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EXTENDING FUNCTIONS FROM PRODUCTS WITH A METRIC FACTOR AND ABSOLUTES

TEODOR C. PRZYMUSIŃSKI

Extendability of continuous functions from products with a metric or a paracompact p-space factor is studied. We introduce and investigate completions mX and pX of a completely regular space X defined as "largest" spaces Y containing X as a dense subspace such that every continuous real-valued function extends continuously from $X \times Z$ over $Y \times Z$ where Z is a metric or a paracompact p-space, respectively. We study the relationship between mX (resp. pX) and the Hewitt realcompactification υX (resp. the Dieudonné completion μX) of X. We show that for normal and countably paracompact spaces $mX = \upsilon X$ and $pX = \mu X$, but neither normality nor countable paracompactness alone suffices. The relationship between completions mX and pX and the absolute EX of X is discussed.

1. Introduction. All spaces are completely regular and all functions and mappings are continuous. Symbols $\boldsymbol{F}, \boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{P}$ denote classes of finite spaces, metrizable spaces, compact spaces and paracompact p-spaces, respectively. We recall that X is a *paracompact p-space* if it is a closed subspace of a product space $M \times C$, where M is metrizable and C is compact or—equivalently—if X is an inverse image of a metrizable space under a perfect mapping. For all undefined notions the reader is referred to [3].

Let X be a subspace of a space Y and let τ be a cardinal number. We recall the definition of *P^{τ}-embedding* of X in Y. Our definition is equivalent to the original definition of this notion involving the extendability of continuous pseudometrics [see [10] for the proof and for more information].

If τ is infinite, then X is P^{τ}-embedded in Y if every mapping $f: X \to B$ of X into a Banach space B of weight τ can be continuously extended over Y. If τ is finite, then X is P^{τ}-embedded in Y if X is C^*-embedded in Y. Moreover, X is *P-embedded* in Y if X is P^{τ}-embedded in Y for every τ. It is known that $P^{\aleph_0}$-embedding is equivalent to C-embedding [4]. The following theorem gives a product-theoretic characterization of P^{τ}-embedding. ($X \subset_{C^*} Y$ means that X is C^*-embedded in Y, etc.)

THEOREM 0 ([8], [10]). *For a subspace X of Y and a cardinal number τ the following are equivalent*:

(i) $X \subset_{P^{\tau}} Y$;

(ii) $X \times C \subset_{C^*} Y \times C$, *for every* $C \in \boldsymbol{C}$ *of weight* τ;

(iii) *there exists a* $C_0 \in \boldsymbol{C}$ *of weight* τ *such that* $X \times C_0 \subset_{C^*} Y \times C_0$;

(iv) $X \times D^\tau \subset_{C^*} Y \times D^\tau$, *where* D *is the discrete two-point space.* □

COROLLARY 0 ([8], [10]). *For a subspace* X *of* Y *the following are equivalent*:

(i) $X \subset_P Y$;

(ii) $X \times C \subset_{C^*} Y \times C$, *for every* $C \in \boldsymbol{C}$;

(iii) *there exists a* $C_0 \in \boldsymbol{C}$ *of weight* $\tau = |X|$ *such that* $X \times C_0 \subset_{C^*} Y \times C_0$;

(iv) $X \times D^\tau \subset_{C^*} Y \times D^\tau$, *where* $\tau = |X|$. □

The above stated results suggest the following definitions. By $\boldsymbol{Z}$ we denote a nonempty class of spaces.

DEFINITION 1. Let X be a subspace of Y. We say that X is *Π_Z-embedded* in Y if $X \times Z \subset_{C^*} Y \times Z$ for every $Z \in \boldsymbol{Z}$; i.e., if every mapping $f: X \times Z \to I$ can be continuously extended over $Y \times Z$, for $Z \in \boldsymbol{Z}$.

DEFINITION 2. We say that a space X is *Π_Z-complete* if there is no space Y containing X as a proper, dense and Π_Z-embedded subspace, i.e., if X is closed in every space containing it as a Π_Z-embedded subspace.

DEFINITION 3. We say that a space Y is a *Π_Z-completion* of X if Y is a Π_Z-complete space containing X as a dense Π_Z-embedded subspace.

The following fact is easy to prove.

BASIC FACT. Every space X has a uniquely determined Π_Z-completion, denoted by $\pi_Z X$, and $\pi_Z X = \{y \in \beta X: X \subset_{\Pi_Z} X \cup \{y\}\} = \bigcap \{Y: X \subset Y \subset \beta X$ and Y is Π_Z-complete). □

It is the aim of this paper to characterize and investigate Π_Z-embedding, Π_Z-complete spaces and Π_Z-completions $\pi_Z X$ for the classes $\boldsymbol{M}$ and $\boldsymbol{P}$ of metric spaces and paracompact p-spaces, respectively. Let us put

$$mX = \pi_M X \quad \text{and} \quad pX = \pi_P X .$$

The table below illustrates the introduced concepts.

