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# COMPACT OPERATORS AND DERIVATIONS INDUCED BY WEIGHTED SHIFTS

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**In this paper we study the question: which compact operators are contained in $\mathcal{R}(\delta_S)^-$, the norm closure of the range of the derivation $\delta_S(X) = SX - XS$ induced by a weighted shift $S$? We find that $\mathcal{R}(\delta_S)^-$ always contains the lower triangular (with respect to the basis $(e_i)$ on which $S$ is a shift) compact operators. Further, $\mathcal{R}(\delta_S)^-$ contains the $n$-lower triangular (operators $T$ satisfying $(Te_i, e_j) = 0$ for $i - j > n$) compact operators if and only if $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$. We also find necessary and sufficient conditions on the weights of $S$ in order that $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$ and that $\mathcal{K}$, the algebra of compact operators, be contained in $\mathcal{R}(\delta_S)^-$. These results completely answer the question: which essentially normal weighted shifts are $d$-symmetric?**

Let $T \in \mathcal{B}(\mathcal{H})$, the algebra of bounded linear operators on a complex Hilbert space $\mathcal{H}$. The derivation induced by $T$ is the map $\delta_T(X) = TX - XT$ from $\mathcal{B}(\mathcal{H})$ to itself. Let $(e_n)_{n=1}^{\infty}$ (respectively $(e_n)_{n=-\infty}^{\infty}$) be an orthonormal basis for $\mathcal{H}$ and let $S$ be the unilateral (respectively bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ (respectively $n \in \mathbf{Z}$) with nonzero weights $w_n$. By taking a unitarily equivalent weighted shift, we may assume that $w_n = |w_n| > 0$.

Recall that for $f, g \in \mathcal{H}$, the operator $f \otimes g \in \mathcal{B}(\mathcal{H})$ is defined by $(f \otimes g)h = (h, g)f$ for $h \in \mathcal{H}$. In particular, $(e_i \otimes e_j)e_n = e_i$ if $n = j$ and $(e_i \otimes e_j)e_n = 0$ otherwise. In Theorem 2 we show that $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$ if and only if $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{n+k-1} = \infty$. In Corollary 2, we find that this is also equivalent to $\mathcal{R}(\delta_S)^-$ containing all the $n$-lower triangular compact operators.

The above results enable us to characterize those essentially normal weighted shifts that are $d$-symmetric (i.e., satisfy $\mathcal{R}(\delta_S)^- = \mathcal{R}(\delta_S)^{-*}$). Namely, an essentially normal weighted shift is $d$-symmetric if and only if $S$ satisfies the total products condition $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n} = \infty$ for all $n \in \mathbf{N}$. This yields another proof of the fact proved in Corollary 4 of [**8**] that all hyponormal (and hence all subnormal) weighted shifts are all $d$-symmetric.

THEOREM 1. *Let $S$ be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$ $n \in \mathbf{N}$ $(\mathbf{Z})$. Then $e_i \otimes e_j \in \mathcal{R}(\delta_S)$ for all $i, j \in \mathbf{N}$ $(\mathbf{Z})$ with $i > j$.*

*Proof.* Write $i = j + n$ with $n > 0$. Let $a_0 = 1/w_j$, $a_k = w_{j+n} \cdot \cdots \cdot w_{j+n+k-1}/w_j \cdot \cdots \cdot w_{j+k}$ for $k \geq 1$, and $a_k = 0$ for $k < 0$. Then for $k > n$, cancellation is possible and

$$a_k = w_{j+k+1} \cdot \cdots \cdot w_{j+n+k-1}/w_j \cdot \cdots \cdot w_{j+n-1} \le \|S\|^{n-1}/w_j \cdots w_{j+n-1}.$$

Thus the $a_k$'s are uniformly bounded by some constant $B_n$. Also note that $a_k w_{j+n+k} = a_{k+1} w_{j+k+1}$ for $k \neq 1$ so $w_{m+n-1} a_{m-j-1} = a_{m-j} w_m$ for $m - j - 1 \neq -1$.

