PACIFIC JOURNAL OF MATHEMATICS
Vol. 108, No. 1, 1983

AN ORDERING FOR THE BANACH SPACES

G. A. EDGAR

A binary relation will be defined on the class of all Banach spaces. The relation is transitive and symmetric, so it is natural to call it an "ordering". (The definition also makes sense for locally convex spaces with good duality properties, but this will not be pursued here.) Many of the elementary properties of the ordering are spelled out. Although some connections with Pettis integration and unique preduals have been found, the usefulness of this ordering in Banach space theory remains to be determined.

Notation and terminology used in this paper generally matches Dunford and Schwartz [**4**], Chapters IV and VI. More recent results in Banach space theory will usually be quoted from Lindenstraus and Tzafriri [**11**] or from Diestel and Uhl [**3**]. If $\mathfrak{X}$ is a Banach space, its dual will be denoted $\mathfrak{X}^*$, its bidual $\mathfrak{X}^{**}$. The subset of $\mathfrak{X}^{**}$ canonically identified with $\mathfrak{X}$ will simply be written $\mathfrak{X}$.

DEFINITION. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be Banach spaces. Then $\mathfrak{X} \prec \mathfrak{Y}$ means

$$\mathfrak{X} = \bigcap T^{**-1}[\mathfrak{Y}],$$

where the intersection is over all bounded linear operators $T\colon \mathfrak{X} \to \mathfrak{Y}$.

The definition can be rephrased as follows: $\mathfrak{X} \prec \mathfrak{Y}$ if and only if any $\alpha \in \mathfrak{X}^{**}$, such that $T^{**}(\alpha) \in \mathfrak{Y}$ for all bounded linear operators T: $\mathfrak{X} \to \mathfrak{Y}$, must be in $\mathfrak{X}$.

A single operator T with $\mathfrak{X} = T^{**-1}[\mathfrak{Y}]$ has been called a *Tauberian operator* (see [**10**]). If there exists a Tauberian operator $\mathfrak{X} \to \mathfrak{Y}$, then $\mathfrak{X} \prec \mathfrak{Y}$, but not conversely.

Following [**9**], where the case $\mathfrak{Y} = l_1$ is considered, we define the *$\mathfrak{Y}$-frame* (cadre) of $\mathfrak{X}$ by

$$\mathfrak{F}(\mathfrak{X}, \mathfrak{Y}) = \bigcap T^{**-1}[\mathfrak{Y}],$$

with intersection over all operators $T\colon \mathfrak{X} \to \mathfrak{Y}$. Then $\mathfrak{F}(\mathfrak{X}, \mathfrak{Y})$ is a Banach space, $\mathfrak{X} \subseteq \mathfrak{F}(\mathfrak{X}, \mathfrak{Y}) \subseteq \mathfrak{X}^{**}$. One extreme possibility is $\mathfrak{F}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}^{**}$, which means that every operator from $\mathfrak{X}$ to $\mathfrak{Y}$ is weakly compact. The other extreme is $\mathfrak{F}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}$, which is the relation $\mathfrak{X} \prec \mathfrak{Y}$ considered here.

Many properties of the relation $\prec$ are stated below. When they are stated without proof, they can be verified by straightforward diagram chasing from the definition.

The relation is transitive (if $\mathfrak{X}_1 \prec \mathfrak{X}_2$ and $\mathfrak{X}_2 \prec \mathfrak{X}_3$, then $\mathfrak{X}_1 \prec \mathfrak{X}_3$) and reflexive ($\mathfrak{X} \prec \mathfrak{X}$ for any Banach space $\mathfrak{X}$). So it defines a partial order on the equivalence classes defined by the equivalence relation $\mathfrak{X} \sim \mathfrak{Y}$ iff $\mathfrak{X} \prec \mathfrak{Y}$ and $\mathfrak{Y} \prec \mathfrak{X}$. This partial order is not a total order (see remarks following Proposition 11).

If $\mathfrak{X}$ is isomorphic to a closed subspace of $\mathfrak{Y}$ (below I will say "$\mathfrak{X}$ embeds in $\mathfrak{Y}$"), then $\mathfrak{X} \prec \mathfrak{Y}$. This can be seen using the isomorphic embedding $\mathfrak{X} \to \mathfrak{Y}$ in place of T in the definition. The converse is false, however. We have $l_3 \prec l_2$ (Proposition 1) and $l_1 \prec c_0$ (Proposition 2), but l_3 does not embed in l_2 and l_1 does not embed in c_0.

The relation $\prec$ has no relation with "semi-embedding", defined by Rosenthal [12]. The space l_1 semi-embeds in l_2, but $l_1 \not\prec l_2$ (Proposition 1); conversely, $L_1 \prec l_1$ (Proposition 10), but L_1 does not semi-embed in l_1.

There is a *least equivalence class*, namely the class consisting of all reflexive spaces. That is:

1. PROPOSITION. *If $\mathfrak{X}$ is reflexive, then $\mathfrak{X} \prec \mathfrak{Y}$ for all Banach spaces $\mathfrak{Y}$. If $\mathfrak{Y}$ is reflexive, then $\mathfrak{X} \prec \mathfrak{Y}$ if and only if $\mathfrak{X}$ is reflexive.*

Proof. Suppose $\mathfrak{X}$ is reflexive. Since

$$\mathfrak{X} \subseteq \bigcap T^{**-1}[\mathfrak{Y}] \subseteq \mathfrak{X}^{**},$$

we have equality. Now suppose $\mathfrak{Y}$ is reflexive. Then

$$\bigcap T^{**-1}[\mathfrak{Y}] = \mathfrak{X}^{**},$$

so $\mathfrak{X} \prec \mathfrak{Y}$ implies $\mathfrak{X} = \mathfrak{X}^{**}$. □

There is a second-to-least equivalence class, namely the class containing l_1. That is:

2. PROPOSITION. *Let $\mathfrak{Y}$ be a Banach space. Then $l_1 \prec \mathfrak{Y}$ if and only if $\mathfrak{Y}$ is not reflexive.*

Proof. If $l_1 \prec \mathfrak{Y}$, then $\mathfrak{Y}$ is not reflexive by Proposition 1.

