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THE EXISTENCE OF STRONG LIFTINGS FOR TOTALLY ORDERED MEASURE SPACES

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Let X be a totally ordered space, μ a finite Borel measure on X with full support, and $\mathfrak{F}$ the σ-algebra of all μ^*-measurable subsets of X. Then there exists a lifting ρ: $\mathfrak{F} \to \mathfrak{F}$ which satisfies $U \subset \rho(U)$ for every open subset U of X.

Assume that X is a topological space and μ a finite, Borel measure on X with full support. We are interested in finding conditions for the topology of X, which insure the existence of strong liftings for the associated topological measure space. In **[8]** Losert has given an example showing that this is not always possible even if X is compact. On the other hand Graf has proved that strong liftings always exist for measures on second countable spaces **[6]**. Other positive results on the existence of strong liftings are given in **[2]** and **[4]**.

In this paper we show that every totally ordered measure space admits a strong lifting. Moreover we prove that if μ is a Radon, non-atomic measure on a totally ordered space then every lifting of the associated measure space is almost strong, if and only if, the set of all two sided limit points of the support of μ is μ^*-measurable with full measure.

1. Preliminaries and notation. Throughout X will be a set and "$\leq$" a total order on X. If x, y are two points of X, let $x < y$ means that $x \leq y$ and $x \neq y$. Let $(-\infty, x) = \{z \in X\colon z < x\}$, $(x, +\infty) = \{z \in X\colon x < z\}$ and $(x, y) = (-\infty, y) \cap (x, +\infty)$. Assume that $Y \subset X$ and $y \in Y$. We say that y is a left limit point of Y if $(-\infty, y) \cap Y \neq \varnothing$ and there is no z in Y such that $z < y$ and $(z, y) \cap Y = \varnothing$. Analogous is the definition of the right limit point. A point y in Y is said to be a two sided limit point of Y if it is both a left and right limit point of Y.

We say that $(X, \leq)$ is a totally ordered (topological) space if its topology is generated by all the intervals of the form $(-\infty, x)$, $(x, +\infty)$. By a measure on X we mean a finite, non-negative, countably additive set function defined on the Borel sets of X. A measure μ on X is said to be Radon if it is inner approximated by the compact sets of X. The support S_μ of a measure μ on X is defined by

$$S_\mu = \bigcap \{F\colon F \text{ closed subset of } X \text{ such that } \mu(F) = \mu(X)\}.$$

Clearly S_μ is a closed subset of X and if $\mu(X) = \mu(S_\mu)$, S_μ satisfies the countable chain condition.

A totally ordered measure space is the quadriple $(X, \leq, \mathcal{F}, \mu)$, where $(X, \leq)$ is a totally ordered space, μ is a measure on X and $\mathcal{F}$ is the completion of the Borel σ-algebra of X w.r.t. μ.

A map $\rho: \mathcal{F} \to \mathcal{F}$ is called lifting [7] if for all $A, B \in \mathcal{F}$

(i) $\mu(A \Delta \rho(A)) = 0$,
(ii) $\mu(A \Delta B) = 0 \Rightarrow \rho(A) = \rho(B)$,
(iii) $\rho(\varnothing) = \varnothing$, $\rho(X) = X$,
(iv) $\rho(A \cup B) = \rho(A) \cup \rho(B)$,
(v) $\rho(A \cap B) = \rho(A) \cap \rho(B)$.

If ρ satisfies only the properties (i), (ii), (iii), (iv) (resp. (i), (ii), (iii), (vi)) is called an upper (resp. lower) density.

A lifting ρ is said to be

(a) strong, if $U \subset \rho(U)$ for all open subsets U of X,

(b) almost strong if there is a set $N \in \mathcal{F}$ with $\mu(N) = 0$, and $U \subset N \cup \rho(U)$ for all open subsets U of X.

2. The results. We start with the main result of this paper, the proof of which is divided in three steps. The proof of the first step is based on an idea of [[7], Example 3, page 122], and on the arguments used in the proof of Theorem 3.2 in [**10**].

THEOREM 2.1. *Let $(X, \leq, \mathcal{F}, \mu)$ be a totally ordered measure space such that $X = S_\mu$. Then there exists a strong lifting $\rho: \mathcal{F} \to \mathcal{F}$.*

Proof. Without loss of generality we may assume that $\mu(X) = 1$.

Step I. We prove the theorem in the case when S is compact and μ a Radon, non-atomic measure.

