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q-KONHAUSER POLYNOMIALS

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A pair of biorthogonal sets of polynomials suggested by the q-Laguerre polynomials are constructed. These are biorthogonal on $(0, \infty)$ with respect to a continuous or discrete distribution function. Several properties are also given.

1. Introduction. Let $\alpha(x)$ be a distribution function on the interval (finite or infinite) $[a, b]$ with infinitely many points of increase and such that $\int_a^b x^n d\alpha(x) < \infty$ for all $n = 0, 1, 2, \ldots$.

The set of polynomials in x, $\{P_n(x)\}$, and the set of polynomials $\{Q_n(x)\}$, $\deg Q_n(x) = n$ for $n = 0, 1, 2, \ldots$ are said to be biorthogonal with respect to $d\alpha(x)$ on (a, b) if

$$(1.1) \qquad \int_a^b P_n(x) Q_m(x) d_\alpha(x) \begin{cases} = 0 & (n \neq m) \\ \neq 0 & (n = m) \end{cases}.$$

Didon [4] and Deruyts [3] considered this concept in some detail. For example for a given $\{P_n(x)\}$ the set $\{Q_n(x)\}$ is uniquely determined and conversely.

Both Didon and Deruyts paid special attention to the situation in which $P_n(x)$ is a polynomial of degree n in x^k (k fixed). In this case (1.1) is equivalent to

$$(1.2) \qquad \int_a^b x^i P_n(x) d\alpha(x) = 0 \quad \text{and} \quad \int_a^b x^{ik} Q_n(x) d\alpha(x) = 0 \qquad (0 \leq i < n).$$

and both integrals are $\neq 0$ for $i = n$.

Thus if $k = 1$, $\{P_n(x)\}$ and $\{Q_n(x)\}$ collapse to the set of orthogonal polynomials associated with $\alpha(x)$ on (a, b).

Both Didon and Deruyts gave as examples the case in which $d\alpha(x) = x^{\alpha-1}(1-x)^{\beta-1}\,dx$, the distribution for the Jacobi polynomials on $(0, 1)$. Deruyts also gave the case in which $d\alpha(x) = x^\alpha e^{-x} dx$ on $(0, \infty)$, the distribution for the Laguerre polynomials.

More recently these polynomials gained a sudden popularity with the interesting work of Konhauser [7, 8] and Preiser [10] (see also [2]). In particular the biorthogonal system related to the Laguerre distribution is now known as the Konhauser polynomials.

With the recent interest in orthogonal q-polynomials it has become of interest to look for a q-generalization of the Konhauser polynomials.

Our starting point would naturally be the q-Laguerre polynomials which were introduced by Hahn [5]. The polynomials belong to an indeterminate moment problem and thus there is more than one distribution function with respect to which the q-Laguerre polynomials are orthogonal. In particular there is a discrete distribution and a continuous one [9]. This is not a problem in our case since, as one might expect, it is the moments that really determine the orthogonal as well as the biorthogonal sets of polynomials.

2. Preliminaries. In this paper we shall use the following notation. For $|q| < 1$,

$$(a;q)_\infty = \prod_{j=0}^{\infty}(1-aq^j)$$

and, for arbitrary complex n,

$$(a;q)_n = (a;q)_\infty/(aq^n;q)_\infty,$$

so that in particular if $n = 1, 2, \ldots$ we have

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$$

in which case the restriction $|q| < 1$ is not necessary.

For writing economy we shall write $[a]_n$ to mean $(a;q)_n$. If the base is not q but, say p, then we shall mention it explicitly as $(a;p)_n$.

The q-derivative (base q) is $D_q f(x) = \{(x) - f(qx)\}/x$. Its nth iterate is [5]

$$D_q^n f(x) = x^{-n}\sum_{j=0}^{n}\frac{[q^{-n}]_j}{[q]_j}q^j f(xq^j). \tag{2.1}$$

