 \neq 1$ and $\mu(\rho) = 1/2 \text{ trace } (\alpha^2 x \mu(\phi))\}$.

THEOREM 5.2. *Let ψ_1 and $\psi_2 \in \hat{C}^\pi$ have conductors $C^\pi \cap \Gamma_{M_n}$ for $n = 1, 2$ and let $\phi \in \hat{C}^\delta$ have conductor $C^\delta \cap \Gamma_N$. Assume also that $-1 \in (\mathfrak{k}^\times)^2$.*

(a) *Let $\delta = \varepsilon$, $\lambda_0 \in \mathfrak{k}^\times$ and $1 \not\equiv \phi \in \hat{C}^\varepsilon$. Then*

$$m(\varepsilon, \lambda_0, \phi) = \begin{cases} 1 & \text{if } N > \max\{M_1, M_2\} \\ 0 & \text{otherwise .} \end{cases}$$

(b) *$\phi \in \hat{C}^\pi$ such that $\phi^2 \not\equiv 1$ we have*

$$m(\pi, 1, \phi) = \begin{cases} 2 & \text{if } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\bar{\psi}) \cup \mathscr{D}(\bar{\phi}, 1) \text{ and } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\phi) \cup \mathscr{D}(\phi, 1) \\ 1 & \text{if } \phi = \psi_1\psi_2 \text{ and } \psi_1\bar{\psi}_2 \in \mathscr{C}(\phi) \cup \mathscr{D}(\phi, 1) \\ 1 & \text{if } \phi = \psi_1\bar{\psi}_2 \text{ and } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\bar{\phi}) \cup \mathscr{D}(\bar{\psi}, 1) \\ 1 & \text{if } \phi = \psi_1 \text{ and } \psi_2 \equiv 1 \\ 0 & \text{otherwise} \end{cases}$$

$$m(\varepsilon, \pi, \phi) = \begin{cases} 2 & \text{if } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\phi) \cup \mathscr{D}(\phi, 1) \text{ and } \psi_1\psi_2 \in \mathscr{C}(\phi) \cup \mathscr{D}(\phi, 1) \\ 1 & \text{if } \phi = \psi_1\bar{\psi}_2 \text{ and } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\phi) \\ 1 & \text{if } \phi = \bar{\psi}_1\bar{\psi}_2 \text{ and } \psi_1\bar{\psi}_2 \in \mathscr{C}(\phi) \\ 1 & \text{if } \phi = \bar{\psi}_2 \text{ and } \psi_1 \equiv 1 \\ 0 & \text{otherwise .} \end{cases}$$

(c) *For $\delta = \varepsilon\pi$ and $\phi \in \hat{C}^{\varepsilon\pi}$ with $\phi^2 \not\equiv 1$ we have*

$$m(\varepsilon\pi, 1, \phi) = \begin{cases} 1 & \textit{if } N > \max\{M_1, M_2\} \\ 1 & \textit{if } \psi_1\psi_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \\ 0 & \textit{otherwise} \end{cases}$$

$$m(\varepsilon\pi, \theta, \phi) = \begin{cases} 1 & \textit{if } N > \max\{M_1, M_2\} \\ 1 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \\ 0 & \textit{otherwise}\,. \end{cases}$$

(d) $m_0 = \begin{cases} 1 & \textit{if } \psi_1 \equiv \psi_2 \equiv 1 \\ 0 & \textit{otherwise.} \end{cases}$

THEOREM 5.3. *Let $\theta = \pi$ with ϕ, ψ_1, ψ_2 as in Theorem 5.2. Assume $-1 \in (\mathfrak{k}^\times)^2$ and set $\lambda = \lambda' = 1$.*

(a) *Let $\delta = \varepsilon$ with $1 \not\equiv \phi \in \hat{C}^\varepsilon$. Let $\lambda_0 = 1$ or π. Then*

$$m(\varepsilon, \lambda_0, \phi) = \begin{cases} 1 & \textit{if } N > \max\{M_1, M_2\} \\ 0 & \textit{otherwise}\,. \end{cases}$$

(b) *For $\phi \in \hat{C}^\pi$, $\delta = \pi$, $\phi^2 \not\equiv 1$, let $\phi_* = \phi\sigma_\pi$. Then*

$$m(\pi, 1, \phi_*) + m(\pi, 1, \bar{\phi}_*) = \begin{cases} 2 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \\ 2 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \cup \mathscr{C}(\phi) \\ 1 & \textit{if } \psi_1\psi_2 \in \{\phi, \bar{\phi}\} \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \cup \mathscr{C}(\phi) \\ 1 & \textit{if } \psi_1\bar{\psi}_2 \in \{\phi, \bar{\phi}\} \textit{ and } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \cup \mathscr{C}(\phi) \\ 1 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \{\phi, \bar{\phi}\} \textit{ and } \psi_2(-1) = 1\,. \\ 0 & \textit{otherwise} \end{cases}$$

