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ON SINGULARITY OF HARMONIC MEASURE IN SPACE

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We construct a topological ball D in $\mathbf{R}^3$, and a set E on ∂D lying on a 2-dimensional hyperplane so that E has Hausdorff dimension one and has positive harmonic measure with respect to D. This shows that a theorem of Øksendal on harmonic measure in $\mathbf{R}^2$ is not true in $\mathbf{R}^3$. Suppose D is a bounded domain in $\mathbf{R}^m$, $m \geq 2$, $\mathbf{R}^m \setminus D$ satisfies the corkscrew condition at each point on ∂D; and E is a set on ∂D lying also on a BMO_1 surface, which is more general than a hyperplane; then we can prove that if E has $m-1$ dimensional Hausdorff measure zero then it must have harmonic measure zero with respect to D.

Lavrentiev (1936) found a simply-connected domain D in $\mathbf{R}^2$ and a set E on ∂D which has zero linear measure and positive harmonic measure with respect to D [5]. McMillan and Piranian subsequently simplified the example [6]. See also [1] and [3].

In [7], Øksendal proved that if D is a simply-connected domain in $\mathbf{R}^2$, and E is a set on ∂D with vanishing linear measure, and if E is situated on a line, then E has vanishing harmonic measure $\omega(E, D)$ with respect to D. In [3], Kaufman and Wu generalized this result and proved that the theorem still holds if E is situated on a quasi-smooth curve, but no longer holds if E is situated on a quasi-conformal circle. An interesting, perhaps very difficult, question is whether the theorem is true if E lies on a rectifiable curve.

Another question is the higher dimensional generalization: if D is a topological ball in $\mathbf{R}^m$, $m \geq 3$, and E is a set on ∂D, situated also on an $m-1$ dimensional hyperplane, does the vanishing of the $m-1$ dimensional Hausdorff measure, $\Lambda^{m-1}(E) = 0$, imply that $\omega(E, D) = 0$?

We answer this negatively by giving the following example.

EXAMPLE. There exists a topological ball D in $\mathbf{R}^3$, and a set E on ∂D, lying on a 2-dimensional hyperplane so that E has Hausdorff dimension one but has positive harmonic measure with respect to D.

We notice that $\dim E = 1$ is much stronger than $\Lambda^2(E) = 0$; and that 1 is best possible, because if $\dim E < 1$ then E has zero capacity in $\mathbf{R}^3$, hence E has zero harmonic measure with respect to D in $\mathbf{R}^3$.

Also this example suggests that a question left open in [**1**] by Carleson has no analogue in higher dimensions: if E is a set on the boundary of a Jordan domain D, and $\Lambda^{\beta}(E) = 0$ for some $1/2 < \beta < 1$, is it true that $\omega(E, D) = 0$?

The real reason behind the example is that the Carleman-Milloux type estimation of harmonic measure is no longer valid on the boundary of a topological ball in $\mathbf{R}^3$. In order to obtain positive results we require the complement of the domain to be "big" near each boundary point, and allow E to lie on a surface more general than a hyperplane.

THEOREM. *Suppose D is a bounded domain in $\mathbf{R}^m$, $m \geq 2$, whose complement $\mathbf{R}^m \setminus D$ satisfies the corkscrew condition. Let Γ be a topological sphere in $\mathbf{R}^m$, whose interior Ω_1 and exterior Ω_2 are both* NTA *domains, and on Γ,*

$$(0.1) \qquad \Lambda^{m-1}(E) = 0 \Rightarrow \omega(E, \Omega_i) = 0 \quad \textit{for } i = 1 \textit{ and } 2.$$

Then a set on $\partial D \cap \Gamma$ having zero Λ^{m-1} measure must have zero harmonic measure with respect to D.

The definitions of corkscrew condition and NTA domain are introduced by Jerison and Kenig in [**2**] and are given below.

Examples of Γ that satisfy the conditions in Theorem 2 are quasi-smooth curves ($m = 2$) and boundaries of BMO_1 domains ($m \geq 3$); BMO_1 domains are domains whose boundaries are given locally as the graph of a function ϕ with $\nabla\phi \in \text{BMO}$, see [**2**] for more discussions. In these examples, the harmonic measures ω_i on Γ and Λ^{m-1} are mutually absolutely continuous, in fact, $\omega_i \in A_\infty(\Lambda^{m-1})$.