The q-gamma function may be defined (see Askey [1] for an interesting treatment) by

$$\Gamma_q(x) = \frac{[q]_\infty}{[q^x]_\infty}(1-q)^{1-x}, \qquad 0 < q < 1.$$

The q-Laguerre polynomials

$$L_n^{(\alpha)}(x\,|\,q) = \frac{[q^{1+\alpha}]_n}{[q]_n}\sum_{j=0}^{n}\frac{[q^{-n}]_j q^{\frac{1}{2}j(j+1)+j(\alpha+n)}}{[q]_j[q^{1+\alpha}]_j}x^j$$

are orthogonal on $(0, \infty)$ with respect to the continuous distribution

$$d\Omega(\alpha, x) = \frac{Ax^\alpha}{[-x]_\infty}dx, \qquad (\alpha > -1) \tag{2.2}$$

where $A = \Gamma_q(-\alpha)/\Gamma(-\alpha)\Gamma(1+\alpha)(1-q)^{1+\alpha}$ or the discrete distribution $d\beta(\alpha, x)$ which has jumps $Bx^{\alpha+1}/[-x]_\infty$ at $x = q^k$, $k = 0, \pm 1, \pm 2, \ldots$. Where

$$B = \frac{2[q^{1+\alpha}]_\infty \{[-q]_\infty\}^2}{[-q^{1+\alpha}]_\infty [-q^{-\alpha}]_\infty [q]_\infty}.$$

The moments in either case are (see Moak [**9**])

$$\mu_n = [q^{1+\alpha}]_n q^{-\frac{1}{2}n(2\alpha+n+1)}, \qquad n = 0, 1, 2, \ldots. \tag{2.3}$$

The q-binomial theorem is

$$\sum_{n=0}^{\infty} \frac{[a]_n}{[q]_n} x^n = \frac{[ax]_\infty}{[x]_\infty}, \qquad (|x| < 1). \tag{2.4}$$

3. The q-Konhauser polynomials. We define for $n = 0, 1, 2, \ldots$

$$\begin{aligned} &Z_n^{(\alpha)}(x, k \mid q) \\ &= \frac{[q^{1+\alpha}]_{nk}}{(q^k; q^k)_n} \sum_{j=0}^{n} \frac{(q^{-nk}; q^k)_j q^{\frac{1}{2}kj(kj-1)+kj(n+\alpha+1)}}{(q^k; q^k)_j [q^{1+\alpha}]_{jk}} x^{kj} \end{aligned} \tag{3.1}$$

and

$$\begin{aligned} Y_n^{(\alpha)}(x, k \mid q) = \frac{1}{[q]_n} &\sum_{r=0}^{n} \frac{x^r q^{\frac{1}{2}r(r-1)}}{[q]_r} \\ &\times \sum_{j=0}^{r} \frac{[q^{-r}]_j (q^{1+\alpha+j}; q^k)_n}{[q]_j} q^j \end{aligned} \tag{3.2}$$

and prove that

$$\int_0^\infty Z_n^{(\alpha)}(x, k \mid q) Y_m^{(\alpha)}(x, k \mid q) d\Omega(\alpha, x) = k_n \delta_{nm}, \tag{3.3}$$

where

$$K_n = \frac{[q^{1+\alpha}]_{nk}}{[q]_n} q^{-nk}.$$

Formula (3.1), (3.2) and (3.3) reduce for $k = 1$ to the q-Laguerre polynomials $L_n^{(\alpha)}(x \mid q)$ and its orthogonality relation (2.2). To prove (3.3) it is necessary and sufficient to show

$$(3.3(\mathrm{a}))\quad I_{n,m} \equiv \int_0^\infty x^m Z_n^{(\alpha)}(x, k \mid q)\, d\Omega(\alpha, x) \begin{cases} = 0, & 0 \le m < n, \\ \ne 0, & m = n, \end{cases}$$

and

$$(3.3(\mathrm{b}))\quad J_{n,m} \equiv \int_0^\infty x^{km} Y_n^{(\alpha)}(x, k \mid q)\, d\Omega(\alpha, x) \begin{cases} = 0, & 0 \le m < n, \\ \ne 0, & m = n. \end{cases}$$

Proof of (3.3(a)). in the left hand side of (3.3(a)) substituting for $Z_n^{(\alpha)}(x, k \mid q)$ from (3.1) and integrating term by term and using (2.3), we get

$$(3.4)\qquad I_{n,m} = \frac{\left[q^{1+\alpha}\right]_{nk} q^{-\frac{1}{2}m(m+2\alpha+1)}}{\left(q^k; q^k\right)_n} \times \sum_{j=0}^{n} \frac{\left(q^{-nk}; q^k\right)_j \left[q^{1+\alpha+kj}\right]_m}{\left(q^k; q^k\right)_j} q^{kj(n-m)}$$

$$= \frac{(-1)^m \left[q^{1+\alpha}\right]_{nk}}{\left(q^k; q^k\right)_n} \left[D_{p^k}^n \left(xp^{1+\alpha}; p\right)_m\right]_{x=1}$$

where $p = q^{-1}$. The last equality is obtained by replacing $p = 1/q$ in the summation that appears in (3.4), simplifying and then comparing with (2.1).