Now define $T = \sum_{k=0}^{\infty} a_k e_{j+n+k} \otimes e_{j+k+1}$. Then $\|T\| = \sup_k a_k \le B_n$ so $T \in \mathfrak{B}(\mathcal{H})$. Further,

$$\begin{aligned}(ST - TS)(e_m) &= Sa_{(m-j-1)} e_{j+n+(m-j-1)} \otimes e_{j+(m-j-1)+1}(e_m) \\ &\quad - a_{(m-j)} e_{j+n+(m-j)} \otimes e_{j+(m-j)+1}(w_m e_{m+1}) \\ &= Sa_{m-j-1} e_{m+n-1} - a_{m-j} w_m e_{m+n} \\ &= (w_{m+n-1} a_{m-j-1} - a_{m-j} w_m) e_{m+n} \\ &= \begin{cases} 0 & m - j - 1 \neq -1 \\ 0 - a_0 w_j e_{j+n} & m - j = 0 \end{cases} \\ &= \begin{cases} 0 & m \neq j \\ -e_i & m = j. \end{cases}\end{aligned}$$

Thus $ST - TS = -e_i \otimes e_j$ and $\delta_S(-T) = e_i \otimes e_j$. □

LEMMA 1. *If $Se_n = w_n e_{n+1}$ $n \in \mathbf{N}$ $(\mathbf{Z})$ is a unilateral (bilateral) weighted shift and $f \in \mathfrak{B}(\mathcal{H})^*$ is in the annihilator of $\mathfrak{R}(\delta_S)$, then*

$$f(e_{i+k} \otimes e_{j+k}) = \frac{w_j \cdot w_{j+1} \cdot \cdots \cdot w_{j+k-1}}{w_i \cdot w_{i+1} \cdot \cdots \cdot w_{i+k-1}} f(e_i \otimes e_j)$$

*for $i, j \in \mathbf{N}$ $(\mathbf{Z})$ and $k \in \mathbf{N}$.*

*Proof.* Since $f$ annihilates $\mathfrak{R}(\delta_S)$,

$$0 = f(S(e_i \otimes e_{j+1}) - (e_i \otimes e_{j+1})S) = w_i f(e_{i+1} \otimes e_{j+1}) - w_j f(e_i \otimes e_j).$$

Thus $f(e_{i+1} \otimes e_{j+1}) = (w_j/w_i) f(e_i \otimes e_j)$ for all $i$, $j$ and the lemma follows by induction. □

COROLLARY 1. *If $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ $(\mathbf{Z})$ is a unilateral (bilateral) weighted shift and $e_n \otimes e_m \in \mathfrak{R}(\delta_S)^-$, then $e_i \otimes e_j \in \mathfrak{R}(\delta_S)^-$ for all $i, j \in \mathbf{N}$ $(\mathbf{Z})$ satisfying the condition $m - n = j - i$.* □

THEOREM 2. *Let $S$ be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ $(\mathbf{Z})$. For $i \in \mathbf{N}$ $(\mathbf{Z})$ and $n \in \mathbf{N}$, we have $e_i \otimes e_{i+n} \in \mathcal{R}(\delta_S)^-$ if and only if $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{K+n-1} = \infty$ where the sum is taken over $\mathbf{N}$ or $\mathbf{Z}$ as $S$ is unilateral or bilateral.*

*Proof.* By Corollary 1, it suffices to consider $e_1 \otimes e_{n+1}$.

Suppose that $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$. If $J$ is a trace class operator that commutes with $S$, the equation

$$\begin{aligned}\operatorname{trace}((SA - AS)J) &= \operatorname{trace}(SAJ - AJS) \\ &= \operatorname{trace}(SAJ) - \operatorname{trace}(SAJ) = 0\end{aligned}$$

shows that trace$(\cdot J)$ annihilates $\mathcal{R}(\delta_S)^-$. Since $S^n$ commutes with $S$ and $\operatorname{trace}(S^n(e_1 \otimes e_{n+1})) = \operatorname{trace}(w_1 \cdot w_2 \cdot \cdots \cdot w_n e_{n+1} \otimes e_{n+1}) = w_1 \cdot w_2 \cdot \cdots \cdot w_n \neq 0$, it follows that $S^n$ cannot be of trace class. Hence $\infty = \Sigma_k(|S^n| e_k, e_k) = \Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1}$.