When $m = 2$, the theorem by Kaufman and Wu [**3**] mentioned before is not comparable to Theorem 2. There, D is only simple-connected; however, Γ has a stronger property, namely, quasi-smooth.

From the Example, we see that the corkscrew condition on $\mathbf{R}^m \setminus D$ cannot be discarded even when D is a topological ball. Also condition (0.1) is necessasry as one can see in the case $D = \Omega_1$ or Ω_2. However, we do not know whether the geometric condition on $\Gamma : \Omega_i$ are NTA domains, can be weakened, or whether Γ can be replaced by a simple rectifiable curve in $\mathbf{R}^2$.

1. An example. We call D a *topological ball* in $\mathbf{R}^m$ if it is the image of a ball under a homeomorphism of R^m. And the boundary of a topological ball is called topological sphere. For $A \in \mathbf{R}^m$, $r > 0$, we let $B(A, r) = \{P \in \mathbf{R}^m : |A - P| < r\}$.

For a domain D in $\mathbf{R}^m$, $E \subseteq \partial D$, we denote by $\omega^X(E, D)$ the harmonic measure of E at X with respect to D.

LEMMA 1. *In* $\mathbf{R}^2$, *there exists a simply-connected Jordan domain* Ω, *satisfying*

(1) $\Omega \cap \{x\colon x_1 > 0\} \subseteq \{x\colon |x| < 2\}$
$\Omega \cap \{x\colon x_1 < 0\} = \{x\colon x_1 < 0, |x| < 3\}$;

(2) $\partial_2\Omega$ *has Hausdorff dimension* 1;

(3) $\operatorname{cap}_3(\partial_2\Omega) > 0$;

(4) $\operatorname{cap}_3(\Omega_\varepsilon) \to 0$ *as* $\varepsilon \to 0$;

where $\Omega_\varepsilon = \{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) < \varepsilon\}$, $\partial_2\Omega$ *is the boundary of* Ω *relative to* $\mathbf{R}^2$, *and* cap_3 *is the capacity with respect to the kernel* $1/|x|$.

Lemma 1 is proved at the end of this section; some readers may prefer to supply their own construction. The next lemma is the key to our example.

LEMMA 2. *Let* Ω *be a domain in* $\mathbf{R}^2$ *with all the properties in Lemma* 1. *We identify it with the set* $\{(x, 0)\colon x \in \Omega\}$ *in* $\mathbf{R}^3$. *Then*

$$\omega\left(\partial_2\Omega, B(0,20) \setminus \overline{\Omega}\right) > 0.$$

Proof. Choose $\varepsilon_0 > 0$ so that

$$\operatorname{cap}_3\left(\Omega_{2\varepsilon_0}\right) < \frac{1}{100}\operatorname{cap}_3(\partial_2\Omega). \tag{1.1}$$

Let $\Omega_{\varepsilon_0,\eta} = \Omega_{\varepsilon_0} \setminus \overline{\Omega}_\eta$, for $0 < \eta < \varepsilon_0$, let μ and ν be the capacitary measures corresponding to $\partial_2\Omega$ and $\overline{\Omega}_{\varepsilon_0,\eta}$, with respect to the kernel $1/|x|$, respectively. Let U and V be the corresponding equilibrium potentials:

$$U(x) = \int_{\partial_2\Omega} \frac{1}{|x-y|}\, d\mu(y), \tag{1.2}$$

$$V(x) = \int_{\overline{\Omega}_{\varepsilon_0,\eta}} \frac{1}{|x-y|}\, d\nu(y). \tag{1.3}$$

We recall from [4] that U and V are positive superharmonic on $\mathbf{R}^3$ bounded by 1 and are harmonic off the supports of their respective capacitary measures; moreover $U = 1$ on $\partial_2\Omega$ except possibly on a set S with $\operatorname{cap}_3(S) = 0$ and $V = 1$ on $\overline{\Omega}_{\varepsilon_0,\eta}$ except possibly on a set T with $\operatorname{cap}_3(T) = 0$; $\mu(\partial_2\Omega) = \operatorname{cap}_3(\partial_2\Omega)$ and $\nu(\overline{\Omega}_{\varepsilon_0,\eta}) = \operatorname{cap}_3(\overline{\Omega}_{\varepsilon_0,\eta})$.