Conversely, suppose $\mathfrak{Y}$ is not reflexive. I must show $l_1 \prec \mathfrak{Y}$. If l_1 embeds in $\mathfrak{Y}$, then clearly $l_1 \prec \mathfrak{Y}$. So we may assume l_1 does not embed in $\mathfrak{Y}$. Now since $\mathfrak{Y}$ is not reflexive, there is a bounded sequence, no subsequence of which converges in the weak topology [**4**, Theorems V.4.7 and V.6.1]. Then by a theorem of Rosenthal [**11**, Theorem 2.e.5] there is a subsequence (y_n) which is either equivalent to the unit vector basis of l_1 or a weak Cauchy sequence. But l_1 does not embed in Y, so (y_n) is a weak Cauchy sequence. Then $\alpha(f) = \lim_n f(y_n)$, $f \in \mathfrak{Y}^*$, defines $\alpha \in \mathfrak{Y}^{**}$ and $\alpha \notin \mathfrak{Y}$.

Now consider $\mu \in l_1^{**}$ such that $T^{**}(\mu) \in \mathfrak{Y}$ for all operators T: $l_1 \to \mathfrak{Y}$. We must show that $\mu \in l_1$. Now l_1^{**} is canonically identified with the space $ba(\mathbf{N}, \mathcal{P}(\mathbf{N}))$ of bounded, finitely additive set functions on $\mathbf{N}$ [**4**, Theorem IV.8.16]. Any such set function μ can be written as a sum of a purely finitely additive set function (vanishing on all finite sets) and an element of l_1 itself [**3**, p. 30]. So to show $\mu \in l_1$, we may assume μ is purely finitely additive and show $\mu = 0$.

So assume $\mu \in l_1^{**}$ is a purely finitely additive set function on $\mathbf{N}$, and $T^{**}(\mu) \in \mathfrak{Y}$ for all T. Consider an infinite set $A \subseteq \mathbf{N}$. Define T: $l_1 \to \mathfrak{Y}$ by: $T(e_n) = y_n$ if $n \in A$, $T(e_n) = 0$ if $n \notin A$. (Here e_n is the vector $(0,0,\ldots,0,1,0,0,\ldots)$ with 1 in the nth place.) To compute $T^{**}(\mu)$, let $f \in \mathfrak{Y}^*$ and $\varepsilon > 0$. Choose N large enough that $|f(y_n) - \alpha(f)| \le \varepsilon$ for all $n \ge N$. Then $|f(T(e_n)) - \alpha(f)\chi_A(n)| \le \varepsilon$ except for finitely many n. But μ vanishes on finite sets, so $|T^{**}(\mu)(f) - \alpha(f)\mu(A)| \le \varepsilon$. Thus $T^{**}(\mu) = \mu(A)\alpha$. But $T^{**}(\mu) \in \mathfrak{Y}$ and $\alpha \notin \mathfrak{Y}$, so $\mu(A) = 0$. The set A was arbitrary so $\mu = 0$. This shows $l_1 \prec \mathfrak{Y}$. □

More information on the class of l_1 is given below (Propositions 10 and 13).

A Banach space X satisfies the *condition of Mazur* iff any $\alpha \in \mathfrak{X}^{**}$ which is sequentially continuous on $(\mathfrak{X}^*, \text{weak*})$ is actually continuous there (and hence is an element of $\mathfrak{X}$).

3. PROPOSITION. *A Banach space $\mathfrak{X}$ satisfies the condition of Mazur if and only if $\mathfrak{X} \prec c_0$.*

Proof. Suppose $\mathfrak{X} \prec c_0$. Let $\alpha \in \mathfrak{X}^{**}$, and suppose α is sequentially continuous on $(\mathfrak{X}^*, \text{weak}^*)$. Let $T\colon \mathfrak{X} \to c_0$ be any operator. Define $f_n \in \mathfrak{X}^*$ by $f_n(x) = T(x)(n)$. Now $\lim_n f_n(x) = 0$ for all x, so $f_n \to 0$ (weak*). By hypothesis, $\alpha(f_n) \to 0$. But $T^{**}(\alpha) \in l_\infty$ is given by $T^{**}(\alpha)(n) = \alpha(f_n)$. Thus $T^{**}(\alpha) \in c_0$. This holds for all $T\colon \mathfrak{X} \to c_0$. Therefore $\alpha \in \mathfrak{X}$. So $\mathfrak{X}$ satisfies the condition of Mazur.

Conversely, suppose $\mathfrak{X}$ satisfies the condition of Mazur. Let $\alpha \in \mathfrak{F}(\mathfrak{X}, c_0)$. I claim α is sequentially continuous on $(\mathfrak{X}^*, \text{weak}^*)$. Indeed, let $f_n \to f$ (weak*) in $\mathfrak{X}^*$. Then the operator $T\colon \mathfrak{X} \to c_0$ defined by $T(x)(n) = (f_n - f)(x)$ has $T^{**}(\alpha) \in l_\infty$ given by $T^{**}(\alpha)(n) = \alpha(f_n - f)$. But $\alpha \in T^{**-1}[c_0]$, so $\alpha(f_n) \to \alpha(f)$. Thus α is weak* sequentially continuous. Thus $\alpha \in X$. This proves $\mathfrak{X} \prec c_0$. □

There is a largest class containing separable Banach spaces. Namely:

4. Corollary. *If $\mathfrak{X}$ is separable, then $\mathfrak{X} \prec c_0$.*

Proof. Suppose $\mathfrak{X}$ is separable, and $\alpha \in \mathfrak{X}^{**}$ is sequentially continuous on $(\mathfrak{X}^*, \text{weak}^*)$. Then the unit hall $B_{\mathfrak{X}^*}$ of $\mathfrak{X}^*$ has metrizable weak* topology [**4**, Theorem V.5.1], so α is continuous on $(B_{\mathfrak{X}^*}, \text{weak}^*)$, and therefore [**4**, Theorem V.5.6] $\alpha \in \mathfrak{X}$. So $\mathfrak{X} \prec c_0$ by Proposition 3. □

5. Proposition. *Let $\mathfrak{Y}$ be a Banach space. Then $c_0 \prec \mathfrak{Y}$ if and only if c_0 embeds in $\mathfrak{Y}$.*

Proof. If c_0 embeds in $\mathfrak{Y}$, then clearly $c_0 \prec \mathfrak{Y}$. Conversely, suppose $c_0 \prec \mathfrak{Y}$. Then there is an operator $T\colon c_0 \to \mathfrak{Y}$ that is not weakly compact. But then T is an isomorphism on some subspace of c_0 isomorphic to c_0 [**3**, Theorem 15, p. 159]. Thus c_0 embeds in $\mathfrak{Y}$. □

If S is an infinite compact metric space, then $C(S) \sim c_0$. The reader may find it interesting to write down exactly what $\mathfrak{X} \prec C(S)$ means, using [**4**, Theorem VI.7.1]. Then observe that the result is equivalent to the condition of Mazur (by Proposition 3).