Conversely, suppose that $\Sigma_k w_k w_{k+1} \cdots w_{k+n-1} = \infty$ and that $f \in \mathcal{B}(\mathcal{H})^*$ annihilates $\mathcal{R}(\delta_S)^-$. Then $\Sigma_{k=1}^{\infty} w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} = \infty$ or (in the bilateral case) $\Sigma_{k=0}^{-\infty} w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} = \infty$. In the first case define $T_N = \Sigma_{k=n}^{N+n} e_k \otimes e_{n+k}$. Then $\|T_N\| = 1$ and using Lemma 1,

$$\begin{aligned}\|f\| \geq |f(T_N)| &= \left|\sum_{k=n}^{N+n} \frac{w_{n+1} \cdot w_{n+2} \cdot \cdots \cdot w_{n+k-1}}{w_1 \cdot w_2 \cdot \cdots \cdot w_{k-1}} f(e_1 \otimes e_{n+1})\right| \\ &= \left|\sum_{k=n}^{N+n} \frac{w_k \cdot \cdots \cdot w_{n+k-1}}{w_1 \cdot \cdots \cdot w_n} f(e_1 \otimes e_{n+1})\right| \\ &= \frac{|f(e_1 \otimes e_{n+1})|}{w_1 \cdot \cdots \cdot w_n} \sum_{k=n}^{N+n} w_k w_{k+1} \cdot \cdots \cdot w_{K+n-1}.\end{aligned}$$

Since $\Sigma_{k=n}^{N+n} w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} \to \infty$ as $N \to \infty$, we see that $f(e_1 \otimes e_{n+1}) = 0$ and $e_1 \otimes e_{n+1} \in \mathcal{R}(\delta_S)^-$.

Now suppose that $\Sigma_{k=0}^{-\infty} w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} = \infty$. If $l < 0$, we can apply Lemma 1 to $k = -l + 1$ to show that

$$f(e_1 \otimes e_{n+1}) = \frac{w_{n+l} \cdot \cdots \cdot w_n}{w_l \cdot \cdots \cdot w_0} f(e_l \otimes e_{n+l})$$

or

$$f(e_l \otimes e_{n+l}) = \frac{w_l \cdot \cdots \cdot w_0}{w_{n+l} \cdot \cdots \cdot w_n} f(e_1 \otimes e_{n+1}).$$

Defining $R_N = \sum_{l=-n}^{-N-n} e_l \otimes e_{n+l}$, we see that

$$\|f\| \geq |f(R_N)| = \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdot \cdots \cdot w_0}{w_{n+l} \cdot \cdots \cdot w_n} f(e_1 \otimes e_{n+1}) \right|$$

$$= \left| \sum_{l=-n}^{-N-n} \frac{w_l \cdot \cdots \cdot w_{n+l-1}}{w_1 \cdot \cdots \cdot w_n} f(e_1 \otimes e_{n+1}) \right|$$

$$= \frac{|f(e_1 \otimes e_{n+1})|}{w_1 \cdot \cdots \cdot w_n} \sum_{l=-n}^{-N-n} w_l \cdot \cdots \cdot w_{n+l-1}.$$

As before, the fact that $\sum_{l=-n}^{-N-n} w_l \cdots w_{n+l-1} \to \infty$ implies that $f(e_1 \otimes e_{n+1}) = 0$ and $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$. □

REMARK. Note that if we take $n = 0$ in the proof of Theorem 1 then the $a_n$ become $1/w_n$. Thus $e_i \otimes e_i \in \mathfrak{R}(\delta_S)$ if the $w_n$ are bounded away from zero. If the weights are not bounded away from zero, then taking $n = 0$ in the proof of Theorem 2 we find that $\|f\| \geq \sum_{k=0}^{N} |f(e_1 \otimes e_1)|$ and thus $e_i \otimes e_i \in \mathfrak{R}(\delta_S)^-$.

COROLLARY 2. *Let $S$ be the unilateral (bilateral) weighted shift $Se_n = w_n e_{n+1}$, $n \in \mathbf{N}$ ($\mathbf{Z}$). Then the following are equivalent.*

(a) *$\mathfrak{R}(\delta_S)^-$ contains the $n$-lower triangular compact operators.*

(b) $e_1 \otimes e_{1+n} \in \mathfrak{R}(\delta_S)^-$

(c) $e_i \otimes e_{i+n} \in \mathfrak{R}(\delta_S)^-$ *for some $i \in \mathbf{N}$ ($\mathbf{Z}$).*

(d) $\sum_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} = \infty$.