Z	Π_Z-embedding	Π_Z-complete spaces	Π_Z-completion $\pi_Z X$
F	C^*-embedding	compact spaces	βX Cech-Stone compactification
$M \cap C$	C-embedding	realcompact spaces	υX Hewitt real-compactification
C	P-embedding	Dieudonné-complete spaces	μX Dieudonné completion
M	?	?	?
P	?	?	?

Clearly, the following inclusions hold for any space X:

$$\begin{array}{ccc} & \pi_F X & \\ & \cup & \\ & \pi_{M \cap C} X & \\ \swarrow\!\!\cup & & \cup\!\!\searrow \\ \pi_C X & & \pi_M X \\ \cup & & \cup \\ & \pi_P X & \\ & \cup & \\ & X & \end{array} \qquad \text{or equivalently:} \qquad \begin{array}{ccc} & \beta X & \\ & \cup & \\ & \upsilon X & \\ \cup & & \cup \\ \mu X & & mX \\ \cup & & \cup \\ & pX & \\ & \cup & \\ & X & \end{array}$$

It follows from well-known facts and the results proved in this paper that if measurable cardinals exist then all inclusions in the above diagram are in general proper and no other inclusions are generally valid. On the other hand, if the nonexistence of measurable cardinals is assumed, then the above diagram can be simplified as follows:

$$\begin{array}{c} \beta X \\ \cup \\ \upsilon X = \mu X \\ \cup \\ mX = pX \\ \cup \\ X \end{array}$$

REMARK 1. It is pointless to investigate Π_Z-completions for too broad classes of spaces. For example, if the class Z contains all spaces with one non-isolated point (in particular, if it contains all paracompact spaces), then $\pi_Z X = X$ for every X (cf. [7]; Theorem 5.2). $\square$

This paper consists of four sections. In §2 we present characterizations of Π_M- and Π_P-embeddings, Π_M- and Π_P-complete spaces and Π_M- and Π_P-completions mX and pX. In §3 we give an example of a normal space X such that $mX \neq \upsilon X$ and $pX \neq \mu X$. Section 4 is devoted to a discussion of the relationship existing between the above introduced concepts and absolutes of topological spaces. Several problems are raised.

2. Characterization theorems. Theorems 1 and 2 below give characterizations of Π_M- and Π_P-embeddings (for dense subsets X of Y). By $J(\tau)$ we denote the *hedgehog with τ spikes* (see [3], Example 4.1.5). A set A is *regularly open* (*regularly closed*) if $A = \operatorname{Int} \bar{A}$ ($A = \overline{\operatorname{Int} A}$).

THEOREM 1. *For a dense subspace X of Y the following are equivalent*:

(i) $X \subset_{\Pi_M} Y$;

(ii) $X \times Z \subset_{C^*} Y \times Z$, *for every first countable* Z;

(iii) $X \times J(\tau) \subset_{C^*} Y \times J(\tau)$, *where* $\tau = |X|$;

(iv) $X \subset_{C^*} Y$ *and every regularly open increasing cover* $\{U_n\}_{n<\omega}$ *of X can be extended over Y.*

Proof. Implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are obvious.

(iii) $\Rightarrow$ (iv). Let $\{U_n\}_{n<\omega}$ be an increasing regularly open cover of X and for every n let F_n denote a closed set in X such that $U_n = \operatorname{Int} F_n$. Since X is completely regular there exist families $\boldsymbol{W}_n = \{W_{n,\alpha}\}_{\alpha<\tau}$ and $\boldsymbol{F}_n = \{F_{n,\alpha}\}_{\alpha<\tau}$ of cozero and zero sets, respectively, such that $F_n \subset F_{n,\alpha} \subset W_{n,\alpha}$ and $\bigcap_{\alpha<\tau} W_{n,\alpha} = F_n$. (Notice, that if the sets F_n are zero sets, then we can require that the families $\boldsymbol{W}_n$ and $\boldsymbol{F}_n$ be countable.) For every $n < \omega$ and $\alpha < \tau$ let $f_{n,\alpha}: X \to I$ be such that $f_{n,\alpha} | F_{n,\alpha} \equiv 0$ and $f_{n,\alpha} | (X \setminus W_{n,\alpha}) \equiv 1$. Represent $J(\tau)$ as the set $\{(t, \alpha): t \in I, \alpha < \tau\}$ with points $\{(0, \alpha): \alpha < \tau\}$ identified to a point θ and define a mapping $f: X \times J(\tau) \to I$ as follows. If $t = 0$ then we put $f(x, (t, \alpha)) = 0$. If $t \in (0, 1]$, then we can find an integer $n = 1, 2, \cdots$ such that $t \in [(1/n + 1), (1/n)]$. There exists a unique $s \in [0, 1]$ such that $t = s(1/(n + 1)) + (1 - s)(1/n)$. Define for each $x \in X$ and $\alpha < \tau$, $f(x, (t, \alpha)) = s.\ f_{n+1,\alpha}(x) + (1 - s) \cdot f_{n,\alpha}(x)$. Note that if $t = 1/n$ for some integer n, the two possible values for $f(x, (t, \alpha))$ given by the above formula agree. Thus f is well defined and is obviously continuous except perhaps at points of the form (x, θ). We now verify the continuity of f at such points. Let $x \in X$.