$$m(\pi, \varepsilon, \phi_*) + m(\pi, \varepsilon, \bar{\phi}_*) = \begin{cases} 2 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \textit{ and } \\ & \quad \psi_1\bar{\psi}_2 \in \mathscr{C}(\phi) \cup \mathscr{C}(\bar{\phi}) \cup \mathscr{D}(1, \phi) \\ 2 & \textit{if } \psi_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \textit{ and } \\ & \quad \bar{\psi}_1\bar{\psi}_2 \in \mathscr{C}(\phi) \cup \mathscr{C}(\bar{\phi}) \cup \mathscr{D}(1, \phi) \\ 1 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \in \{\phi, \bar{\phi}\} \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \\ 1 & \textit{if } \psi_1\bar{\psi}_2 \in \{\phi, \bar{\phi}\} \textit{ and } \bar{\psi}_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \\ 0 & \textit{otherwise}\,. \end{cases}$$

(c) *Let $\delta = \varepsilon\pi$ and let $\phi \in \hat{C}^{\varepsilon\pi}$, $\phi^2 \not\equiv 1$. Then*

$$m(\varepsilon\pi, 1, \phi) = m(\varepsilon\pi, \varepsilon, \phi) = \begin{cases} 1 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(1, \phi) \\ 1 & \textit{if } \bar{\psi}_1\bar{\psi}_2 \textit{ and } \psi_1\bar{\psi}_2 \in \mathscr{D}(\xi, \phi) \\ 0 & \textit{otherwise}\,. \end{cases}$$

(d) $m_0 = \begin{cases} 1 & \textit{if } \psi_1 \equiv \psi_2 \equiv 1 \\ 0 & \textit{otherwise.} \end{cases}$

Proof of 5.1, 5.2, 5.3. Observe that Φ^* sends $C_c^\infty(z\Gamma, U)$ to

$R(B, \bar{U}^z \otimes U)$. Thus, using the data from the propositions in §4 and the formula in Proposition 3.5 we obtain dimension of the various spaces $H_z(\psi_1, \psi_2) \cap C_c^\infty(z\Gamma, U)$ where $U = U(\delta, \phi)$. By Lemma 2.2, these numbers are in turn the multiplicities in $H(\psi_1, \psi_2)$ of representations $T(\delta, \lambda, \phi\sigma_\delta)$ whose representation space in $C_c^\infty(D, U)$ includes functions nonzero on z. By the remarks at the beginning of this section we need consider only the cases $z = 1$ and $z \notin \mathfrak{f}(\sqrt{\delta})$ in D. These choices will produce $m(\delta, 1, \phi)$ and $m(\delta, \lambda_*, \phi)$ respectively where $\lambda_* \notin \nu_\delta(\mathfrak{f}(\sqrt{\delta}))$.

There are a number of points requiring further comment. First of all, when $\theta = \varepsilon$ and $\phi \in \hat{C}^\varepsilon$ we have the relation $\rho \in \mathscr{C}(\phi)$ if and only if $\phi \in \mathscr{C}(\rho)$. Thus the multiplicities given in 5.1(b) can be expressed neatly using this inversion.

In Theorem 5.3(b) we use sums of the form $m(\pi, x, \phi_*) + m(\pi, x, \bar{\phi}_*)$. This is more expedient because in fact $U(\pi, \phi)$ and $U(\pi, \bar{\phi})$ are identical. The values given here are obtained from considering $U(\pi, \phi)$ and $U(\pi', \phi')$ together. No information is lost since $T(\pi, x, \phi_*)$ and $T(\pi, x, \bar{\phi}_*)$ are equivalent.

Finally, in Theorem 5.2(c) we find $m(\varepsilon\pi, 1, \phi)$ and $m(\varepsilon\pi, \varepsilon, \phi)$ considered together. The corresponding values for z are 1 and ζ. The question comes down to seeing whether $\mathscr{D}(1, \phi) = \mathscr{D}(1, \phi')$ in this case. We may assume $\mu(\phi) = x\zeta$; $\mu(\phi') = x\xi\zeta$ where $x \in \mathfrak{f}$. We then check to see that the sets $\{\text{trace } (\alpha^2\xi^m\zeta) \colon \alpha \in C^\varepsilon \cap S\}$ are equal for $m = 0$ or 1. (Recall that $\zeta^{q-1} = \xi$; ξ generates $C^\varepsilon \cap S$.)

We now consider the remaining question of what happens when $T(\theta, \lambda, \psi)$ is reducible. The case when $\theta = \varepsilon$ and $-1 \in (\mathfrak{f}^x)^2$ is reasonably typical. For simplicity of notation we confine our discussion to this case.