Let $u = \omega(\partial_2\Omega, B(0,20)\setminus\partial_2\Omega)$ and $v = \omega(\overline{\Omega}_{\varepsilon_0,\eta}, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0,\eta})$. We observe from the last paragraph that

$$(1.4)\qquad u(X) \geq U(X) - \int_{|Y|=20} U(Y)\, d\omega^X(Y, B(0,20))$$

for $X \in B(0,20)\setminus\partial_2\Omega$; and clearly $U \geq u$ and $V \geq v$ in their common domains.

For $6 \leq |X| \leq 20$ it follows from Lemma 1, (1.1), (1.2) and (1.3) that

$$(1.5)\qquad \begin{aligned} V(X) &\leq \frac{1}{3}\operatorname{cap}_3\left(\overline{\Omega}_{\varepsilon_0,\eta}\right) < \frac{1}{300}\operatorname{cap}_3(\partial_2\Omega) \\ &< \frac{23}{300}U(X) < \frac{1}{10}U(X); \end{aligned}$$

for $|X| = 6$, it follows from (1.2), (1.4) and (1.5) that

$$(1.6)\qquad \begin{aligned} u(X) &\geq \frac{1}{3}U(X) + \frac{2}{3}U(X) - \frac{1}{17}\operatorname{cap}_3(\partial_2\Omega) \\ &\geq \frac{10}{3}V(X) + \frac{2}{27}\operatorname{cap}_3(\partial_2\Omega) - \frac{1}{17}\operatorname{cap}_3(\partial_2\Omega) > 3v(X). \end{aligned}$$

From the maximum principle, it follows that for $|X| = 6$ and $0 < \eta < \varepsilon_0$,

$$(1.7)\qquad \begin{aligned} \omega^X\left(\partial_2\Omega, B(0,20)\setminus\left(\overline{\Omega}_{\varepsilon_0,\eta} \cup \partial_2\Omega\right)\right) &> u - v(X) > \frac{2}{3}u(X) \\ &> \frac{1}{100}\operatorname{cap}_3(\partial_2\Omega) > 0, \end{aligned}$$

by the estimation in (1.6).

From (1.7) and the maximum principle, we obtain for $|X| = 6$,

$$\begin{aligned} \omega^X\left(\partial_2\Omega, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}\right) &= \inf_{0<\eta<\varepsilon_0} \omega^X\left(\Omega_\eta \cup \partial_2\Omega, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}\right) \\ &\geq \inf_{0<\eta<\varepsilon_0} \omega^X\left(\partial_2\Omega, B(0,20)\setminus\left(\overline{\Omega}_{\varepsilon_0,\eta/2} \cup \partial_2(\Omega)\right)\right) \\ &> \frac{1}{100}\operatorname{cap}_3(\partial_2\Omega) > 0. \end{aligned}$$

Let $\alpha = \sup\{\omega^X(\partial_2\Omega, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}) \colon x \in \Omega\setminus\Omega_{\varepsilon_0}\}$. Because $\Omega\setminus\Omega_{\varepsilon_0}$ has positive distance from $\partial_2\Omega$, we have $0 < \alpha < 1$. Choose β, $\alpha < \beta < 1$, and a point P in $B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}$ so that $\omega^P(\partial_2\Omega, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}) > \beta$. By the maximum principle,

$$\omega^P\left(\partial_2\Omega, B(0,20)\setminus\overline{\Omega}\right) \geq \omega^P\left(\partial_2\Omega, B(0,20)\setminus\overline{\Omega}_{\varepsilon_0}\right) - \alpha > \beta - \alpha > 0.$$

This completes the proof.