There exists an n such that $x \in U_n$. Therefore, $x \in U_n \subset U_k \subset F_k \subset F_{k,\alpha}$ for $k \geqq n$ and $\alpha < \tau$ and $f | U_n \times B_n \equiv 0$, where $B_n = \{(t, \alpha) \in J(\tau): t < 1/n\}$.

By (iii) there exists a continuous extension $\tilde{f}: Y \times J(\tau) \to I$. To prove (iv) it is enough to check that $\bigcap_{n<\omega} \overline{X \setminus U_n}^Y = \emptyset$. Let $y_0 \in Y$. Then $\tilde{f}(y_0, 0) = 0$ and there exists a neighborhood W of y_0 in Y and $n \geqq 1$ such that $\tilde{f}(W \times B_{n-1}) \subset [0, 1)$. We shall show that $W \cap X \subset U_n$. Suppose otherwise. Then $(W \cap X) \setminus U_n \neq \emptyset$ and thus $(W \cap X) \setminus F_n \neq \emptyset$. Choose $x_0 \in (W \cap X) \setminus F_n$ and $\alpha < \tau$ such that $x_0 \in X \setminus W_{n,\alpha}$. Then $f(x_0, ((1/n), \alpha)) = f_{n,\alpha}(x_0) = 1$, but $x_0 \in W$ and $(1/n, \alpha) \in B_{n-1}$. Contradiction.

(iv) $\Rightarrow$ (ii). Let Z be a arbitrary first countable space and let $f: X \times Z \to I$. For every $y \in Y$ and $z \in Z$ put $\tilde{f}(y, z) = \tilde{f}_z(y)$, where $\tilde{f}_z$ is the continuous extension over Y of the function $f_z: X \to I$ defined by $f_z(x) = f(x, z)$. We shall show that the mapping $\tilde{f}: Y \times Z \to I$ is continuous. Let $y_0 \in Y$, $z_0 \in Z$, $\varepsilon > 0$ and $\tilde{f}(y_0, z_0) = s_0$. There exists a neighborhood U of the point y_0 in Y such that $\tilde{f}(U \times \{z_0\}) \subset (s_0 - (\varepsilon/2), s_0 + (\varepsilon/2))$. Let K be a zero subeset of Y such that $Y \setminus U \subset \operatorname{Int} K$ and $y_0 \notin K$ and let A be a dense subset of Z. For every $a \in A$ put

$$K_a = \left\{x \in X: \left| f(x, a) - s_0 \right| \leqq \frac{\varepsilon}{2}\right\}$$

and for every $n < \omega$ define

$$F_n = (K \cap X) \cup \bigcap \{K_a: a \in A \cap B_n\} ,$$

where $\{B_n\}_{n<\omega}$ is a decreasing neighborhood base at z_0 in Z. The sets F_n are closed in X and nondecreasing. (Notice, that if A is countable, then the sets F_n are zero subsets of X.) The sets $U_n = \operatorname{Int} F_n$ are regularly open and nondecreasing. We shall show first that $X = \bigcup_{n<\omega} U_n$. If $x \in \operatorname{Int} K$, then $x \in U_n$ for every n. Otherwise, $x \in U$ and there exists a neighborhood U_x of x and $n < \omega$ such that

$$f(U_x \times B_n) \subset (s_0 - \varepsilon, s_0 + \varepsilon) .$$

Then $U_x \subset K_a$, for all $a \in A \cap B_n$ and thus $U_x \subset F_n$.

By (iv) there exist open sets $\tilde{U}_n$ in Y such that $\bigcup_{n<\omega} \tilde{U}_n = Y$ and $\tilde{U}_n \cap X = U_n$. Let n be such that $y_0 \in \tilde{U}_n$. Since $y_0 \notin K$ there exists a neighborhood V of y_0 such that $V \cap X \subset U_n \setminus K$. By the continuity of f we have

$$f((V \cap X) \times B_n) \subset \left[s_0 - \frac{\varepsilon}{2}, s_0 + \frac{\varepsilon}{2}\right] ,$$

and therefore by the continuity of functions $\tilde{f}_z$ for $z \in Z$ and the density of $V \cap X$ in V we get

$$\tilde{f}(V \times B_n) \subset \left[s_0 - \frac{\varepsilon}{2}, s_0 + \frac{\varepsilon}{2}\right] \subset (s_0 - \varepsilon, s_0 + \varepsilon) . \qquad \square$$

The following variant of Theorem 1 will be used in §3.