*Proof.* The equivalence of (b), (c) and (d) has already been established and (b) follows from (a) since $e_1 \otimes e_{1+n}$ is compact and $n$-lower triangular. It remains to be shown that (b) implies (a). From the proof of Theorem 2, we see that if $e_1 \otimes e_{n+1} \in \mathfrak{R}(\delta_S)^-$, then $S^n$ is not trace class. Hence $S^m$ is not trace class for $0 \leq m < n$. Thus $\sum_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+m-1} = \infty$ and all operators of the form $e_i \otimes e_{i+m}$ are elements of $\mathfrak{R}(\delta_S)^-$. *Since by Theorem 1, and the above remark,* $e_i \otimes e_{i+m} \in \mathfrak{R}(\delta_S)^-$ *for* $m \leq 0$, it follows that $\mathfrak{R}(\delta_S)^-$ contains the closed linear span of $\{e_i \otimes e_{i+m} : m \leq n\}$ (i.e., the $n$-lower triangular compact operators). □

REMARK. It is not true that if $\mathfrak{R}(\delta_S)^-$ contains an $n$-lower triangular *compact operator which is not $(n-1)$-lower triangular then* $\mathfrak{R}(\delta_S)^-$ contains all $n$-lower triangular compact operators. In fact $\mathfrak{R}(\delta_S)^-$ will always contain such an operator; namely $\delta_S(e_1 \otimes e_{n+2}) = w_1 e_2 \otimes e_{n+2} - w_{n+1} e_1 \otimes e_{n+1}$.

DEFINITION. A weighted shift satisfies the *total products condition* if $\Sigma_k w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n} = \infty$ for all $n \in \mathbf{N}$.

COROLLARY 3. *Let $Se_n = w_n w_{n+1}$, $n \in \mathbf{N}$ $(\mathbf{Z})$ be a unilateral (bilateral) weighted shift. Then $\mathcal{K} \subseteq \mathcal{R}(\delta_S)^-$ if and only if $S$ satisfies the total products condition.* □

We now make application to the question: which weighted shifts are $d$-symmetric? Recall that an operator $T$ is $d$-symmetric if $\mathcal{R}(\delta_T)^- = \mathcal{R}(\delta_T)^{-*}$. In [2] it is proved that an operator $T$ is $d$-symmetric if and only if $TT^* - T^*T \in \mathcal{C}(T) = \{C \in \mathcal{B}(\mathcal{K}) : C\mathcal{B}(\mathcal{K}) + \mathcal{B}(\mathcal{K})C \subseteq \mathcal{R}(\delta_T)^-\}$.

THEOREM 3. *The weights of a d-symmetric weighted shift S satisfy the total products condition.*

*Proof.* By Theorem 1, $e_i \otimes e_j \in \mathcal{R}(\delta_S)^-$ for $i \geq j$. By the $d$-symmetry of $S$, we see that $e_j \otimes e_i = (e_i \otimes e_j)^* \in \mathcal{R}(\delta_S)^-$ for $j \leq i$. Thus $\mathcal{K}$, the linear span of all $e_i \otimes e_j$, is contained in $\mathcal{R}(\delta_S)^-$ and so by Corollary 3, the weights of $S$ satisfy the total products condition. □

The total products condition is not sufficient for $d$-symmetry else any weighted shift with weights bounded away from zero would be $d$-symmetric. However the weighted shift with weights alternating between 1 and 2 has an irreducible representation as the operator $\left(\begin{smallmatrix} 0 & 2 \\ 1 & 0 \end{smallmatrix}\right)$ on $\mathbf{C}^2$, while in [2] it is shown that any irreducible representation of a $d$-symmetric operator must be over a Hilbert space of dimension 1 or $\aleph_0$. There are, however, natural conditions under which the total products condition is sufficient.

THEOREM 4. *An essentially normal weighted shift S is d-symmetric if and only if it satisfies the total products condition.*

*Proof.* The necessity of the total products condition follows from Theorem 3 and sufficiency follows from the facts that $SS^* - S^*S$ is compact and that $\mathcal{K} \subseteq \mathcal{R}(\delta_S)^-$ implies $\mathcal{K} \subseteq \mathcal{C}(S)^-$. □

COROLLARY 4. *A hyponormal (in particular subnormal) weighted shift $Se_n = w_n e_{n+1}$ is d-symmetric.*

*Proof.* If $S$ is hyponormal, then its weights are increasing and bounded. Thus

$$SS^* - S^*S = \operatorname{diag}\left(w_{i-1}^2 - w_i^2\right)$$

is compact and $\Sigma_{k=1}^{\infty} w_k \cdot w_{k+1} \cdot \cdots \cdot w_{k+n-1} \geq \Sigma_{k=1}^{\infty} w_1^n = \infty$ for all $n \in \mathbf{N}$. □

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