LEMMA 3. *Let Ω be the domain in Lemma* 1. *Let $g(x)$ be a strictly positive continuous function on Ω, defined by*

$$g(x) = \tfrac{1}{4}\operatorname{dist}(x, \partial_2\Omega). \tag{1.8}$$

Let

$$G = \{(x_1, x_2, x_3) \colon (x_1, x_2) \in \Omega \text{ and } |x_3| < g(x_1, x_2)\}.$$

Then

$$\omega(\partial_2\Omega, B(0,20) \setminus \overline{G}) > 0.$$

Proof. Suppose otherwise, we have

$$\omega(\partial_2\Omega, B(0,20) \setminus \overline{G}) = 0. \tag{1.9}$$

Let $X \in \overline{G} \setminus \overline{\Omega}$, Δ_X be the disk on $\{x_3 = 0\}$ with center $(X_1, X_2, 0)$ and of radius $|X_3|$ and B_X be the ball in $\mathbf{R}^3$ with center $(X_1, X_2, 0)$ and of radius $2|X_3|$. By (1.8) and the maximum principle, we have for $X \in \overline{G} \setminus \overline{\Omega}$,

$$\omega^X(\partial_2\Omega, B(0,20) \setminus \overline{\Omega}) \le \omega^X(\partial B_X, B_X \setminus \overline{\Delta(X)}) = C < 1, \tag{1.10}$$

where C is an absolute constant. Let A be any point in $B(0,20) \setminus \overline{G}$. Because of (1.9) and (1.10) we have

$$\begin{aligned} &\omega^A(\partial_2\Omega, B(0,20) \setminus \overline{\Omega}) \\ &\quad = \omega^A(\partial_2\Omega, B(0,20) \setminus \overline{G}) \\ &\qquad + \int_{\partial G \setminus \partial_2\Omega} \omega^X(\partial_2\Omega, B(0,20) \setminus \overline{\Omega})\, d\omega^A(X, B(0,20) \setminus \overline{G}) \\ &\quad = 0 + C < 1. \end{aligned} \tag{1.11}$$

From (1.10) and (1.11) we see that

$$\omega(\partial_2\Omega, B(0,20) \setminus \overline{\Omega}) < C < 1$$

everywhere in $B(0,20) \setminus \overline{\Omega}$. Therefore, $\omega(\partial_2\Omega, B(0,20) \setminus \overline{\Omega}) = 0$. This contradicts Lemma 2 and proves Lemma 3.

Finally, we let Ω and G be the domains in Lemma 1 and Lemma 3,

$$D = \{(x_1, x_2, x_3) \colon x_1^2 + x_2^2 < 8 \text{ and } |x_3| < 4\} \setminus \overline{G}$$

and

$$E = \partial_2\Omega \cap \{x \colon |x| \le 2\}.$$

From the constructions of Ω and G, the domain D is a topological ball; from properties (1) and (2) in Lemma 1, $\dim E = 1$ and

$$\operatorname{cap}_3(\partial_2\Omega \cap \{x \colon |x| > 2\}) = 0.$$

Therefore by Lemma 3,

$$\omega(E, B(0,20)\setminus\overline{G}) > 0.$$

Arguing as in the last paragraph of the proof of Lemma 2, we conclude

$$\omega(E, D) > 0.$$

Consequently all the properties of D and E in our example are justified.

It remains to prove Lemma 1.

Proof of Lemma 1. All line segments considered below are closed. Let $l_{0,1}$ be the line segment with end points $(0,-1)$ and $(0,1)$. Let $l_{1,m}$, $m = 1,2$, be two horizontal line segments with left endpoints $(0, -\frac{1}{2})$ and $(0, \frac{1}{2})$ respectively and of length 1.

Suppose $\{l_{n-1,m}: 1 \le m \le 2^{n(n-1)/2}\}$ have been selected for some $n \ge 2$, so that length of $l_{n-1,m}$ is $2^{-(n-1)(n-2)/2}$. Subdivide each $l_{n-1,m}$ into 2^n equal subintervals, each of length $2^{-1-n(n-1)/2}$. Let $\{l_{n,j}: 1 \le j \le 2^{(n+1)n/2}\}$ be horizontal (if n is odd) or vertical (if n is even) line segments of length $2^{-n(n-1)/2}$, with left (if n is odd) or lower (if n is even) endpoints coinciding with those of the subintervals of $l_{n-1,m}$ and disjoint from any $l_{n-2,m'}$. We notice that the distance between two disjoint line segments $l_{n,m}$ and $l_{n',m'}$ $(n \ge n')$ is at least $2^{-1-n(n-1)/2}$.