Assume $\psi_1 \in \hat{C}^\varepsilon$ is the unique character of order 2. Then $T(\varepsilon, 1, \psi_1) = T^1(\varepsilon, 1, \psi_1) \oplus T^\varepsilon(\varepsilon, 1, \psi_1)$. In general we may write $T^x(\varepsilon, 1, \psi_1) \otimes T(\varepsilon, \pi, \psi_2)$ as

$$\coprod_{\substack{\lambda=1,\pi \\ y=1,\varepsilon}} m_y^x(\varepsilon, \lambda, \psi_1) T^y(\varepsilon, \lambda, \psi_1) + \coprod m^x(\delta, \lambda, \phi) T(\delta, \lambda, \phi) ,$$

where δ, λ, and ϕ range over the same sets as given in the beginning of this section. Thus, for $\phi \neq \psi_1 \in \hat{C}^\varepsilon$, we have $m^1(\delta, \lambda, \phi) + m^\varepsilon(\delta, \lambda, \phi) = m(\delta, \lambda, \phi)$ and also $m(\varepsilon, \lambda, \psi_1) = \sum m_y^x(\delta, \lambda, \psi_1)$.

THEOREM 5.4. *Here we describe how summands are distributed between the two* $T^x(\varepsilon, 1, \psi_1) \otimes T(\varepsilon, \lambda, \psi_2)$.

(a) *For* $\phi \neq \psi_1$, $m^x(\varepsilon, 1, \phi) = 0$

(b) $m_y^x(\delta, 1, \psi_1) = \delta_{xy} m(\varepsilon, 1, \psi_1)$

(c) $m^x(\delta, x, \phi) = m(\delta, x, \phi)$ *for* $x = 1$ *or* ε, $\delta \neq \varepsilon$, $\phi \in \hat{C}^\delta$

(d) $m^x(\varepsilon, \pi, \phi) = (1/2) m(\varepsilon, \pi, \phi)$ *for* $\phi \in \hat{C}^\varepsilon$.

Proof. First of all, we note that it was valid to exclude m_0 since ψ_1 was not trivial (Theorem 5.2(d)).

The space of $T^x(\varepsilon, 1, \psi_1)$ consists of functions supported on $\sqrt{x}\,\mathfrak{k}C^\varepsilon$. Thus the space of functions in the tensor product is contained in $H^x \oplus H^\pi \oplus H^{\varepsilon\pi}$. (This can be verified by considering what elements are represented by the various 3-dimensional anisotropic quadratic forms over $\mathfrak{k}$.) This is sufficient for parts (a), (b), and (c).

Let $X: C_c^\infty(D) \to C_c^\infty(D)$ be given by $(Xf)(z) = f(zi)$. While X is not an $SL_2(\mathfrak{k})$ isomorphism, it does take invariant subspaces to invariant subspaces. In particular, if $U = U(\varepsilon, \phi)$, we have $U^i = U$ so that X sends $H^\pi(U) \oplus H^{\varepsilon\pi}(U)$ to itself. Moreover, X interchanges the spaces of $T^x(\varepsilon, 1, \psi_1) \otimes T(\varepsilon, \pi, \psi_2)$ for $x = 1$ and ε. Since the number of irreducible G-spaces of $H^\pi(U) \oplus H^{\varepsilon\pi}(U)$ is finite, the number of such components accounted for by each $T^x(\varepsilon, 1, \psi_1) \otimes T(\varepsilon, \pi, \psi_2)$ must be the same. Hence part (d).

6. The above methods apply only to the specific case of the tensor product of two supercuspidals $T(\theta, \lambda, \psi_1)$, $T(\theta, \lambda', \psi_2)$ belonging to the same quadratic extension $\mathfrak{k}(\sqrt{\theta})$ and with λ and λ' related as in Proposition 2.1. It is the purpose of this section to describe what is known about tensor products of other pairs of supercuspidal representations.

First we note that the contragredient of $T(\theta, \lambda, \phi)$ is $T(\theta, -\lambda, \bar{\phi})$, which is the same as $T(\theta, \lambda, \bar{\phi})$ if and only if $-1 \in \nu_\theta(\mathfrak{k}(\sqrt{\theta}))$, which is always true for $\theta = \varepsilon$, and otherwise is true if and only if $-1 \in (\mathfrak{k}^\times)^2$. Then we remark that the principal series representations are all self-contragredient.

We also note (see [8], Corollary 3.4) that the tensor product of two supercuspidals can only contain a direct sum of supercuspidals and (possibly) copies of the special representation and (possibly) continuous direct integrals of principal series representations.

We apply the results of [6] and [7] to find, for example, which such tensor products contain direct integrals of principal series representations. The results depend on whether or not $-1 \in (\mathfrak{k}^\times)^2$; let us therefore consider first the case when $-1 \in (\mathfrak{k}^\times)^2$.