Let f be the distribution function of μ defined by $f(x) = \mu((-\infty, x))$ for every $x \in X$. Then f is an ordered preserving, continuous function from X onto the unit interval $[0, 1]$, such that $f^{-1}f(B) - B$ is at most countable for every Borel subset B of X (see proof of Theorem 3.2 in [**10**]). Moreover if λ denotes the Lebesgue measure on $[0, 1]$ we can easily verify that $\lambda(B) = \mu(f^{-1}(B))$ for every Borel subset B of $[0, 1]$.

We show that for every $F \in \mathcal{F}$

$$(*) \qquad f(F) \text{ is Lebesgue measurable and } \lambda(f(F)) = \mu(F).$$

Indeed since X is hereditary Lindelöf [[3], Theorem 2.2], every Borel subset F of X is κ-analytic and so $f(F)$ is an absolutely measurable subset of $[0, 1]$. Thus $f(F)$ is Lebesgue measurable and since $f^{-1}f(F) - F$ is at most countable we deduce that $\lambda(f(F)) = \mu(F)$. This shows that $(*)$ is true for Borel subsets of X and can easily be extended for every F in $\mathcal{F}$.

Now let L_1 (resp. L_2) be the set of all left (resp. right) limit points of X. Clearly since X has no isolated points we have that $X = L_1 \cup L_2$. We consider each L_i ($i = 1,2$) with the order topology, given by the restriction of the ordering on L_i. Let f_i be the restriction of f on L_i. Then since f_1 (resp. f_2) is an ordered preserving, one-to-one map from L_1 (resp. L_2) on $(0,1]$ (resp. $[0,1)$), we may easily conclude that f_1 (resp. f_2) is an homeomorphism.

Let λ_i, be the measure on L_i ($i = 1,2$) defined by $\lambda_i(E) = \lambda(f_i(E))$ for every Borel $E \subset L_i$, and $\mathfrak{F}_i$ the completion of the Borel σ-algebra of L_i w.r.t. λ_i. Then using $(*)$ we can easily deduce that

(1) If $F \in \mathfrak{F}$ then $F \cap L_i \in \mathfrak{F}_i$ for $i = 1,2$.

(2) For every $A \subset X$, $\mu^*(A) = 0$ if and only if $\lambda_i^*(A \cap L_i) = 0$ for every $i = 1,2$.

(3) $\lambda_1^*(A) = \lambda_2^*(A)$ for every $A \subset L_1 \cap L_2$.

Now by [[7], Theorem 6, page 123] there are strong liftings ρ_1, ρ_2 of $\mathfrak{F}_1$, $\mathfrak{F}_2$ respectively, such that

$$A \subset \rho_1(A) \text{ for every left-open right-closed interval } A \text{ of } L_1$$

and

$$B \subset \rho_2(B) \text{ for every left-closed right-open interval } B \text{ of } L_2.$$

We define θ on $\mathfrak{F}$ by

$$\theta(F) = \rho_1(F \cap L_1) \cup \rho_1(F \cap L_2).$$

Then using (1), (2) and (3) we may easily check that θ is an upper density of $\mathfrak{F}$. Let θ' be the associated to θ lower density of $\mathfrak{F}$ defined by $\theta'(F) = X - \theta(X - F)$. Then

$$\theta'(F) = (\rho_1(F \cap L_1) \cup X - L_1) \cap (\rho_2(F \cap L_2) \cup X - L_2).$$

We show that $U \subset \theta'(U)$, where U is either of the form $(-\infty, x)$ or $(x, +\infty)$. Indeed if $U = (-\infty, x)$, then $U \cap L_1$ is either an open interval of L_1 (if $x \in L_1$) or a left-open right-closed interval of L_1 (if $x \in L_2$). It follows that $U \cap L_1 \subset \rho_1(U \cap L_1)$. On the other hand $U \cap L_2$ is always a left-closed right-open interval of L_2 and so $U \cap L_2 \subset \rho_2(U \cap L_2)$. Thus $U \subset \theta'(U)$. In the same way we show that $(x, +\infty) \subset \theta'((x, +\infty))$. By [[7], corollary page 58] there exists a lifting ρ of $\mathfrak{F}$ such that $\theta'(F) \subset \rho(F) \subset \theta(F)$ for every $F \in \mathfrak{F}$. It follows that ρ is a strong lifting of $\mathfrak{F}$.

Step II. Here we assume that X is compact and μ Radon.