The condition (b) described in the next result appeared first in [**6**, Proposition 4.4] in connection with the Pettis integral.

6. PROPOSITION. *Let $\mathfrak{X}$ be a Banach space. Then the following are equivalent.*

(a) $\mathfrak{X} \prec l_\infty$.

(b) *If $\alpha \in \mathfrak{X}^{**}$ is weak* continuous on all bounded weak*-separable subsets of $\mathfrak{X}^*$, then $\alpha \in \mathfrak{X}$.*

Proof. Suppose $\mathfrak{X} \prec l_\infty$. Let $\alpha \in \mathfrak{X}^{**}$, and suppose α is continuous on all bounded separable subsets A of $(\mathfrak{X}^*, \text{weak}^*)$. Let T: $\mathfrak{X} \to l_\infty$ be any operator. Then (since the ball of l_∞^* is separable), $A = T^*(B_{l_\infty^*})$ is a bounded weak*-separable set. By hypothesis, α is weak*-continuous on A. Then $T^{**}(\alpha) = \alpha \circ T^*$ is weak* continuous on $B_{l_\infty^*}$, so $T^{**}(\alpha) \in l_\infty$. Thus $\alpha \in T^{**-1}[l_\infty]$. But T was any operator, so $\alpha \in \mathfrak{F}(\mathfrak{X}, l_\infty)$. Since $\mathfrak{X} \prec l_\infty$, we have $\alpha \in \mathfrak{X}$.

Conversely, assume (b). Let $\alpha \in \mathfrak{F}(\mathfrak{X}, l_\infty)$. Let A be a bounded, weak*-separable set in $\mathfrak{X}^*$, say $\{f_1, f_2, \ldots\}$ is weak*-dense in A. Define an operator T: $\mathfrak{X} \to l_\infty$ by $T(x)(n) = f_n(x)$. Then by hypothesis $\alpha \in T^{**-1}[l_\infty]$, i.e., $T^{**}(\alpha) \in l_\infty$, or $\alpha \circ T^*$ is weak*-continuous on $B_{l_\infty^*}$. But then I claim that α is weak*-continuous on $T^*(B_{l_\infty^*})$. Indeed, suppose (g_ξ) is a net in $T^*(B_{l_\infty^*})$ and $g_\xi \to g$. Choose $g'_\xi \in B_{l_\infty^*}$ with $T^*(g'_\xi) = g_\xi$. There is a subnet $g'_{\xi'}$ such that $g'_{\xi'}$ converges, say to $g' \in B_{l_\infty^*}$. Then $T^*(g') = \lim T^*(g'_{\xi'}) = \lim g_{\xi'} = g$. Thus, $\alpha(g_{\xi'}) = \alpha(T^*(g'_{\xi'})) \to \alpha(T^*(g')) = \alpha(g)$. So α is weak*-continuous on $T^*(B_{l_\infty^*}) \supseteq A$. So $\alpha \in \mathfrak{X}$. □

If the word "bounded" is omitted in condition (b), the resulting condition characterizes Banach spaces whose weak topology is real compact [**2**, Lemma 9]. Talagrand [**13**] has shown that the two conditions (with and without the word "bounded") are not equivalent.

7. PROPOSITION. *Let $\mathfrak{Y}$ be a Banach space. Then $l_\infty \prec \mathfrak{Y}$ if and only if l_∞ embeds in $\mathfrak{Y}$.*

Proof. If l_∞ embeds in $\mathfrak{Y}$, then clearly $l_\infty \prec \mathfrak{Y}$. Conversely, suppose $l_\infty \prec \mathfrak{Y}$. Then there is an operator T: $l_\infty \to \mathfrak{Y}$ that is not weakly compact. But then T is an isomorphism on a subspace of l_∞ isomorphic to l_∞ [**3**, Theorem 10, p. 156]. Thus l_∞ embeds in $\mathfrak{Y}$. □

My original interest in this relation can be traced to the following proposition. Background for this result can be found in [**5**], [**6**].

8. PROPOSITION. *Let $\mathfrak{X}$, $\mathfrak{Y}$ be Banach spaces and let $(\Omega, \mathfrak{F}, \mu)$ be a finite measure space. If $\mathfrak{Y}$ has the μ-Pettis integral property and $\mathfrak{X} \prec \mathfrak{Y}$, then $\mathfrak{X}$ also has the μ-PIP.*

Proof. Let $\varphi: \Omega \to \mathfrak{X}$ be a bounded scalarly measurable function, and let $A \in \mathfrak{F}$. Define $\alpha \in X^{**}$ by $\alpha(f) = \int_A f \circ \varphi \, d\mu$, $f \in \mathfrak{X}^*$. I must show that $\alpha \in \mathfrak{X}$. This will be done by showing that $\alpha \in \mathfrak{Z}(\mathfrak{X}, \mathfrak{Y})$. Let T: $\mathfrak{X} \to \mathfrak{Y}$ be an operator. Then $T \circ \varphi$ is bounded and scalarly measurable $\Omega \to \mathfrak{Y}$. By hypothesis, the Pettis integral $y = \int_A T \circ \varphi \, d\mu$ exists in $\mathfrak{Y}$. Now for $g \in \mathfrak{Y}^*$, we have

$$T^{**}(\alpha)(g) = \alpha(T^*(g)) = \int_A T^*(g)(\varphi(\omega))\mu(d\omega)$$

$$= \int_A g(T(\varphi(\omega)))\mu(d\omega) = g(y).$$

So $T^{**}(\alpha) = y \in \mathfrak{Y}$. Thus $\alpha \in \mathfrak{Z}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}$. This shows that the Pettis integral $\int_A \varphi \, d\mu$ exists in $\mathfrak{X}$. Thus $\mathfrak{X}$ has the μ-PIP. □

Corollaries of this result include: If $\mathfrak{X}$ satisfies the condition of Mazur, then $\mathfrak{X}$ has the PIP [**6**]. If $\mathfrak{X} \prec l_\infty$ then $\mathfrak{X}$ has the μ-PIP for all μ such that l_∞ has the μ-PIP [**6**, Proposition 4.4].