Consider the representation $T(\varepsilon, \lambda_1, \psi)$, $\psi^2 \not\equiv 1$. From [6], Theorem 5, we see that for any principal series representation T_σ, $T_\sigma \otimes T(\varepsilon, \lambda_1, \psi)$ contains 2 copies of $T(\varepsilon, \lambda_1, \phi)$ for all ϕ such that $\sigma(-1)\psi(-1) = \phi(-1)$, plus one copy of $T(\alpha, \lambda, \phi)$, for each $\alpha \neq \varepsilon$ and for all ϕ such that $\phi(-1) = \sigma(-1)\psi(-1)$ and for *each* choice of λ; and if $\sigma(-1)\psi(-1) = -1$, one copy of $T(\varepsilon, \lambda_1, \psi_1)$, where ψ_1 is the square-trivial character and this last representation contains two irreducible components. By the main theorem of [7], and using the above remarks about contragredients, we see that if ψ, $\phi \neq \psi_1$, then

$T(\varepsilon, \lambda_1, \psi) \otimes T(\varepsilon, \lambda_1, \phi)$ contains 2 copies of the direct integral (with respect to Lebesgue measure) of the principal series of appropriate parity, $T(\varepsilon, \lambda_1, \psi) \otimes T(\alpha, \lambda, \phi)$ contains 1 copy of the direct integral of the right parity for either choice of $\alpha \neq \varepsilon$ and either choice of λ, and the tensor product of $T(\varepsilon, \lambda_1, \psi)$ with either irreducible piece of $T(\varepsilon, \lambda_1, \psi_1)$ also contains 1 copy of the appropriate direct integral. On the other hand, if $\lambda_1/\lambda \notin \nu_\varepsilon(\mathfrak{k}(\sqrt{\varepsilon}))$, then $T(\varepsilon, \lambda_1, \psi) \otimes T(\varepsilon, \lambda, \phi)$ does not contain any continuous part (including the case $\phi = \psi_1$).

A similar analysis, using [6], Theorem 6, shows that if $\{\theta_1, \theta\} = \{\pi, \varepsilon\pi\}$ then $T(\theta_1, \lambda_1, \phi_1) \otimes T(\theta, \lambda, \phi)$ contains one copy of the direct integral of the principal series of the appropriate parity, for any λ, ϕ $(\phi_1, \phi \neq \psi_1)$, and $T(\theta, \lambda, \phi_1) \otimes T(\theta, \lambda, \phi)$ contains 2 copies of the appropriate direct integral, while $T(\theta, \lambda_1, \phi_1) \otimes T(\theta, \lambda, \phi)$ does not, if $\lambda_1/\lambda \notin \nu_\theta(\mathfrak{k}(\sqrt{\theta}))$.

If $-1 \notin (\mathfrak{k}^\times)^2$, the results are similar, except that for $\theta \neq \varepsilon$, it is $T(\theta, \lambda, \phi_1) \otimes T(\theta, -\lambda, \phi)$ which contains 2 copies of the direct integral and $T(\theta, \lambda, \phi_1) \otimes T(\theta, \lambda, \phi)$ which does not.

The results involving the square-trivial characters can also be read off without difficulty. Moreover, by [2] and [6], the special representation occurs in any of these tensor products with the same multiplicity (0, 1, or 2) as the direct integral of the even parity principal series.

In addition, similar considerations allow us to compute the multiplicities of certain supercuspidal components of tensor products of pairs of supercuspidal representations. Indeed, if λ, λ' are related as in Proposition 2.1, the results of §5 tell how to decompose $T(\theta, \lambda, \phi) \otimes T(\theta, \lambda', \psi)$ so reciprocity considerations give us the multiplicity of $T(\theta, -\lambda', \bar{\psi})$ in the tensor product of $T(\theta, \lambda, \phi)$ with *any* supercuspidal representation. The calculation is altogether trivial, so we omit the details.

Our results so far are far from complete, but they are substantial. To summarize: In §5 we decomposed *completely* tensor products of the form $T(\theta, \lambda, \phi) \otimes T(\theta, \lambda', \psi)$; then we found the principal series and special constituents of the tensor products of *any* pair of supercuspidal representations; and we have just seen how to calculate *some* of the supercuspidal constituents of any tensor product of two supercuspidals (those belonging to the same θ as either of the factors and, in each case, to the other choice of λ, i.e., $-\lambda'$).

As yet we have no way to treat completely those tensor products not covered in §5. We cannot calculate the multiplicities of those supercuspidals associated to the other θ (or θ's) or to the other choice of λ.

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TEXAS A & M UNIVERSITY
COLLEGE STATION, TX 77843
AND
UNIVERSITY OF TORONTO
TORONTO, CANADA M5S 1A1