Set $Y = \{x \in X\colon \mu(\{x\}) > 0\}$. Clearly Y is at most countable. Let ν be the Radon, non-atomic measure on X defined by $\nu(B) = \mu(B \cap X - Y)$ for every Borel $B \subset X$. Since the support S_ν of ν is a compact totally order space, under the subspace topology, by step I there exists a strong lifting for the restriction of ν on S_ν. Let $Z = S_\nu \cap (X - Y)$. Clearly $\nu(Z) = \nu(S_\nu)$ and since μ, ν coincide on the Borel subsets of Z, there exists a strong lifting ρ' for the restriction of μ on Z.

Let $T = Z \cup Y$. We define ρ on $\mathfrak{F}_T$, the class of all μ^*-measurable subsets of T, by

$$\rho(E) = \rho'(E \cap Z) \cup (E \cap Y).$$

It follows that ρ is a strong lifting for the restriction of μ on T and since $\mu(T) = \mu(X)$ using a standard argument ([7], Remark 1, page 127) we can find a strong lifting of $\mathfrak{F}$.

Step III. The general case.

Let X^* be the totally ordered compactification of X (see [5], §6). We define a measure $\bar{\mu}$ on X^* by $\bar{\mu}(B) = \mu(B \cap X)$, for every Borel $B \subset X^*$. We note that $\bar{\mu}$ has full support in X^* and thus X^* is hereditary Lindelöf [3]. It follows easily that $\bar{\mu}$ is a Radon measure on X^*.

Let $\overline{\mathfrak{F}}$ be the σ-algebra of all $\bar{\mu}^*$-measurable subsets of X^*. Then we have $F \in \mathfrak{F}$ if and only if there is $E \in \overline{\mathfrak{F}}$ such that $F = E \cap X$. By Step II, there exists a strong lifting $\bar{\rho}$ of $\overline{\mathfrak{F}}$. We define ρ on $\mathfrak{F}$ by

$$\rho(F) = \bar{\rho}(E) \cap X$$

where $E \in \overline{\mathfrak{F}}$ such that $F = E \cap X$. It follows that ρ is a strong lifting of $\mathfrak{F}$ and hence the theorem is completely proved.

As measures on totally ordered spaces share a number of properties with measures on metric spaces [**10**], it is natural to ask whether every lifting of $(X, \leq, \mathfrak{F}, \mu)$ is almost strong. The answer is given by using the next theorem, the proof of which uses an argument of [[1], Theorem 3.1] and some already familiar techniques from Theorem 2.1.

THEOREM 2.2. *Let $(X, \leq, \mathfrak{F}, \mu)$ be a totally ordered measure space, where μ is a Radon, non-atomic measure on X. Then the following are equivalent*

(i) *Every lifting of $\mathfrak{F}$ is almost strong.*

(ii) *The set L of all two sided limit points of S_μ is μ^*-measurable with full measure.*

Proof. We first prove the theorem in the case when X is compact. Also in this case without loss of generality we may assume that μ is a probability measure with full support.

(i) $\Rightarrow$ (ii) As in the Step I of Theorem 2.1 we denote by f the distribution function of μ. We define ρ on $\mathfrak{F}$ by

$$\rho(E) = f^{-1}(\rho'(f(E)))$$

where ρ' is a lifting of the Lebesgue measure on [0, 1]. Clearly ρ is a lifting of $\mathfrak{F}$ and so there exists a G_δ subset N of X such that $\mu(N) = 0$ and $U \subset \rho(U) \cup N$ for every open subset U of X.

We will prove that $X - N$ contains at most countable many non-two sided limit points of X. Assume that this is not true, without loss of generality let $\{x_i\}_{i \in I}$ be an incountable set of non-left limit points of $X - N$. Let $y_i \in X$ such that $y_i < x_i$ and $(y_i, x_i) = \varnothing$. Then $y_i \in N$ for each i, because otherwise $y_i \in \rho((-\infty, x_i))$ and if $t_i = f(x_i) = f(y_i)$ we have that

$$t_i \in \rho'(f(-\infty, x_i)) \cap \rho'(f(y_i, +\infty)),$$

while

$$\rho'(f(-\infty, x_i)) \cap \rho'(f(y_i, +\infty)) = \rho'(\{t_i\}) = \varnothing .$$

Now since N is a G_δ subset of X we may find an open subset U of X such that $N \subset U$ and $x_i \notin U$ for uncountable many i. Further since X is hereditary Lindelöf there is a sequence $\{(a_n, b_n)\}$ of open intervals in X such that $U = \bigcup_{n=1}^{\infty}(a_n, b_n)$. Then if we pick an i such that $x_i \notin U$ and $x_i \neq b_n$ for all n, we get that $y_i \notin U$, which is a contradiction. This shows that $L \in \mathfrak{F}$ and $\mu(L) = \mu(X)$.