9. PROPOSITION. *If $\mathfrak{Y}$ is weakly sequentially complete and $\mathfrak{X} \prec \mathfrak{Y}$, then $\mathfrak{X}$ is weakly sequentially complete.*

Proof. Let $(x_n) \subseteq \mathfrak{X}$ be a weakly Cauchy sequence. Define $\alpha \in \mathfrak{X}^{**}$ by $\alpha(f) = \lim f(x_n)$. In order to show $\alpha \in X$, we will show $\alpha \in \mathfrak{Z}(\mathfrak{X}, \mathfrak{Y})$. Let T: $\mathfrak{X} \to \mathfrak{Y}$ be an operator. Then $(T(x_n))$ is a weakly Cauchy sequence in $\mathfrak{Y}$, so it converges weakly, say to $y \in \mathfrak{Y}$. Then for $g \in \mathfrak{Y}^*$,

$$T^{**}(\alpha)(g) = \alpha(T^*(g)) = \lim T^*(g)(x_n)$$

$$= \lim g(T(x_n)) = g(y).$$

So $T^{**}(\alpha) = y \in \mathfrak{Y}$. This shows $\alpha \in \mathfrak{Z}(\mathfrak{X}, \mathfrak{Y}) = \mathfrak{X}$. So (x_n) converges weakly to $\alpha \in \mathfrak{X}$. □

A condition on a separable Banach space $\mathfrak{X}$ stronger than weak sequential completeness is the following: if $\alpha \in \mathfrak{X}^{**}$ is Borel measurable on $(\mathfrak{X}^*, \text{weak*})$, then $\alpha \in \mathfrak{X}$. An argument like the preceding one shows that if $\mathfrak{X} \prec \mathfrak{Y}$ and $\mathfrak{Y}$ satisfies this condition, then so does $\mathfrak{X}$. It seems to be unknown whether this condition is *equivalent* to weak sequential completeness (in a Banach space). Any counterexample would have to be a space that fails $\mathfrak{X} \prec l_1$. (See note added in proof.)

Godefroy and Talagrand [9] say that a Banach space $\mathfrak{X}$ has *property* (X) iff any $\alpha \in \mathfrak{X}^{**}$ such that $\alpha(\sum f_n) = \sum \alpha(f_n)$, for every sequence $(f_n) \subset \mathfrak{X}^*$ with $\sum |f_n(x)| < \infty$ for all $x \in \mathfrak{X}$, must be in $\mathfrak{X}$. (The sum $\sum f_n$ is taken in the weak* topology of $\mathfrak{X}^*$.)

10. Proposition. *Let $\mathfrak{X}$ be a Banach space. Then $\mathfrak{X} \prec l_1$ if and only if $\mathfrak{X}$ has property* (X).

Proof. Suppose $\mathfrak{X} \prec l_1$. Let $\alpha \in \mathfrak{X}^{**}$, and suppose $\alpha(\sum f_n) = \sum \alpha(f_n)$ for every sequence $(f_n) \subset \mathfrak{X}^*$ with $\sum |f_n(x)| < \infty$ for all $x \in \mathfrak{X}$. (This is a "weakly unconditionally Cauchy" series.) To show that $\alpha \in \mathfrak{X}$, we show that $\alpha \in \mathfrak{F}(\mathfrak{X}, l_1)$. Let $T\colon \mathfrak{X} \to l_1$ be an operator. Write e_n for the canonical unit vectors in $l_1^* = l_\infty$. Let $f_n = T^*(e_n)$. Then for any $x \in \mathfrak{X}$, we have

$$\sum |f_n(x)| = \sum |e_n(T(x))| = \|T(x)\| < \infty.$$

Thus $\alpha(\sum f_n) = \sum \alpha(f_n)$. If (a_n) is a bounded sequence of scalars, the same argument shows $\alpha(\sum a_n f_n) = \sum a_n \alpha(f_n)$. Define $u\colon \mathbf{N} \to \mathbf{R}$ by $u(n) = \alpha(f_n)$. Then for any $g = \sum a_n e_n \in l_\infty$, we have

$$\begin{aligned} T^{**}(\alpha)(g) &= \alpha(T^*(g)) = \alpha\left(T^*\left(\sum a_n e_n\right)\right) \\ &= \alpha\left(\sum a_n f_n\right) = \sum a_n \alpha(f_n) = g(u). \end{aligned}$$

So $T^{**}(\alpha) = u \in l_1$. Thus $\alpha \in \mathfrak{F}(\mathfrak{X}, l_1) = \mathfrak{X}$. This shows that property (X) holds.

Conversely, suppose property (X) holds. Let $\alpha \in \mathfrak{X}^{**}$, and suppose $T^{**}(\alpha) \in l_1$ for all operators $T\colon \mathfrak{X} \to l_1$. Let $(f_n) \subset \mathfrak{X}^*$ with $\sum |f_n(x)| < \infty$ for all $x \in \mathfrak{X}$. Then an operator $T\colon X \to l_1$ is defined by $T(x)(n) = f_n(x)$. Since $T^{**}(\alpha) \in l_1$, we have

$$T^{**}(\alpha)\left(\sum e_n\right) = \sum T^{**}(\alpha)(e_n),$$

or

$$\alpha\left(\sum f_n\right) = \sum \alpha(f_n).$$

So by property (X), we have $\alpha \in \mathfrak{X}$. This shows $\mathfrak{X} \prec l_1$. $\square$

Recall that $\mathfrak{X} \prec l_1$ implies that either $\mathfrak{X}$ is reflexive or $\mathfrak{X} \sim l_1$. Godefroy and Talagrand **[9]** show that the following Banach spaces have property (X) [assuming that there are no measurable cardinals, or that the spaces are small enough (e.g., separable) that measurable cardinals do not matter].

(i) A subspace of $L_1(\mu)$, where μ is any measure.

(ii) L_1/H_1.

(iii) A weakly sequentially complete direct summand of a Banach lattice.

(iv) Sequentially complete subspace of an order continuous Banach lattice.

(v) Predual of a W^*-algebra.

(vi) A space with local unconditional structure not containing l_∞^n uniformly.

Godefroy and Talagrand studied property (X) in connection with uniqueness of preduals. The ordering $\prec$ can be used in the same way. The following proof can be imitated with many other spaces in place of the James space J.

11. THEOREM. *Let J be the James quasireflexive space, and let $\mathfrak{X} \prec J$. Then $\mathfrak{X}$ is the unique isometric predual of $\mathfrak{X}^*$.*

Proof. First note that any predual of $\mathfrak{X}^*$ is canonically identified with a subspace of $\mathfrak{X}^{**}$.