Let $R_{0,1}$ be the semidisk $\{x: x_1 < 0, |x| < 3\}$ in R^2. We shall attach a thin rectangle to each $l_{n,m}$, $n \ge 1$. Let $a_n = 2^{-2^{3n}}$ and consider, for $n \ge 1$, the rectangle with one side coinciding with $l_{n,m}$, two opposite sides of length a_n, and interior disjoint from any $l_{n',m'}$. Let $R_{n,m}$ be the interior of this rectangle together with the open line segment $S_{n,m}$ which is the side of length a_n and lies on some $l_{n-1,m'}$.

Let

$$\Omega = \bigcup_{n=0}^{\infty} \bigcup_{m=1}^{2^{n(n+1)/2}} R_{n,m}, \quad \Omega_N = \bigcup_{n=0}^{N} \bigcup_{m=1}^{2^{n(n+1)/2}} R_{n,m}.$$

We claim that Ω is simply-connected Jordan. Using induction and the fact that

$$|l_{n+1,m}| = 2^{-(n+1)n/2} < 2^{-1-n(n-1)/2} = \operatorname{dist}(l_{n,m}, l_{n,m'}) \quad \text{for } m \ne m',$$

we see that Ω_n is Jordan simply-connected for each n. Since the distance between two disjoint $l_{n,m}$ and $l_{n',m'}$ $(n \ge n')$ is at least $2^{-1-n(n-1)/2}$ and

$$\sum_{k=n+1}^{\infty} |l_{k,1}| < 2^{-1-n(n-1)/2} - a_n, \quad \text{for } n \ge 3,$$

it follows from the construction of Ω that Ω is simply connected Jordan.

Property (1) in Lemma 1 can be verified easily.

We claim that $\partial_2\Omega$ has Hausdorff dimension one. Let $\delta > 0$ and $r = 2^{-1-n(n-1)/2}$, which is the distance between two disjoint $l_{n,m}$ and $l_{n,m'}$. From the construction, we see that $\partial_2\Omega$ can be covered by a family of K squares, each of side length r, and K no greater than

$$C\left(2^{n(n+1)/2} + \sum_{k=0}^{n-1} \sum_{j=1}^{2^{(k+1)k/2}} |l_{k,j}|/2^{-1-n(n-1)/2}\right) \leq C2^{n(n+1)/2}.$$

Therefore the $(1+\delta)$-dimensional Hausdorff measure satisfies

$$\Lambda^{1+\delta}(\partial_2\Omega) \leq C \limsup_{n\to\infty} 2^{n(n+1)/2}(2^{-1-n(n-1)/2})^{1+\delta},$$

which approaches zero as $n \to \infty$. Thus $\Lambda^{1+\delta}(\partial_2\Omega) = 0$ for every $\delta > 0$, and $\partial_2\Omega$ has dimension at most 1. It is clear $\partial_2\Omega$ has dimension at least 1.

Next, we claim that $\text{cap}_3(\partial_2\Omega)$ is positive. Recall that $\partial_2\Omega$ is a Jordan curve and $S_{n,m}$ is a particular side of $R_{n,m}$ that is situated on some $l_{n-1,m'}$. Let $A_{n,m}$ and $B_{n,m}$ be the endpoints of $S_{n,m}$; from the construction of Ω, one sees that $A_{n,m}$ and $B_{n,m}$ are on $\partial_2\Omega$. Let μ be the probability measure on $\partial_2\Omega$ satisfying, for $n \geq 1$,

$$\mu(E_{n,m}) = 2^{-n(n+1)/2}, \tag{1.12}$$

where $E_{n,m}$ is the subarc of $\partial_2\Omega$ with endpoints $A_{n,m}$ and $B_{n,m}$ which does not contain the point $(-3, 0)$.