Now $(xp^{1+\alpha}; p)_m$ is a polynomial in x of degree m. Hence for $m = 0, 1, \dots, n-1$ its q-difference is zero, whereas for $m = n$ we get

$$(3.5)\qquad I_{n,n} = (-1)^n \left[q^{1+\alpha}\right]_{nk} q^{-\frac{1}{2}nk(n+1) - \frac{1}{2}n(2\alpha+n+1)}.$$

This completes the proof of (3.3(a)).

To prove (3.3(b)) we require the following formula which is a q-analog of a result of Carlitz [**2**].

$$(3.6)\qquad \left(q^{-ki}; q^k\right)_m = \sum_{r=0}^{m} \frac{\left(q^{1+\alpha+ki}\right)_r}{[q]_r} q^{-r(1+\alpha+ki)} \times \sum_{s=0}^{r} q^s \frac{\left[q^{-r}\right]_s \left(q^{1+\alpha+s}; q^k\right)_m}{[q]_s}.$$

Formula (3.6) can be proved by using Jackson's q-analog of Taylor's theorem **[6]** for polynomials of degree $\leq m$,

$$f(x) = \sum_{r=0}^{m} D_q^r f(x)\,|_{x=1} \frac{x^r[1/x]_r}{[q]_r}. \tag{3.7}$$

Put $f(x) = (xq^{1+\alpha}; q^k)_m$ in (3.7) to get

$$(xq^{1+\alpha}; q^k)_m = \sum_{r=0}^{m} \frac{x^r[1/x]_r}{[q]_r} \sum_{s=0}^{r} \frac{[q^{-r}]_s}{[q]_s} q^s (q^{1+\alpha+s}; q^k)_m \tag{3.8}$$

which for $x = q^{-1-\alpha-ki}$ reduces to (3.6).

Proof of (3.3(b)). Substitute for $Y_n^{(\alpha)}(x, k \mid q)$ from (3.2) in the left hand side of (3.3(b)), integrating term by term, then using (3.6) we get

$$J_{n,m} = \frac{[q^{1+\alpha}]_{km}}{[q]_n} q^{-\frac{1}{2}km(2\alpha+1+km)} (q^{-km}; q^k)_n. \tag{3.9}$$

Since $(q^{-km}; q^k)_n = 0$ for $m = 0, 1, 2, \ldots, n-1$ and

$$J_{n,n} = (-)^n \frac{[q^{1+\alpha}]_{kn}}{[q]_n} (q^{-km}; q^k)_n q^{-\frac{1}{2}nk(nk+2\alpha+n+2)}, \tag{3.10}$$

hence the proof of (3.3(b)) is complete.

Furthermore (3.10), put together with the fact that the leading coefficient of $Z_n^{(\alpha)}(x; k \mid q)$ is $(-)^n q^{\frac{1}{2}kn(kn+2\alpha+n)}$, yields (3.3).

4. Properties of $Z_n^{(\alpha)}(x, k \mid q)$ **and** $Y_n^{(\alpha)}(x, k \mid q)$. We devote this section to some of the interesting properties of the polynomials $Z_n^{(\alpha)}(x, k \mid q)$ and $Y_n^{(\alpha)}(x, k \mid q)$ introduced in §3. We mention below first some of the properties of $Z_n^{(\alpha)}(x, k \mid q)$

$$\sum_{n=0}^{\infty} \frac{Z_n^{(\alpha)}(x, k \mid q)}{[q^{1+\alpha}]_{nk}} t^n = \frac{f(tx^k)}{(t; q^k)_\infty}, \tag{4.1}$$

where

$$f(u) = \sum_{j=0}^{\infty} \frac{q^{\frac{1}{2}kj(kj+j+2\alpha)}}{(q^k; q^k)_j [q^{1+\alpha}]_{kj}} (-u)^j.$$

$$\begin{aligned} &Z_n^{(\alpha)}(xy, k \mid q) \\ &\quad = \sum_{j=0}^{n} \frac{[q^{1+\alpha}]_{kn}}{[q^{1+\alpha}]_{kn-kj}} \frac{(1/y^k; q^k)_j}{(q^k; q^k)_j} y^{kj} Z_{n-j}^{(\alpha)}(x, k \mid q). \end{aligned} \tag{4.2}$$

