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VECTOR BUNDLES OVER $(8k + 3)$-DIMENSIONAL MANIFOLDS

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Let M be a closed, connected, smooth and 3-connected mod 2 (i.e. $H_i(M; \mathbf{Z}_2) = 0$, $0 < i \leq 3$) manifold of dimension $3 + 8k$ with $k > 1$. We obtain some necessary and sufficient condition for the span of a $(3 + 8k)$-plane bundle η over M to be greater than or equal to 7 or 8. We obtain, for M 4-connected mod 2 and satisfying $\mathrm{Sq}^2\,\mathrm{Sq}^1\,H^{n-8}(M) = \mathrm{Sq}^2\,H^{n-7}(M)$, where $n = \dim M \equiv 11 \mod 16$ with $n > 11$, that span $M \geq 8$ if and only if $\chi_2(M) = 0$. Some applications to product manifolds and immersion are given.

1. Introduction. Let M be a closed, connected and smooth manifold whose dimension n is congruent to 3 mod 8 with $n \geq 11$. Let η be an n-plane bundle over M. Recall span(η) is defined to be the maximal number of linearly independent cross sections of η. When η is the tangent bundle of M we simply write span(M) for span(η). Recall that the Kervaire mod 2 semi-characteristic of M $\chi_2(M)$, is defined by

$$\chi_2(M) = \sum_{2i<n} \dim_{\mathbf{Z}_2} H^i(M; \mathbf{Z}_2) \mod 2.$$

Suppose M is a 1-connected mod 2 spin manifold. Then according to Thomas [14] and Randall [11], span$(M) \geq 4$ if and only if $\chi_2(M) = 0$ and $w_{n-3}(M) = 0$, where $w_j(M)$ is the jth-mod 2 Stiefel-Whitney class of the tangent bundle of M. In this paper we shall derive some necessary and sufficient condition for span$(M) \geq 7$ or 8 when M is 3-connected mod 2.

For the rest of the paper we shall assume that M is 3-connected mod 2. Then from [14] and the methods of Mahowald [4] we have:

THEOREM 1.1 (*Thomas-Mahowald*). Span$(M) \geq 5$ *if and only if* $\delta w_{n-5}(M) = 0$ *and* $\chi_2(M) = 0$. *Here δ is the Bockstein coboundary homomorphism associated with the exact sequence* $0 \to \mathbf{Z} \to \mathbf{Z} \to \mathbf{Z}_2 \to 0$.

We shall consider the modified Postnikov tower for the fibration $B\mathrm{spin}_{n-k} \to B\mathrm{spin}_n$ for $n \geq 19$ and $k = 7$ or 8 where $B\mathrm{spin}_j$ is the classifying space of spin j-plane bundles. Then for an n-plane bundle η over M, η is classified by a map g from M into $B\mathrm{spin}_n$. Then η admits k-linearly independent cross-sections if and only if g lifts to $B\mathrm{spin}_{n-k}$.

For the remainder of the paper we shall assume that $n \geq 19$. All cohomology will be ordinary mod 2 cohomology unless otherwise specified. Let $\mathfrak{A}$ denote the mod 2 Steenrod algebra.

Let ϕ_3 be the stable secondary cohomology operation associated with the relation in $\mathfrak{A}$ also denoted by the same symbol:

$$\phi_3\colon \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^1(\mathrm{Sq}^2\mathrm{Sq}^1) = 0.$$

Then ϕ_3 is spin-trivial in the sense of Thomas [13].

Then η admits 6 linearly independent cross sections if and only if $w_{n-5}(\eta) = 0$, $0 \in k_1^2(\eta)$ and $0 \in k_4^2(\eta)$ where k_i^2, $i = 1, 4$ are the k-invariants for the n-modified Postnikov tower for $\pi\colon B\mathrm{spin}_{n-6} \to B\mathrm{spin}_n$ defined by the following relations

$$k_1^2\colon \mathrm{Sq}^2 w_{n-5} = 0;$$
$$k_4^2\colon (\mathrm{Sq}^4 + w_4)w_{n-3} = 0.$$

Let $B\mathrm{spin}_{n-6} \xrightarrow{q} E_1 \xrightarrow{p} B\mathrm{spin}_n$ be the first stage n-MPT for the fibration π. Let $j\colon B\mathrm{spin}_{n-7} \to B\mathrm{spin}_{n-6}$ be the obvious inclusion. Then since ϕ_3 is spin-trivial $0 \in \phi_3(w_{n-7}) \subset H^{n-4}(B\mathrm{spin}_{n-7})$. Since j^* is an epimorphism in dimensions $\leq n$ and a monomorphism in dimension $n-4$, we see that $0 \in \phi_3(w_{n-7}) \subset H^{n-4}(B\mathrm{spin}_{n-6})$. Since $\mathrm{Sq}^2 w_{n-7} = w_{n-5}$ and $\mathrm{Sq}^2\mathrm{Sq}^1 w_{n-7} = 0$ we conclude that the class w_{n-7} in $H^{n-7}(B\mathrm{spin}_n)$ is a generating class (see [12]) for k_1^2 in $H^{n-4}(E_1)$ relative to the operation ϕ_3. Suppose now $\mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M)$. We have by the generating class theorem that $0 \in k_1^2(\eta)$ if and only if $0 \in \phi_3(w_{n-7}(\eta))$.