We shall prove that

$$\mu(\partial_2\Omega \cap \Delta(P,t)) \leq Ct\left(\log\frac{1}{t}\right)^{-2} \tag{1.13}$$

for every $P \in \mathbf{R}^2$ and $0 < t < t_0$. Once (1.13) is proved, we have for any $P \in \mathbf{R}^2$,

$$\begin{aligned}\int_{\partial_2\Omega} \frac{1}{|P-X|}\, d\mu(X) &= \int_0^\infty \mu(\Delta(P,t) \cap \partial_2\Omega)\,\frac{dt}{t^2}\\ &\leq \int_{t_0}^1 \frac{dt}{t^2} + \int_0^{t_0} \frac{1}{t\log^2(1/t)}\,dt < C(t_0) < \infty.\end{aligned}$$

Therefore $\text{cap}_3(\partial_2\Omega) > 0$.

To prove (1.13), we assume

$$2^{-n(n-1)/2} \leq t < 2^{-(n-1)(n-2)/2}.$$

For any $P \in \mathbf{R}^2$, $\Delta(P,t)$ meets at most $Ct2^{n(n-1)/2}$ arcs of the form $E_{n,m}$. Therefore by (1.12),

$$\begin{aligned}\mu(\Delta(p,t) \cap \partial_2\Omega) &\leq Ct2^{n(n-1)/2}2^{-n(n+1)/2}\\ &\leq Ct2^{-n} < Ct\left(\log\frac{1}{t}\right)^{-2}\end{aligned}$$

if $0 < t < t_0$.

Finally we prove that $\mathrm{cap}_3(\Omega_\varepsilon) \to 0$ as $\varepsilon \to 0^+$. Because $\mathrm{cap}_3(\Omega_\varepsilon)$ decreases as ε decreases, we need only to show that $\mathrm{cap}_3(\Omega_{a_N}) \to 0$ as $N \to \infty$. We observe, by the relative narrowness of a_N to the distance between $R_{n,m}$ and $R_{n',m'}$ $(n, n' < N)$, that

$$\Omega_{a_N} \subseteq \bigcup_{n=0}^{N-1} \bigcup_{m=1}^{2^{n(n+1)/2}} R_{n,m,a_N} \cup \bigcup_{n=N}^{\infty} \bigcup_{m=1}^{2^{n(n+1)/2}} R_{n,m}$$

where $R_{n,m,a_N} = \{x \in R_{n,m}, \mathrm{dist}(x, \partial R_{n,m}) < a_N\}$. By a variation of Lemma 4 below, we have the following estimation:

$$\begin{aligned}
&\mathrm{cap}_3(\Omega_{a_N}) \\
&\leq C\left(\sum_{n=0}^{N-1} 2^{n(n+1)/2} \frac{|l_{n,1}|}{\log(|l_{n,1}|/a_N)} + \sum_{n=N}^{\infty} 2^{n(n+1)/2} \frac{|l_{n,1}|}{\log(|l_{n,1}|/a_n)} \right) \\
&\leq C\left(\sum_{n=0}^{N-1} \frac{2^{n(n+1)/2} 2^{-n(n-1)/2}}{\log(2^{-n(n-1)/2} 2^{2^{2N}})} + \sum_{n=N}^{\infty} \frac{2^{n(n+1)/2} 2^{-n(n-1)/2}}{\log(2^{-n(n-1)/2} 2^{2^{3n}})} \right) \\
&\leq \sum_{n=0}^{N-1} 2^n 2^{-2N} + \sum_{n=N}^{\infty} 2^{-n},
\end{aligned}$$

which approaches 0 as $N \to \infty$. This completes the proof of Lemma 1.

LEMMA 4 [4; p. 165]. *Let E be an elongated ellipsoid of revolution with axes a, b $(b < a)$. Then*

$$\mathrm{cap}_3(E) = \frac{2}{\pi} \frac{\sqrt{a^2 - b^2}}{\log\left[\left(a + \sqrt{a^2 - b^2}\right)/\left(a - \sqrt{a^2 - b^2}\right)\right]}.$$

2. Proof of the Theorem. Following the definition in [**2**], we say a domain Ω in $\mathbf{R}^m$ is a *non-tangentially accessible* (NTA) domain if there exist fixed constants $M = M(\Omega) > 10$ and $r_0 = r_0(\Omega) > 0$ such that the following conditions are satisfied.