THEOREM 1*. *For a dense subspace X of Y the following are equivalent*:

(i) $X \times M \subset_{C^*} Y \times M$, *for every separable* $M \in \boldsymbol{M}$;

(ii) $X \times Z \subset_{C^*} Y \times Z$, *for every separable first countable space* Z;

(iii) *there exists a non-locally compact metric space* M_0 *such that* $X \times M_0 \subset_{C^*} Y \times M_0$;

(iv) $X \subset_{C^*} Y$ *and every increasing open cover* $\{U_n\}_{n<\omega}$ *of* X, *such that* $U_n = \operatorname{Int} F_n$ *for some zero sets* F_n, *can be extended over* Y.

Proof. Implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are obvious. The proof of implications (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (ii) is analogous to the proof of the corresponding implications in Theorem 1 (see the remarks in parentheses). One should only notice that every non-locally compact metric space contains as a closed subspace the subspace $J^*(\omega) = \{(t, \alpha) \in J(\omega): t = 0$ or $t = (1/n)$ for some $n = 1, 2, \cdots\}$ of the hedgehog $J(\omega)$ and use the fact that for any space T, a closed subspace F of a metric space M and any mapping $h: F \times T \to I$ there exists a continuous extension $\tilde{h}: M \times T \to I$ [13]. $\square$

THEOREM 2. *For a dense subspace X of Y the following are equivalent*:

(i) $X \subset_{\Pi P} Y$;

(ii) $X \times Z \subset_{C^*} Y \times Z$, *for every space Z of point-countable type*[1];

(iii) $X \times J(\tau) \times D^\tau \subset_{C^*} Y \times J(\tau) \times D^\tau$, *where* $\tau = |X|$;

(iv) $X \subset_{\Pi_M} Y$ *and* $X \subset_{\Pi_C} Y$.

Proof. The implications (ii) $\Rightarrow$ (i) and (i) $\Rightarrow$ (iii) are obvious. The implication (iii) $\Rightarrow$ (iv) follows from Theorems 0 and 1.

(iv) $\Rightarrow$ (ii) Let Z be a space of point-countable type and $f: X \times Z \to I$. As in the proof of Theorem 1 we define $\tilde{f}: Y \times Z \to I$ by putting $\tilde{f}(y, z) = \tilde{f}_z(y)$. We have to show that $\tilde{f}$ is continuous. Let $y_0 \in Y$, $z_0 \in Z$, $\varepsilon > 0$ and $\tilde{f}(y_0, z_0) = s_0$. Let C be a compact set of countable character in Z containing z_0. By (iv) $\tilde{f} | Y \times C$ is continuous. Let $G = \{(y, z) \in Y \times C: \tilde{f}(y, z) \in (s_0 - (\varepsilon/2), s_0 + (\varepsilon/2)\}$. The set G is open in $Y \times C$ and contains (y_0, z_0). Let us put $L = \{z \in C: \tilde{f}(y_0, z) = s_0\}$.

[1] A space Z is of *point-countable type* if for every $z_0 \in Z$ there exists a compact $F \ni z_0$ of countable character in Z.

The set L is a zero set in C and thus L is of countable character in Z and $z_0 \in L$. Moreover, $\{y_0\} \times L \subset G$ and therefore there exists a neighborhood U of y_0 in Y such that $U \times L \subset G$. Let K be a closed set in Y such that $Y \backslash U \subset \operatorname{Int} K$ and $y_0 \notin K$ and let $\{B_n\}_{n<\omega}$ be a decreasing base of neighborhoods of L in Z. Since f is continuous and L is compact, for every $x \in U \cap X$ there exists a neighborhood U_x and n such that $f(U_x \times B_n) \subset (s_0 - (\varepsilon/2), s_0 + (\varepsilon/2))$. Put $H_n = \{x \in X: f(\{x\} \times B_n)) \subset [s_0 - (\varepsilon/2), s_0 + (\varepsilon/2\}]$. Of course, the sets H_n are closed. Define

$$U_n = \operatorname{Int}(H_n \cup (K \cap X)) .$$

The sets U_n are regularly open, nondecreasing and cover X, hence by (iv) and Theorem 1 there exists an n and an open set $V \ni y_0$ in Y such that $V \cap X \subset U_n$ and $V \cap K = \varnothing$. Then $f((V \cap X) \times B_n) \subset [s_0 - (\varepsilon/2), s_0 + (\varepsilon/2)]$ and consequently $\tilde{f}(V \times B_n) \subset [s_0 - (\varepsilon/2), s_0 + (\varepsilon/2)] \subset (s_0 - \varepsilon, s_0 + \varepsilon)$. □

COROLLARY 1. *For every X we have $pX = mX \cap \mu X$.*

Proof. $pX = \pi_P X = \pi_M X \cap \pi_C X = mX \cap \mu X$. □

COROLLARY 2. *If there are no measurable cardinals, then $pX = mX$ for every X.* □

COROLLARY 3. *For every X the following are equivalent*:
(i) $\mu(X \times M) = \mu X \times M$, *for every* $M \in \boldsymbol{M}$;
(ii) $\mu(X \times P) = \mu X \times P$, *for every* $P \in \boldsymbol{P}$;
(iii) $\mu X = pX$.