(A) We first prove the following: Let $\mathfrak{X}_1 \subseteq \mathfrak{X}^{**}$ be a predual of $\mathfrak{X}^*$, and let S: $J^* \to \mathfrak{X}^*$ be an operator. Then S is $\sigma(J^*, J) - \sigma(\mathfrak{X}^*, \mathfrak{X})$ continuous if and only if S is $\sigma(J^*, J) - \sigma(\mathfrak{X}^*, \mathfrak{X}_1)$ continuous.

The James space J will be considered to be (as in [7]) the set of all functions f on the ordinal space $[0, \omega]$ satisfying

(i) $f(0) = 0$,

(ii) $\lim_{k<\omega} f(k) = f(\omega)$,

(iii) $\|f\| = \sup(\sum_{i=1}^n |f(k_i) - f(k_{i-1})|^2)^{1/2} < \infty$,

the sup is over all finite sequences $k_0 < k_1 < \cdots < k_n$ in $[0, \omega]$. The evaluation functionals are defined by $e_\alpha(f) = f(\alpha)$, $\alpha \in (0, \omega]$.

So suppose S is $\sigma(J^*, J) - \sigma(X^*, X)$ continuous. Then $S(e_k) \to S(e_\omega)$ in $\sigma(\mathfrak{X}^*, \mathfrak{X})$. To show that S is $\sigma(J^*, J) - \sigma(\mathfrak{X}^*, \mathfrak{X}_1)$ continuous, it suffices to show $S(e_k) \to S(e_\omega)$ in $\sigma(\mathfrak{X}^*, \mathfrak{X}_1)$. Now since J^{**} can be identified with the set of all f on $[0, \omega]$ satisfying (i) and (iii), above, we know $S(e_k)$ converges in $\sigma(\mathfrak{X}^*, \mathfrak{X}_1)$. Write h for the limit of $S(e_k) - S(e_\omega)$ in $\sigma(\mathfrak{X}^*, \mathfrak{X}_1)$. I must show $h = 0$.

Fix n, and let $k_0 < k_1 < \cdots < k_n$ be positive integers. In J^*, consider

$$u = \sum_{i=1}^{n} \frac{(-1)^{n-1}}{\sqrt{n}} (e_{k_i} - e_{k_{i-1}}).$$

Then $\|u\| \leq 1$, so $\|S(u)\| \leq \|S\|$. Thus:

$$\left\| \sum_{i=1}^{n} (-1)^{n-i} \frac{1}{\sqrt{n}} \left(S(e_{k_i} - e_\omega) - S(e_{k_{i-1}} - e_\omega) \right) \right\| \leq \|S\|$$

or

$$\frac{1}{\sqrt{n}} \left\| S(e_{k_n} - e_\omega) + (-1)^{n+1} S(e_{k_0} - e_\omega) + 2 \sum_{i=1}^{n-1} (-1)^{n-i} S(e_{k_i} - e_\omega) \right\| \leq \|S\|.$$

Now let k_n increase; we have $S(e_{k_n} - e_\omega) \to 0$ in $\sigma(\mathfrak{X}^*, \mathfrak{X})$ as $k_n \to \omega$ and $k_0, \ldots, k_{n-1}$ remain fixed, so

$$\frac{1}{\sqrt{n}} \left\| (-1)^{n+1} S(e_{k_0} - e_\omega) + 2 \sum_{i=1}^{n-1} (-1)^{n-i} S(e_{k_i} - e_\omega) \right\| \leq \|S\|.$$

Then let k_{n-1} increase; we have $S(e_{k_{n-1}} - e_\omega) \to h$ in $\sigma(\mathfrak{X}^*, \mathfrak{X}_1)$, so

$$\frac{1}{\sqrt{n}} \left\| (-1)^{n+1} S(e_{k_0} - e_\omega) + 2 \sum_{i=1}^{n-2} (-1)^{n-i} S(e_{k_i} - e_\omega) - 2h \right\| \leq \|S\|.$$

Then let k_{n-2} increase; we have $S(e_{k_{n-2}} - e_\omega) \to 0$ in $\sigma(\mathfrak{X}^*, \mathfrak{X})$. Then let k_{n-3} increase; we have $S(e_{k_{n-3}} - e_\omega) \to h$ in $\sigma(\mathfrak{X}^*, \mathfrak{X}_1)$. We get in the end

$$\frac{1}{\sqrt{n}} \|nh\| \leq \|S\|.$$

But this holds for all n, so $h = 0$. This shows that S is $\sigma(J^*, J) - \sigma(\mathfrak{X}^*, \mathfrak{X}_1)$ continuous.

(B) We next prove the following: if $\mathfrak{X}$ and $\mathfrak{X}_1$ have isometric dual spaces, then $\mathfrak{F}(\mathfrak{X}, J) = \mathfrak{F}(\mathfrak{X}_1, J)$.

We may assume $\mathfrak{X}_1 \subseteq \mathfrak{X}^{**}$. Now

$$\begin{aligned}\mathfrak{F}(\mathfrak{X}, J) &= \bigcap \{T^{**-1}(J) \mid T\colon \mathfrak{X} \to J\} \\ &= \bigcap \{S^{*-1}(J) \mid S\colon J^* \to \mathfrak{X}^*, S \text{ is weak* continuous}\}\end{aligned}$$

depends only on which operators $S\colon J^* \to \mathfrak{X}^*$ are weak* continuous, and by the above observation this is the same for $\mathfrak{X}_1$ and $\mathfrak{X}$.

(C) Finally, suppose $\mathfrak{X} \prec J$ and $\mathfrak{X}_1^*$ is isometric to $\mathfrak{X}^*$. If $\mathfrak{X}_1$ is identified with the appropriate subspace of $\mathfrak{X}^{**}$, we have $\mathfrak{X}_1 \subseteq \mathfrak{F}(\mathfrak{X}_1, J) = \mathfrak{F}(\mathfrak{X}, J) = \mathfrak{X}$. Similarly $\mathfrak{X} \subseteq \mathfrak{X}_1$. So $\mathfrak{X}$ is the unique predual of $\mathfrak{X}^*$. □

The relation $\prec$ is not a total order on the equivalence classes. The spaces c_0 and $J(\omega_1)$ of [7] are not comparable. An example using only separable spaces can be obtained using the James quasireflexive space J and Bourgain's $\mathfrak{L}_\infty$ space with the Schur property $\mathfrak{X}$ [**1**]. Then $J \prec \mathfrak{X}$ fails by Proposition 9, since $\mathfrak{X}$ is weakly sequentially complete, but J is not. And $\mathfrak{X} \prec J$ fails since J^{**} contains no copy of c_0, and $\mathfrak{X}^{**}$ is isomorphic to an $L_\infty(\mu)$ space, so all operators $\mathfrak{X}^{**} \to J^{**}$ are weakly compact [**3**, VI.1.2], and hence all operators $\mathfrak{X} \to J$ are weakly compact.