(2.1) *corkscrew condition*: for any $Q \in \partial\Omega$, $r < r_0$, there exists $A = A_r(Q) \in \Omega$ such that $M^{-1}r < |A - Q| < r$ and $\mathrm{dist}(A, \partial\Omega) > M^{-1}r$;

(2.2) $\mathbf{R}^m \setminus \Omega$ satisfies the corkscrew condition;

(2.3) *Harnack chain condition*: if X_1 and $X_2 \in \Omega$, $\mathrm{dist}(X_i, \partial D) > \varepsilon > 0$, $i = 1, 2$, and $|X_1 - X_2| \leq K\varepsilon$, then there exist balls $B_j = B(Y_j, r_j)$, $1 \leq j \leq L$, L depending only on K, but not on ε, so that $Y_1 = X_1$ and

$Y_L = X_2$; and the balls B_j satisfy

$$(2.4) \qquad M^{-1}r_j < \operatorname{dist}(B_j, \partial\Omega) < Mr_j, \qquad 1 \le j \le L;$$

and

$$(2.5) \qquad B(Y_j, r_j/2) \cap B(Y_j, r_{j+1}/2) \neq \varnothing, \qquad 1 \le j \le L-1.$$

($\{B_j\}$ is called a Harnack chain from X_1 to X_2 of length L.)

Assuming $F \subseteq \partial D \cap \Gamma$ and $\Lambda^{m-1}(F) = 0$, we want to show $\omega(F, D) = 0$.

We claim that it is enough to prove that there exists $0 < \beta < 1$, so that

$$(2.6) \qquad \omega^Q(F, D) < \beta \quad \text{for every } Q \in D \cap \Gamma.$$

In fact, for $X \in D \cap \Omega_i$, it follows from (0.1) that

$$\omega^X(F, D \cap \Omega_i) \le \omega^X(F, \Omega_i) = 0;$$

hence

$$(2.7) \quad \omega^X(F, D) = \omega^X(F, D \cap \Omega_i) + \int_{\Gamma \cap D} \omega^Q(F, D)\, d^X(Q, D \cap \Omega_i)$$

$$= \int_{\Gamma \cap D} \omega^Q(F, D)\, d\omega^X(Q, D \cap \Omega_i).$$

After (2.6) is proved, we may conclude

$$\omega^X(F, D) < \beta < 1 \quad \text{for every } X \in D.$$

This is possible only when $\omega(F, D) = 0$. Therefore we need only to show (2.6).

Since Ω_i, $i = 1, 2$, are NTA domains and $\mathbf{R}^m \setminus D$ satisfies the corkscrew condition, we let

$$M = \max\{M(\Omega_1), M(\Omega_2), M(D)\}$$

and

$$r_0 = \min\{r_0(\Omega_1), r_0(\Omega_2), r_0(D)\}$$

from their respective definitions.

For a fixed $Q \in D \cap \Gamma$, let

$$r = \min\{r_0, \operatorname{dist}(Q, \partial D)\}.$$

From the corkscrew condition on Ω_i, we can find

$$U_i = B(A_i, r/4M) \subseteq \Omega_i$$

so that

$$|A_i - Q| < r/2 \quad \text{and} \quad \operatorname{dist}(U_i, \Gamma) > r/4M. \tag{2.8}$$

Notice that $U_1 \cup U_2 \subseteq B(Q, r) \subseteq D$. Therefore we can find α, $0 < \alpha < 1$, depending on M only so that

$$\omega^Q(F, D) \leq 1 - \alpha + \alpha \sup_{X \in \overline{U}_i} \omega^X(F, D), \quad \text{for } i = 1 \text{ or } 2. \tag{2.9}$$

Because of (2.7) and (2.9), in order to prove (2.6), we need only to show there exists $\eta < 1$ so that

$$\min\left\{ \sup_{X \in \overline{U}_i} \omega^X(\Gamma \cap D, D \cap \Omega_i) : i = 1, 2 \right\} < \eta. \tag{2.10}$$

We claim that there exists a ball

$$V \equiv B\left(A, (4M)^{-2} r\right)$$

whose closure is completely in $\Omega_1 \setminus D$ or completely in $\Omega_2 \setminus D$, and

$$|A - Q| < Kr \quad \text{and} \quad \operatorname{dist}(V, \Gamma) > (4M)^{-2} r, \tag{2.11}$$

where $K = 2 + (\operatorname{diam} D)/r_0$.