Proof. The implication (ii) $\Rightarrow$ (i) is obvious. If (i) holds, then $X \times M$ is C^*-embedded in $\mu X \times M$ for every $M \in \boldsymbol{M}$ and therefore $\mu X \subset mX$ and $pX = \mu X \cap mX = \mu X$.

If (iii) holds, then $X \times P \times P'$ is C^*-embedded in $\mu X \times P \times P'$ for every $P, P' \in \boldsymbol{P}$ which implies that $X \times P$ is P-embedded in $\mu X \times P$. □

The following three corollaries can be easily derived from Theorem 1.

COROLLARY 4. *A point $y \in \beta X$ belongs to mX if and only if for every decreasing sequence $\{F_n\}_{n<\omega}$ of regularly closed subsets of X with empty intersection $y \notin \bigcap_{n<\omega} \bar{F}_n^{\beta X}$.* □

COROLLARY 5. *A space X is Π_M-complete if and only if for* *every* $y \in \beta X \backslash X$ *there exists a decreasing sequence* $\{F_n\}_{n<\omega}$ *of regularly closed subsets of* X *such that* $y \in \bigcap_{n<\omega} \bar{F}_n^{\beta X} \subset \beta X \backslash X$. □

COROLLARY 6. *A normal space* X *is* Π_M*-complete if and only if every closed ultrafilter in* X*, such that every decreasing sequence of its regularly closed elements has a nonempty intersection, converges to a point of* X. □

REMARK 2. Since characterizations of P-embedding, Dieudonné-complete spaces and Dieudonné completions are well known, Corollary 1 and Corollaries 4, 5, and 6 yield immediately characterizations of Π_P-complete spaces and the Π_P-completion pX.

It is easy to verify that the assumption in Theorem 1 that the sequence $\{U_n\}_{n<\omega}$ is increasing is essential. □

In [2] N. Dykes introduced the concept of *c-realcompact spaces* and *c-realcompactification* uX of a space X. Later, these concepts were inverstigated by K. Hardy and R. Woods in [5] and [14], where new characterizations of uX were obtained and the relationship between the c-realcompactification uX and the absolute of X was established. It follows from Corollary 4 and Lemma 1.1 from [5] that the concepts of c-realcompact spaces and Π_M-complete spaces are identical and that $uX = mX$ for every X. (We shall discuss the relationship between completions mX and pX and the absolute of X in § 4). The following two results were known for c-realcompactification uX (see [2] and [5]).

COROLLARY 7. *Suppose that* X *is normal and countably paracompact. Then*:

$$mX = \upsilon X \quad and \quad pX = \mu X\,.$$

In particular, X *is* Π_M*-complete* iff X *is realcompact and* X *is* Π_P*-complete* iff X *is Dieudonné-complete.*

Proof. Always $mX \subset \upsilon X$ and $\mu X \subset pX$. Let $y \in \upsilon X \subset \beta X$ and let $\{F_n\}_{n<\omega}$ be a decreasing sequence of regularly closed subsets of X such that $\bigcap_{n<\omega} F_n = \varnothing$. There exists a sequence $\{K_n\}_{n<\omega}$ of zero subsets of X such that $F_n \subset K_n$ and $\bigcap_{n<\omega} K_n = \varnothing$. Let $f_n\colon X \to I$ be functions such that $f_n^{-1}(0) = K_n$ and let $\tilde{f}_n\colon \beta X \to I$ be continuous extensions. Then the function $\tilde{f} = \sum_{n<\omega} (1/2^n)\tilde{f}_n\colon \beta X \to I$ is continuous and $\tilde{f}^{-1}(0) = \bigcap_{n<\omega} \bar{K}_n^{\beta X} \subset \beta X \backslash X$. Therefore $\tilde{f}^{-1}(0) \cap \upsilon X = \varnothing$ and $y \notin \bigcap_{n<\omega} \bar{F}_n^{\beta X} \subset \bigcap_{n<\omega} \bar{K}_n^{\beta X}$, which in view of Corollary 4 shows that $y \in mX$.

Since $pX = \mu X \cap mX = \mu X \cap \upsilon X$ and $\upsilon X \supset \mu X$, we have $pX = \mu X$. □

COROLLARY 8. *The following are equivalent:*
(i) *X is psedocompact;*
(ii) $mX = \beta X$;
(iii) $pX = \beta X$.

Proof. Implications (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) are obvious. If X is pseudocompact, then clearly $\mu X = \beta X$ and since $pX = mX \cap \mu X$, it suffices to show that $mX = \beta X$, but every decreasing sequence of regularly closed subsets of a pseudocompact space is finite and thus $\beta X \subset mX$ by Corollary 4. □

Let us finish this section with two problems.