(ii) $\Rightarrow$ (i) Let ρ be a lifting of $\mathfrak{F}$. We define an upper density θ of the Lebesgue measure space by

$$\theta(E) = f\big(\rho\big(f^{-1}(E)\big)\big)$$

where f again denotes the distribution function of μ. Clearly since every lifting of the Lebesgue measure space is almost strong we can find a set $N \subset [0, 1]$ such that $\lambda(N) = 0$ and $V \subset \theta(V) \cup N$ for every open $V \subset [0, 1]$. It follows easily that

$$U \subset \rho(U) \cup f^{-1}(N) \cup (X - L)$$

for every open subset U of X. This shows that ρ is almost strong.

We now consider the general case. Let X^* be the totally ordered compactification of X and $\bar{\mu}$ the induced by μ Radon measure on X^*. Clearly $\bar{\mu}$ is non-atomic and $S_\mu = S_{\bar{\mu}} \cap X$.

Let $\bar{L}$ be the set of all two sided limit points of $S_{\bar{\mu}}$ and $\bar{\mathfrak{F}}$ the completion of the Borel σ-algebra of X^* w.r.t. $\bar{\mu}$. Clearly since μ is Radon, X is $\bar{\mu}^*$-measurable with full $\bar{\mu}$-measure and so every lifting of $\bar{\mathfrak{F}}$ is almost strong, if and only if, every lifting of $\mathfrak{F}$ is almost strong. It remains to prove that L is μ^*-measurable with full measure, if and only if, $\bar{L}$ is $\bar{\mu}^*$-measurable with full measure. Indeed assume that $L \in \mathfrak{F}$ and $\mu(L) = \mu(X)$. Then using similar arguments as before we can show that $K \cap (X^* - \bar{L})$ is at most countable, for every compact $K \subset L$. It follows by the

Radon property of μ that $\bar{L}$ is $\bar{\mu}^*$-measurable with full $\bar{\mu}$-measure. On the other hand if $\bar{L}$ is $\bar{\mu}^*$-measurable and $\bar{\mu}(\bar{L}) = \bar{\mu}(X^*)$ since $\bar{L} \cap X \subset L$, we deduce that $L \in \mathfrak{F}$ and $\mu(L) = \mu(X)$. Hence the proof of the theorem is complete.

REMARKS 2.3. (i) Since every measure μ (with full support on a totally ordered space) has separable measure algebra, under the continuum hypothesis Theorem 2.1 follows also by Theorem 9 in [**9**].

(ii) Let X be the space we obtain from [0, 1] by replacing each point $t \in (0, 1)$ by two points say $(t, 0)$ and $(t, 1)$. We define an order "$<$" on X by

$$(t, i) < (t', j) \quad \text{iff} \quad t < t' \text{ or } t = t' \text{ and } i = 0, j = 1.$$

Clearly $(X, <)$ is a compact totally ordered space which supports non-atomic, Radon measures [[**10**, Example 4.4]. Clearly since X has no two sided limit points, by Theorem 2.2, every non-zero, non-atomic Radon measure on X provides an example of a totally ordered measure space which admits not almost strong liftings.

(iii) Theorem 2.2 is not valid for atomic measures. For example let X be a compact totally ordered space which supports a non-atomic Radon measure ν such that the set L of all two sided limit points is ν^*-measurable with full measure and $X \neq L$. Then if $x \in X - L$ and $\mu = \nu + \delta_x$, we see that Theorem 2.2 is not applicable for μ. (Here δ_x denotes the Dirac measure at x.)

(iv) In Theorem 2.2 the Radon property of μ cannot be omitted. To see this let Y be the unit interval with the topology generated by all left-closed right-open intervals. It is known that Y is a closed subspace of a totally ordered space X (see Example 2.2(b) and Theorem 2.9 in [**9**]). Moreover, since the Borel sets of Y and (0, 1) are the same, the Lebesgue measure λ is a Borel measure on Y. Let μ be the extension of λ on X. Then μ is a non-atomic measure on X and $S_\mu = Y$. Further every point of Y is a two sided limit point of Y, but μ admits not almost strong liftings (see Example 10 in [2]).

(v) The proofs of Theorems 2.1 and 2.2 apply unchanged for measures on generalized ordered spaces [**9**]

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