In fact, let P be a point on ∂D so that $|P - Q| = \operatorname{dist}(Q, \partial D)$. Since $\mathbf{R}^m \setminus D$ satisfies the corkscrew condition, we can find a ball

$$W = B\left(Y, (2M)^{-1} r\right) \subseteq \mathbf{R}^m \setminus D$$

so that

$$|Y - P| < r \quad \text{and} \quad \operatorname{dist}(W, \partial D) > (2M)^{-1} r.$$

If $B(Y, (4M)^{-1} r) \cap \Gamma = \varnothing$ then $B(Y, (4M)^{-1} r)$ lies completely in $\Omega_1 \setminus D$ or completely in $\Omega_2 \setminus D$; we let

$$A \equiv Y \quad \text{and} \quad V \equiv B\left(Y, (4M)^{-2} r\right),$$

and can verify (2.11) easily.

If $B(Y, (4M)^{-1} r) \cap \Gamma$ contains some point Z, by the corkscrew condition on Ω_1, we can find

$$V \equiv B\left(A, (4M)^{-2} r\right) \subseteq \Omega_1$$

so that

$$(8M^2)^{-1} r < |A - Z| < (8M)^{-1} r \quad \text{and} \quad \operatorname{dist}(V, \Gamma) > (4M)^{-2} r.$$

Because $|A - Y| \leq |A - Z| + |Z - Y| \leq 3r(8M)^{-1}$, we see $V \subseteq W \subseteq \mathbf{R}^m \setminus D$. Therefore $V \subseteq \Omega_1 \setminus D$. Again (2.11) can be verified easily. This proves our claim.

From now on we assume V is contained in $\Omega_1 \setminus D$, and shall prove

$$\sup_{X \in \overline{U}_1} \omega^X(\Gamma \cap D, D \cap \Omega_1) < \eta < 1. \tag{2.12}$$

When V is in $\Omega_2 \setminus D$, we argue similarly.

From (2.8) and (2.11) and the assumption that Ω_1 is an NTA domain, we can find a Harnack chain $\{B_j\}_{j=1}^L$ in Ω_1, whose length L depends on r_0, M and diam D only, that connects A to A_1; moreover, we may choose

$$B_1 \equiv B\left(A, 3r(32M^2)^{-1}\right) \supseteq B\left(A, r(4M)^{-2}\right) = V, \tag{2.13}$$

$$B_L \equiv B\left(A_1, 3r(8M)^{-1}\right) \supseteq B\left(A_1, r(4M)^{-1}\right) = U_1, \tag{2.14}$$

so that (2.4) is still satisfied with a bigger constant M' dependent only on M, r_0 and diam D.

Let $B = \bigcup_{j=1}^L B_j$ and

$$w = \begin{cases} \omega(\Gamma \cap D, D \cap \Omega_1) & \text{on } D \cap \Omega_1, \\ 0 & \text{on } \mathbf{R}^m \setminus (D \cap \Omega_1). \end{cases}$$

Since $\{B_j\}$ is a Harnack chain, $\overline{B} \subseteq \Omega_1$; hence w is subharmonic on B; and because $\overline{V} \cap D = \varnothing$, $w = 0$ on $\overline{V}$. Therefore by the maximum principle, for $X \in \overline{U}_1 \subseteq D \cap \Gamma_1$

$$\omega^X(\Gamma \cap D, D \cap \Omega_1) \leq \omega^X\left(\partial B, B \setminus \overline{V}\right).$$

By (2.13), (2.14), properties (2.4) and (2.5) of the Harnack chain condition, and the Harnack principle, we can find $\eta < 1$, depending on r_0, M, diam D, so that

$$\omega^X\left(\partial D, B \setminus \overline{V}\right) < \eta \quad \text{for every } X \in \overline{U}_1.$$

Therefore (2.12) is proved, and thus (2.6) follows.

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