PROBLEM 1. Characterize closed Π_M-embedded (Π_P-embedded) subspaces of a space X. Is it true that a closed subset of a space X is Π_P-embedded iff it is Π_M-embedded and Π_C-embedded ($=P$-embedded) in X?

PROBLEM 2. Investigate the Π_L-completion $\pi_L X$ of a space X for the class L of Lindelöf spaces.

3. An example. As yet no example was given of a space X such that $mX \neq \upsilon X$ or $pX \neq \mu X$. In view of Corollary 7, such a space cannot be normal and countably paracompact. It follows from the properties of the example in [6], the identity $uX = mX$ and Theorem 1.11 from [5] that there exists a countably paracompact space X such that $\upsilon X = \mu X \neq mX = pX$. (Moreover, the space X is locally compact and υX is σ-compact.) Below we shall give an example of a normal space with analogous properties, thus showing that normality of X is not sufficient in Corollary 7. Our example will be a modification of M. E. Rudin's example [12].

EXAMPLE 1. There exists a collectionwise normal space X such that the space $\upsilon X = \mu X$ is paracompact and for every metric space M we have:

$$(*) \qquad X \times M \underset{C^*}{\subset} \upsilon X \times M \quad \text{iff } M \text{ is locally compact}\,.$$

In particular, $\upsilon X = \mu X \neq mX = pX$ and for every metric M we have:

$$(**) \qquad \mu(X \times M) = \mu X \times M \quad \text{iff } M \text{ is locally compact}\,.$$

REMARK 3. The existence of a (nonnormal) space satisfying (**) follows from results of H. Ohta [9]. □

LEMMA 1. *It suffices to construct a collectionwise normal space X such that:*

(a) *the space $\upsilon X = \mu X$ is paracompact;*

(b) *there exists an increasing regularly open cover $\{U_n\}_{n<\omega}$ of X which does not have an open locally finite refinement and such that the sets $\bar{U}_n$ are zero subsets of X.*

Proof. Clearly $X \times M \subset_{C^*} \upsilon X \times M$ for every locally compact M (see [1]). Conversely, suppose that $X \times M_0 \subset_{C^*} \upsilon X \times M_0$ for some non-locally compact space M_0. From Theorem 1* it follows that the open cover $\{U_n\}_{n<\omega}$ can be extended over υX and since υX is paracompact, it must have a locally finite open refinement. Contradiction. □

By [12] there exists a collectionwise normal space Y of non-measurable cardinality such that the space $\upsilon Y = \mu Y$ is paracompact and an increasing open cover $\{V_n\}_{n<\omega}$ of Y, which does not have a locally finite open refinement.

Let Z be a closed subspace of the space $Y \times \omega$ (where ω bears the discrete topology) defined by $Z = \bigcup_{n<\omega}(\bar{V}_n \times \{n\})$ and let $W_n = Z \cap (V_n \times \{1, 2, \cdots, n\})$. One easily sees (cf. [11]), that Z is collectionwise normal, the space $\upsilon Z = \mu Z$ is paracomprct, the sets $\bar{W}_n$ are zero subsets of Z and the increasing open cover $\{W_n\}_{n<\omega}$ of Z does not have a locally finite open refinement. Observe, that the sets W_n need not be regularly open.

Now, let $X = Z \times I$, where points $(z, t) \in X$ are isolated if $t \neq 0$ and have a base of standard product neighborhoods if $t = 0$ (cf. [11]). One easily checks that the space X is collectionwise normal, the sets $U_n = W_n \times I$ form a regularly open covering of X with no locally finite open refinement and the sets $\bar{U}_n$ are zero subsets of X. By Lemma 1 it suffices to show that the space $\upsilon X = \mu X$ is paracompact.

Let $T = \{(y, t) \in \upsilon Z \times I: y \in Z \text{ if } t \neq 0\}$ be considered with the topology in which points $(y, t) \in T$ are isolated if $t \neq 0$ and basic neighborhoods of a point $(y, 0) \in T$ are of the form $\pi^{-1}(U)\backslash K$, where U is a neighborhood of y in υZ, $\pi: T \to \upsilon Z$ is the projection and K is a closed subset of $X = Z \times I$ contained in $Z \times (0, 1]$. It is not difficult to verify that the space T is paracompact and contains X as a dense subspace. To show that $\upsilon X = \mu X = T$ it suffices to show that X is C-embedded in T.