According to Propositions 2, 9, 10, the second-least equivalence class, the class of l_1, contains all nonreflexive spaces in the list after Proposition 10, and is contained in the class of all weakly sequentially complete spaces. However the class of l_1 does not contain all weakly sequentially complete spaces. We next give two illustrations of this.

12. PROPOSITION. *Let Γ be a set. Then $l_1(\Gamma) \prec l_1$ if and only if* card Γ *is not a (real-valued) measurable cardinal.*

Proof. Recall that $l_1(\Gamma)^* = l_\infty(\Gamma)$ [**4**, Theorem IV.8.5 and following Remark] and $l_\infty(\Gamma)^* = ba(\Gamma, \mathcal{P}(\Gamma))$, the space of all finitely additive, bounded, signed measures on the power set $\mathcal{P}(\Gamma)$ of Γ [**4**, Theorem IV.8.16]. We claim that the frame $\mathfrak{F}(l_1(\Gamma), l_1)$ is the subspace $ca(\Gamma, \mathcal{P}(\Gamma))$ of all countably additive signed measures on $\mathcal{P}(\Gamma)$. Since $l_1(\Gamma) = ca(\Gamma, \mathcal{P}(\Gamma))$ if and only if card Γ is not measurable, this will prove the result. Note that $l_1 = ca(\mathbf{N}, \mathcal{P}(\mathbf{N}))$. If $\mu \in ba(\Gamma, \mathcal{P}(\Gamma))$ and $T\colon l_1(\Gamma) \to l_1$, then $T^{**}(\mu) \in ba(\mathbf{N}, \mathcal{P}(\mathbf{N}))$. If μ is countably additive, so is $T^{**}(\mu)$: If $A_n \subseteq \mathbf{N}$, $A_n \downarrow \varnothing$, then $\chi_{A_n} \to 0$ (weak*) in l_∞, so $T^*(\chi_{A_n}) \to 0$ (weak*) in $l_\infty(\Gamma)$, and thus $T^*(\chi_{A_n}) \to 0$ pointwise, so by the Lebesgue dominated convergence theorem, $T^{**}(\mu)(\chi_{A_n}) \to 0$. This shows that $\mathfrak{F}(l_1(\Gamma), l_1) \supseteq ca(\Gamma, \mathcal{P}(\Gamma))$. Conversely, suppose $\mu \in \mathfrak{F}(l_1(\Gamma), l_1)$, and let $B_n \subseteq \Gamma$, B_n disjoint. Define $T: l_1(\Gamma) \to l_1$ by: $T(f)(n) = \sum_{\gamma \in B_n} f(\gamma), f \in l_1(\Gamma), n \in \mathbf{N}$. If $e_n \in l_\infty$ are the usual unit vectors and $u \in l_\infty$ is $u(n) = 1$, then $\sum e_n = u$ (weak*) in l_∞. But $T^{**}(\mu) \in l_1 = ca(N, \mathcal{P}(\mathbf{N}))$, so $\mu(\bigcup A_n) = T^{**}(\mu)(u) = \sum T^{**}(\mu)(e_n) = \sum \mu(A_n)$. Thus $\mu \in ca(\Gamma, \mathcal{P}(\Gamma))$. This shows that $\mathfrak{F}(l_1(\Gamma), l_1) = ca(\Gamma, \mathcal{P}(\Gamma))$. $\square$

Bourgain and Delbaen [1] give an example of a space $\mathfrak{X}$ which is a separable $\mathcal{L}_\infty$-space but has the property of Schur (so it is weakly sequentially complete). As noted above, $\mathfrak{X} \not\sim l_1$. Another reason for this is the following. Since $\mathfrak{X}$ is a $\mathcal{L}_\infty$-space, so is any complemented subspace, hence no complemented subspace is isomorphic to l_1. By the next result, $\mathfrak{X}$ is not in the equivalence class of l_1.

13. THEOREM. *Let $\mathfrak{X}$ be a Banach space, and suppose $\mathfrak{X} \prec l_1$. Then every bounded sequence in $\mathfrak{X}$ that is not relatively weakly compact has a subsequence equivalent to the unit vector basis of l_1 with closed span complemented in $\mathfrak{X}$.*

Proof. Write $(e_n)_{n=1}^\infty$ for the unit vector basis in l_1, and $(e_n^*)_{n=1}^\infty$ for the biorthogonal sequence in l_∞.

First recall that if $S: \mathfrak{X} \to l_1$ is an operator, and $(y_j)_{j=1}^\infty$ is a bounded sequence in $\mathfrak{X}$ with $S(y_j) = e_j$ for all j, then the span of $\{y_j\}$ is isomorphic to l_1, since S is an isomorphism there with inverse $R: l_1 \to \mathfrak{X}$ defined by $R(\sum a_j e_j) = \sum a_j y_j$, and the span of $\{y_j\}$ is complemented in $\mathfrak{X}$ with projection $P = RS$.

Next recall that if $(A_j)_{j=1}^\infty$ are disjoint finite sets in $\mathbf{N}$, then the span of $(\chi_{A_j})_{j=1}^\infty$ in l_1 is a complemented subspace isomorphic to l_1, since S: $l_1 \to l_1$ defined by $S(\sum_{n=1}^\infty a_n e_n) = \sum_{j=1}^\infty (\sum_{n \in A_j} a_n) e_j$ is a map as required above, with $y_j = \chi_{A_j}/\|\chi_{A_j}\|$.

So suppose $\mathfrak{X} \prec l_1$. Let (x_n) be a bounded sequence in $\mathfrak{X}$. Let $\alpha \in \mathfrak{X}^{**}$ be a fixed cluster point of (x_n). Assume $\alpha \notin \mathfrak{X}$.