If $Z_n^{(\alpha)}(x, k \mid q) = \Sigma_{m=0}^n c(n, m) Z_m^{(\beta)}(x, k \mid q)$ then

$$(4.3)\quad c(n, m) = \frac{\left[q^{1+\alpha}\right]_{nk} q^{km(\alpha-\beta)}}{\left[q^{1+\alpha}\right]_{mk} \left(q^k; q^k\right)_{n-m}} \times \sum_{j=0}^{n-m} \frac{\left(q^{-nk+mk}; q^k\right)_j \left[q^{1+\beta+km}\right]_{jk}}{\left(q^k; q^k\right)_j \left[q^{1+\alpha+km}\right]_{kj}} \cdot q^{kj(n-m+\alpha-\beta)}$$

for $k = 1$ this reduces to the connection coefficient for the q-Laguerre polynomials.

$$(4.4)\quad \left\{D_p^k x^{\alpha+1} D_p\right\} Z_n^{(\alpha)}(x, k \mid q) = (-)^k \frac{\left[q^{1+\alpha}\right]_{nk}}{\left[q^{1+\alpha}\right]_{nk-k}} x^\alpha Z_{n-1}^{(\alpha)}(x, k \mid q),$$

$$(4.5)\quad q^{\frac{1}{2}k(k+2\alpha+1)} x^k Z_n^{(\alpha+k)}(x, k \mid q) = \left[q^{1+\alpha+kn}\right]_k Z_n^{(\alpha)}(x, k \mid q) - Z_{n+1}^{(\alpha)}(x, k \mid q)\left(1 - q^{k(n+1)}\right).$$

If $x^{kn} = \Sigma_{m=0}^n D(n, m) Z_m^{(\alpha)}(x, k \mid q)$ then

$$(4.6)\quad D(n, m) = \frac{\left[q^{1+\alpha}\right]_{kn} \left(q^{-kn}; q^k\right)_m}{\left[q^{1+\alpha}\right]_{km}} q^{\frac{1}{2}kn(kn+2\alpha+1)}.$$

Proof of (4.1). Substituting from (3.1) in the left hand side of (4.1), changing the order of summations and summing the resulting inner series by q-binomial theorem, we get the right hand side of (4.1).

Proof of (4.2). In (4.1) replacing x by xy and in the right hand side of the resulting identity expanding $(ty^k; q^k)_\infty/(t; q^k)_\infty$ by q-binomial theorem and equating the coefficients of t^n on both sides we get (4.2).

Proof of (4.3). Multiplying both sides of (4.3) by $Y_i^{(\beta)}(x, k \mid q) d\Omega(\beta, x)$ where $0 \leq i \leq n$ and integrating from 0 to ∞, we get the desired value of $c(n, m)$ on using (3.3), (3.3(b)) and (3.9).

Proof of (4.4)–(4.6) follow by routine methods hence the details are omitted. In a similar manner one can obtain the following properties of the polynomials $Y_n^{(\alpha)}(x, k \mid q)$

$$(4.7)\quad Y_n^{(\alpha)}(x; k \mid q) = \sum_{m=0}^n \frac{\left(q^k; q^k\right)_n [q]_m \left(q^{\alpha-\beta}; q^k\right)_{n-m}}{\left(q^k; q^k\right)_m [q]_n \left(q^k; q^k\right)_{n-m}} q^{m(\alpha-\beta)} Y_m^{(\beta)}(x, k \mid q)$$

If $x^n = \sum_{m=0}^{n} D(n, m) Y_m^{(\alpha)}(x, k \mid q)$ then

$$D(n, m) = (-1)^{n+m} q^{-m\alpha - \frac{1}{2}km(m-1)} [q^{-n}]_m \times \sum_{j=0}^{n-m} \frac{[q^{-n+m}]_j (q^{k+km}; q^k)_j}{[q^{1+m}]_j (q^k; q^k)_j} q^{-j(km+\alpha)}, \tag{4.8}$$

$$Y_n^{(\alpha)}(x; k \mid q) = [-x]_\infty x^{k-\alpha-1} \left[D_{q^k}^n \left\{ \frac{x^{\beta+n}}{[-x^{1/k}]_\infty} \right\} \right]_{x^k} \tag{4.9}$$

where $\beta = (1 + \alpha - k)/k$.

Once more we remark that (4.7), (4.8), (4.9) reduce, when $k = 1$, to corresponding properties for the q-Laguerre polynomials.

Other formulas which are q-analogs of known results on the Konhauser polynomials can be easily obtained.

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