Let $f: X \to R$ and let $\tilde{f}: T \to R$ be an extension of f defined by $\tilde{f}(y, 0) = \tilde{g}(y)$ for $y \in \upsilon Z$, where $\tilde{g}: \upsilon Z \to R$ is the extension over υZ of the function $g: Z \to R$ defined by $g(z) = f(z, 0)$. We have to show that $\tilde{f}$ is continuous. Let $y_0 \in \upsilon Z$, $\varepsilon > 0$ and $\tilde{f}(y_0, 0) = s_0$. The set

$K_1 = \{x \in X: |f(x) - s_0| \geqq \varepsilon\}$ is closed in X. Let

$$K_2 = \left\{y \in \upsilon Z: |\tilde{f}(y, 0) - s_0| \leqq \frac{\varepsilon}{2}\right\} \times I .$$

The set $K = K_1 \cap K_2$ is closed in X and contained in $Z \times (0, 1]$. Let $W = \pi^{-1}(\{y \in \upsilon Z: |\tilde{f}(y, 0) - s_0| < \varepsilon/2\})\backslash K$. The set W is an open neighborhood of $(y_0, 0)$ in T and $\tilde{f}(W) \subset (s_0 - \varepsilon, s_0 + \varepsilon)$, which completes the proof. $\square$

4. Relationship with absolutes. For information about absolutes of topological spaces we recommend [15]. Here, we only recall that for every space X there exists a uniquely determined extermally disconnected space EX called the *absolute of* X such that EX can be mapped by a perfect irreducible mapping k_X onto X.

If the space Z is compact, then EZ is the set of all ultrafilters in the Boolean algebra $\boldsymbol{R}(Z)$ of all regularly closed subsets of Z with the topology generated by the base $\{\lambda(F): F \in \boldsymbol{R}(Z)\}$, where $\lambda(F) = \{p \in EZ: F \in p\}$.

The mapping $k_Z: EZ \to Z$ is defined by $k_Z(p) = z$ iff $\{z\} = \cap\, p$. The sets $\lambda(F)$, for $F \in \boldsymbol{R}(Z)$, constitute all clopen subsets of the (compact) space EZ.

In general, EX is the inverse image of X under the mapping $k_{\beta X}: E(\beta X) \to \beta X$ and $k_X = k_{\beta X} | EX$. The space EX is dense in $E(\beta X)$. We put $\lambda^*(K) = \lambda(\bar{K}^{\beta X}) \cap EX$ for all $K \in \boldsymbol{R}(X)$.

It is well-known that $E(\beta X) = \beta(EX)$ for every space X and that always $\upsilon(EX) \subset E(\upsilon X)$ and $\mu(EX) \subset E(\mu X)$. The following result has been proved by Hardy and Woods (we replace everywhere uX by mX). Here k denotes the mapping $k_{\beta X}: E(\beta X) \to \beta X$.

THEOREM 3 [5], [14]. *The following are equivalent*

(i) $\upsilon(EX) = E(\upsilon X)$.

(ii) $\upsilon X = mX$.

More precisely, mX is the largest subspace T of βX such that $k^{-1}(T) \subset \upsilon(EX)$. $\square$

We were unable to establish if the analogous fact holds for μX and pX. However, the following two propositions are true. We denote by sX the largest subspace T of βX such that $k^{-1}(T) \subset \mu(EX)$.

PROPOSITION 1. *A point $y \in \beta X$ belongs to sX if and only if for every locally finite regularly closed cover $\{F_s\}_{s \in S}$ of X there exist $s_1, \cdots, s_n \in S$ such that $y \in \operatorname{Int}_{\beta X} \bigcup_{i=1}^n \bar{F}_{s_i}^{\beta X}$.*

PROPOSITION 2. *Always $sX \subset pX$.*

Before proving Propositions 1 and 2 let us note that $sX = \mu X$ if and only if $\mu(EX) = E(\mu X)$ and thus if $\mu(EX) = E(\mu X)$, then $pX = \mu X$. Two natural problems arise:

PROBLEM 3. Is always $sX = pX$?

PROBLEM 4. Is $pX = \mu X$ equivalent to $\mu(EX) = E(\mu X)$? (Naturally, a positive answer to the first question answers positively the second.)

Proof of Proposition 1. Suppose that $y \in sX$ and let $\{F_s\}_{s \in S}$ be a locally finite regularly closed cover of X. Then the family $\{\lambda^*(F_s): s \in S\}$ is a locally finite clopen cover or EX[2]. Since $k^{-1}(y) \subset \mu(EX)$, there exist indices $s_1, \cdots, s_n \in S$ such that

$$k^{-1}(y) \subset \bigcup_{i=1}^{n} \overline{\lambda^*(F_{s_i})}^{\beta(EX)} .$$

But $\beta(EX) = E(\beta X)$ and therefore $\overline{\lambda^*(F_s)}^{\beta(EX)} = \lambda(\bar{F}_s^{\beta(X)})$ for all $s \in S$. Consequently, $k^{-1}(y) \subset \bigcup_{i=1}^n \lambda(\bar{F}_{s_i}^{\beta X})$ and since k is a closed mapping, there exists an open set $U \in y$ in βX such that $k^{-1}(U) \subset \bigcup_{i=1}^n \lambda(\bar{F}_{s_i}^{\beta X})$. Then $y \in U \subset \bigcup_{i=1}^n \bar{F}_{s_i}^{\beta X}$.