Since $\mathfrak{X} \prec l_1$, there exists an operator $T: \mathfrak{X} \to l_1$ with $T^{**}(\alpha) \notin l_1$. That is, under the identification $l_1^{**} = ba(\mathbf{N}, \mathcal{P}(\mathbf{N}))$, the measure $A \mapsto \alpha(T^*(\chi_A))$ is not countably additive on $\mathcal{P}(\mathbf{N})$. So there exist disjoint sets

$A_k \subseteq \mathbf{N}$ with $\sum_{k=1}^{\infty} \alpha(T^*(\chi_{A_k})) \neq \alpha(T^*(\chi_{\cup A_k}))$. Composing T with a projection on l_1, we may assume without loss of generality that $A_k = \{k\}$, so that

$$\sum_{k=1}^{\infty} \alpha(T^*(e_k^*)) \neq \alpha(T^*(\chi_{\mathbf{N}})).$$

In fact (using another projection to suppress a finite number of coordinates) we may assume, for some $\varepsilon > 0$, that

$$\sum_{k=1}^{\infty} |\alpha(T^*(e_k^*))| < \frac{\varepsilon}{10} \quad \text{and} \quad |\alpha(T^*(\chi_{\mathbf{N}}))| > \varepsilon.$$

We now construct "blocks" $A_j = \{a_j + 1, a_j + 2, \ldots, a_{j+1}\}$ recursively. Let $a_1 = 0$. Choose m_1 so that

$$|T^*(\chi_{\mathbf{N}})(x_{m_1})| > \varepsilon.$$

(This is possible since α is a cluster point of the sequence (x_n).) Then choose $a_2 > a_1$ so that $A_1 = \{1, 2, \ldots, a_2\}$ satisfies

$$|T^*(\chi_{A_1})(x_{m_1})| > \varepsilon \quad \text{and} \quad \sum_{k \notin A_1} |T^*(e_k^*)(x_{m_1})| < \frac{\varepsilon}{10}.$$

This specifies a_2, m_1. Next note that

$$|\alpha(T^*(\chi_{\{a_2+1, a_2+2, \ldots\}}))| > \frac{9}{10}\varepsilon.$$

Choose $m_2 > m_1$ so that

$$|T^*(\chi_{\{a_2+1, \ldots\}})(x_{m_2})| > \frac{9}{10}\varepsilon \quad \text{and} \quad \sum_{k \in A_1} |T^*(e_k^*)(x_{m_2})| < \frac{\varepsilon}{10}.$$

Then choose $a_3 > a_2$ so that $A_2 = \{a_2 + 1, \ldots, a_3\}$ satisfies

$$|T^*(\chi_{A_2})(x_{m_2})| > \frac{9}{10}\varepsilon \quad \text{and} \quad \sum_{k=a_3+1}^{\infty} |T^*(e_k^*)(x_{m_2})| < \frac{\varepsilon}{10}.$$

Continuing in this manner, we get $m_1 < m_2 < m_3 < \cdots$ and $0 = a_1 < a_2 < a_3 < \cdots$ so that if $A_j = \{a_j + 1, \ldots, a_{j+1}\}$, then

$$(1) \qquad |T^*(\chi_{A_j})(x_{m_j})| > \frac{9}{10}\varepsilon \quad \text{for all } j,$$

$$(2) \qquad \sum_{k \notin A_j} |T^*(e_k^*)(x_{m_j})| < \frac{2}{10}\varepsilon \quad \text{for all } j,$$

$$(3) \qquad \sum_{j=1}^{\infty} |\alpha(T^*(\chi_{A_j}))| < \frac{\varepsilon}{10} < \varepsilon < |\alpha(T^*(\chi_{\mathbf{N}}))|.$$

Then (1) and (2) show that $(T(x_{m_j}))_{j=1}^{\infty}$ spans a complemented subspace of l_1, and $(T(x_{m_j}))$ is equivalent to the unit vector basis of l_1. Again composing T with a projection on l_1, we may assume that $T(x_{m_j}) = e_j$. [By (3) we still have $T^{**}(\alpha) \notin l_1$.] Thus we finally have that $(x_{m_j})_{j=1}^{\infty}$ is equivalent to the unit vector basis of l_1 and spans a complemented subspace of $\mathfrak{X}$. $\square$

I do not know whether the converse of this theorem holds. If it does, this is a non-trivial characterization of the equivalence class of l_1. It can be shown that $\mathfrak{X} \prec l_1$ is equivalent to the following more complicated condition: For any bounded sequence $(x_n)_{n=1}^{\infty}$ in $\mathfrak{X}$ that is not relatively weakly compact, and any cluster point α of (x_n) in $\mathfrak{X}^{**}$ but not in $\mathfrak{X}$, there exist a subsequence $(x_{m_j})_{j=1}^{\infty}$ equivalent to the unit vector basis of l_1 and a projection T onto the closed span of (x_{m_j}) such that $T^{**}(\alpha) \notin \mathfrak{X}$.

Since there is a least equivalence class (Proposition 1), is there a greatest equivalence class? There is a greatest class containing separable spaces, namely the class of c_0 (Proposition 3). But there is no greatest class:

14. PROPOSITION. *If $\mathfrak{X}$ is any Banach space, then there is a set Γ so large that $\mathfrak{X} \prec l_\infty(\Gamma)$ but $\mathfrak{X} \nsim l_\infty(\Gamma)$.*

Proof. Let the cofinality of the cardinal of Γ exceed the cardinal of $\mathfrak{X}^*$. Let γ be the least ordinal with the same cardinal as Γ. We will show that $C([0,\gamma]) \prec \mathfrak{X}$ fails. Since $C([0,\gamma])$ embeds in $l_\infty(\Gamma)$, this shows that $l_\infty(\Gamma) \prec \mathfrak{X}$ also fails.

To show that $C([0,\gamma]) \prec \mathfrak{X}$ fails, we will exhibit $\alpha \in \mathfrak{F}(C([0,\gamma]), \mathfrak{X})$ with $\alpha \notin C([0,\gamma])$. The dual $C([0,\gamma])^*$ can be identified with $l_1([0,\gamma])$. Let $\alpha \in l_1([0,\gamma])^*$ be defined by $\alpha(h) = h(\gamma)$, $h \in l_1([0,\gamma])$. Then $\alpha \notin C([0,\gamma])$.

Let T: $C([0,\gamma]) \to \mathfrak{X}$ be any operator. If $f \in \mathfrak{X}^*$, then the function $T^*(f) \in l^1([0,\gamma])$ vanishes outside some countable subset of $[0,\gamma]$. Since γ has cofinality greater than card$(\mathfrak{X}^*)$, there exists $\gamma_0 < \gamma$ such that all $T^*(f)$ vanish on the interval (γ_0, γ). Then we have

$$\begin{aligned} T^{**}(\alpha)(f) &= \alpha(T^*(f)) = T^*(f)(\gamma) \\ &= T^*(f)\big(\chi_{(\gamma_0,\gamma]}\big) = f\big(T\big(\chi_{(\gamma_0,\gamma]}\big)\big) \end{aligned}$$

so that $T^{**}(\alpha) = T(\chi_{(\gamma_0,\gamma]}) \in \mathfrak{X}$. This shows that $\alpha \in \mathfrak{F}(C([0,\gamma]), \mathfrak{X})$, which completes the proof. $\square$

Here is a permanence property of the class of l_1. Recall that $l_1(\Gamma) \prec l_1$ is and only if card Γ is not measurable.