Conversely, suppose that $p \in k^{-1}(y)$ and let $U = \{U_s\}_{s \in S}$ be a locally finite cozero cover of EX. Since EX is extremally disconnected, we can assume that U is pairwise disjoint and we have to show that there exists an $s \in S$ such that $p \in \bar{U}_s^{\beta(EX)}$. Since the sets U_s are clopen, there exist regularly closed sets F_s in X such that $U_s = \lambda^*(F_s)$ and clearly $\bar{U}_s^{\beta(EX)} = \lambda(\bar{F}_s^{\beta X})$. The family $\{F_s\}_{s \in S}$ is a regularly closed cover of X and thus there exist $s_1, \cdots, s_n \in S$ such that $y \in \operatorname{Int}_{\beta X} \bigcup_{i=1}^n \bar{F}_{s_i}^{\beta X}$. Therefore, $\bigcup_{i=1}^n \bar{F}_{s_i}^{\beta X} \in p$ and since p is an ultrafilter, there exists an i such that $\bar{F}_{s_i}^{\beta X} \in p$ which means that $p \in \lambda(\bar{F}_{s_i}^{\beta X}) = \bar{U}_{s_i}^{\beta(EX)}$. $\square$

Proof of Proposition 2. By Theorem 3, $sX \subset mX$ and since $pX = mX \cap \mu X$ it is enough to show that $sX \subset \mu X$. Let $y \in sX$ and let $\{U_s\}_{s \in S}$ be a locally finite cozero cover of X. We have to show that there exists an $s \in S$ such that $y \in \bar{U}_s^{\beta X}$. Let $\{V_s\}_{s \in S}$ be a covering of X such that $V_s \subset \bar{V}_s \subset U_s$ for every $s \in S$. The family $\{\bar{V}_s\}_{s \in S}$ is a locally finite regularly closed cover of X and thus there exist $s_1, \cdots, s_n \in S$ such that $y \in \bigcup_{i=1}^n \bar{V}_{s_i}^{\beta X}$. Therefore, there exists an i such that $y \in \bar{V}_{s_i}^{\beta X} \subset \bar{U}_{s_i}^{\beta X}$. $\square$

Added in proof. Professor H. Ohta proved that a positive answer to Problems 3 and 4 above is equivalent to the non-existence of

[2] To show that this is a cover of EX, we use the local finiteness of $\{F_s\}_{s \in S}$.

measurable cardinals. He also—independently—obtained some of the results in this paper. The interested should consult his paper: (1) The Hewitt real-compactification of products, Trans. AMS 263 (1981), 363–375; (2) Local compactness and Hewitt real-compactifications of products II, to appear; (3) Topological extension properties and projective covers, to appear; and also his Ph. D. Thesis at the University of Tsukuba, 1979.

For new results involving Π_Z-embeddings, the reader is referred to the paper by A. Waśko, Extension of functions defined on product spaces, to appear in Fund. Math.

REFERENCES

1. W. W. Comfort, *On the Hewitt realcompactification of a product space*, Trans. Amer. Math. Soc., **131** (1968), 107–118.

2. N. Dykes, *Generalizations of realcompact spaces*, Pacific J. Math., **33** (1970), 571–581.

3. R. Engelking, *General Topology*, Polish Scientific Publishers, Warsaw. 1977.

4. T. E. Gantner, *Extensions of uniformly continuous pseudometrics*, Trans. Amer. Math. Soc., **132** (1968), 147–157.

5. K. Hardy and R. G. Woods, *On c-realcompact spaces and locally bounded normal functions*, Pacific J. Math., **43** (1972), 647–656.

6. J. E. Mack and D. G. Johnson, *The Dedekind completion of $C(X)$*, Pacific J. Math., **20** (1967), 231–243.

7. W. G. McArthur, *Hewitt realcompactifications of products*, Canad. J. Math., **22** (1970), 645–656.

8. K. Morita and T. Hoshina, *P-embedding and product spaces*, Fund. Math., **93** (1976), 71–80.

9. H. Ohta, *Local compactness and Hewitt realcompactifications of products*, Proc. Amer. Math. Soc., **69** (1978), 339–343.

10. T. C. Przymusinski, *Collectionwise normality and extensions of continuous functions*, Fund. Math., **98** (1978), 75–81.

11. ———, *On locally finite coverings*, Colloq. Math., **38** (1978), 187–192.

12. M. E. Rudin, *A normal space X for which $X \times I$ is not normal*, Fund. Math., **73** (1971), 179–186.

13. M. Starbird, *Extending maps from products*, in: Studies in Topology, Academic Press, 1975, 559–565.

14. R. G. Woods, *A Tychonoff almost real-compactification*, Proc. Amer. Math. Soc., **43** (1974), 200–208.

15. ———, *A survey of absolutes of topological spaces*, in: Topological Structures II, Math. Centre Tracts, **116** (1979), 323–362.

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POLISH ACADEMY OF SCIENCES,
00-950 WARSAW, SNIADECKICH 8,
POLAND

Current address: Math. Dept., Unlversity of Pittsburgh, PA 15260.