15. PROPOSITION. *Let Γ be a set with $l_1(\Gamma) \prec l_1$. For each $\gamma \in \Gamma$, let $\mathfrak{X}_\gamma$ be a Banach space with $\mathfrak{X}_\gamma \prec l_1$. Then the l_1-direct sum*

$$\mathfrak{X} = \Big(\bigoplus_{\gamma\in\Gamma} \mathfrak{X}_\gamma \Big)_1$$

satisfies $\mathfrak{X} \prec l_1$.

Proof. For each $\gamma \in \Gamma$, let J_γ: $\mathfrak{X}_\gamma \to \mathfrak{X}$ be the coordinate embedding, and P_γ: $\mathfrak{X} \to \mathfrak{X}_\gamma$ the coordinate projection. Let $\alpha \in \mathfrak{F}(\mathfrak{X}, l_1)$. For fixed γ, consider $P_\gamma^{**}(\alpha) \in \mathfrak{X}_\gamma^{**}$. If S: $\mathfrak{X}_\gamma \to l_1$ is any operator, then SP_γ maps $\mathfrak{X}$ to l_1, so $(SP_\gamma)^{**}(\alpha) \in l_1$. This holds for all S, so $P_\gamma^{**}(\alpha) \in \mathfrak{F}(\mathfrak{X}_\gamma, l_1) = \mathfrak{X}_\gamma$. Write $u_\gamma = P_\gamma^{**}(\alpha)$.

Now if $\Gamma_0 \subseteq \Gamma$ is finite, then

$$\sum_{\gamma\in\Gamma_0} \|u_\gamma\| = \Big\| \sum_{\gamma\in\Gamma_0} J_\gamma(u_\gamma) \Big\| = \Big\| \sum_{\gamma\in\Gamma_0} J_\gamma P_\gamma^{**}(\alpha) \Big\| \leq \|\alpha\|,$$

so $u = \sum_{\gamma\in\Gamma} J_\gamma(u_\gamma)$ converges in the norm of $\mathfrak{X}$, and $P_\gamma(u) = u_\gamma$. I claim $u = \alpha$. Let $f \in \mathfrak{X}^*$. For each $\gamma \in \Gamma$, let $f_\gamma = J_\gamma^*(f) \in \mathfrak{X}_\gamma^*$. Define the map S: $\mathfrak{X} \to l_1(\Gamma)$ by $S(x)(\gamma) = f_\gamma(P_\gamma(x))$. Since $\mathfrak{X} \prec l_1 \prec l_1(\Gamma)$, we have $S^{**}(\alpha) \in l_1(\Gamma)$. Now if $\Gamma_0 \subseteq \Gamma$ is finite, we have

$$\begin{aligned}\Big(\sum_{\gamma\in\Gamma_0} P_\gamma^*(f_\gamma) \Big)(u) &= \sum_{\gamma\in\Gamma_0} f_\gamma(u_\gamma) = \sum_{\gamma\in\Gamma_0} \alpha(f_\gamma) \\ &= \alpha\big(S^*(\chi_{\Gamma_0})\big) = S^{**}(\alpha)(\chi_{\Gamma_0}).\end{aligned}$$

Take the limit as Γ_0 increases:

$$f(u) = S^{**}(\alpha)(\chi_\Gamma) = \alpha\big(S^*(\chi_\Gamma)\big) = \alpha(f).$$

Thus $\alpha = u \in \mathfrak{X}$. This shows $\mathfrak{X} \prec l_1$. $\square$

The referee suggested that I close the paper with some questions. I do not know the answers to the following, although I have not worked on all of them.

a. Is the converse of Theorem 13 true? I think this is the most interesting question listed here.

b. If $\mathfrak{X} \prec l_1$, does it follows that $L_1([0, 1], \mathfrak{X}) \prec l_1$? This is suggested by Proposition 15, which shows that if $l_1(\Gamma) \prec l_1$ and $\mathfrak{X} \prec l_1$ then $l_1(\Gamma, \mathfrak{X}) \prec l_1$. More generally, one could ask: If $\mathfrak{X} \prec l_1$ and $\mathfrak{Y} \prec l_1$, does it follows that $\mathfrak{X} \hat{\otimes} \mathfrak{Y} \prec l_1$? (Note that some related conjectures are refuted by the observation that $l_2 \hat{\otimes} l_2$ is not reflextive.) This is similar to the old problem: If $\mathfrak{X}$ and $\mathfrak{Y}$ are weakly sequentially complete, does it follows that $\mathfrak{X} \hat{\otimes} \mathfrak{Y}$ is weakly sequentially complete? (Recently refuted by Pisier.)

c. Is there a largest PIP. space? (Refer to Proposition 8.) Is there a space $\mathfrak{X}_0$ such that a Banach space $\mathfrak{X}$ has the PIP. if and only if $\mathfrak{X} \prec \mathfrak{X}_0$? If μ is a fixed measure, is there a space $\mathfrak{X}_\mu$ such that $\mathfrak{X}$ has the μ-PIP. if and only if $\mathfrak{X} \prec \mathfrak{X}_\mu$?

d. Similarly, is there a largest weakly sequentially complete space? (Refer to Proposition 9.) Is there a largest separable weakly sequentially complete space?

e. Is there a third-smallest equivalence class? (Compare with Proposition 2.) Does the collection of all equivalence classes greater than l_1 have a least element? a minimal element?

f. Describe the poset of equivalence classes of separable Banach spaces. How many classes are there? What is the largest cardinality of a chain? of an antichain? Is there an infinite decreasing sequence?

Note added in proof. M. Talagrand has told me that he has constructed a weakly sequentially complete space that fails the condition on Borel measurable functionals mentioned after Proposition 9.

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Received December 16, 1981 and in revised form March 8, 1982. Supported in part by N. S. F. Grant MCS 8003078. AMS 1980 subject classification. 46B20.

THE OHIO STATE UNIVERSITY
COLUMBUS, OH 43210