Let Γ be the unstable cohomology operation of Hughes-Thomas type associated with the following relation in $\mathfrak{A}$ on cohomology classes of dimension $\leq n$

$$\Gamma\colon \mathrm{Sq}^4\mathrm{Sq}^{n-3} + \mathrm{Sq}^2(\mathrm{Sq}^{n-3}\mathrm{Sq}^2) + \mathrm{Sq}^1(\mathrm{Sq}^{n-3}\mathrm{Sq}^3 + \mathrm{Sq}^{n-1}\mathrm{Sq}^1) = 0.$$

Let U be the Thom class of the universal n-plane bundle γ over $B\mathrm{spin}_n$. Let U' be the Thom class of the bundle over $B\mathrm{spin}_{n-6}$ induced by π from γ. Let $T\pi$ be the map between the Thom spaces. Then Γ is defined on $U' = (T\pi)^*U$ and is trivial on U'. Then $w_{n-3} \in H^{n-3}(B\mathrm{spin}_n)$ is an admissible class (see *Ng* [8]) for k_4^2 with respect to the operation Γ. Then by the admissible class theorem since $(T\pi)^*$ is an epimorphism in dimension $2n$, $0 \in k_4^2(\eta)$ if and only if $0 \in \Gamma(U(\eta))$, where $U(\eta)$ is the Thom class of the bundle η. Hence we have proved

THEOREM 1.2. *Suppose* $\mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M)$.

(a) *If* $w_4(\eta) \neq w_4(M)$ *then* $\mathrm{span}(\eta) \geq 6$ *if and only if* $w_{n-5}(\eta) = 0$ *and* $0 \in \phi_3(w_{n-7}(\eta))$.

(b) *If* $w_4(\eta) = w_4(M)$ *then* $\operatorname{span}(\eta) \geq 6$ *if and only if* $w_{n-5}(\eta) = 0$, $0 \in \phi_3(w_{n-7}(\eta))$ *and* $\Gamma(U(\eta)) = 0$ *modulo zero indeterminacy.*

If η is the tangent bundle of M and $w_4(M) = 0$ then it can be easily deduced that for $n \equiv 11 \bmod 16$, $w_{n-5}(M) = w_{n-7}(M) = 0$. According to [14], $\Gamma(U(\eta)) = \chi_2(M) \cdot U(\eta) \cdot \mu$ where $\mu \in H^n(M)$ is a generator. Hence we have

THEOREM 1.3. *Suppose* $\mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M)$.

(a) *If* $n \equiv 3 \bmod 16 > 3$, *then* $\operatorname{span}(M) \geq 6$ *if and only if* $w_{n-5}(M) = 0$, $\chi_2(M) = 0$ *and* $0 \in \phi_3(w_{n-7}(M))$.

(b) *If* $n \equiv 11 \bmod 16 > 11$ *and* $w_4(M) = 0$, *then* $\operatorname{span}(M) \geq 6$ *if and only* $\chi_2(M) = 0$.

2. Statement of results for the case $k = 7$. The n-MPT for $B\mathrm{spin}_{n-7} \to B\mathrm{spin}_n$ is given in Ng [8]. We list the result (in the relevant dimensions) as follows:

TABLE 1. The n-MPT for $B\mathrm{spin}_{n-7} \to B\mathrm{spin}_n$.

k-invariant	Dimension	Defining Relation
k_1^1	$n-6$	$k_1^1 = \delta w_{n-7}$
k_2^1	$n-5$	$k_2^1 = w_{n-5}$
k_3^1	$n-3$	$k_3^1 = w_{n-3}$
k_1^2	$n-5$	$\mathrm{Sq}^2 k_1^1 = 0$
k_2^2	$n-4$	$\mathrm{Sq}^2 k_2^1 = 0$
k_6^2	n	$(\mathrm{Sq}^4 + w_4)k_3^1 = 0$
k_1^3	$n-4$	$\mathrm{Sq}^2 k_1^2 = 0.$

Consider the following stable secondary cohomology operations associated with the following relations in $\mathfrak{A}$ also denoted by the same symbols

$$\phi_4\colon \mathrm{Sq}^2(\mathrm{Sq}^2\mathrm{Sq}^1) = 0$$
$$\phi_5\colon (\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^2\mathrm{Sq}^1) + \mathrm{Sq}^1(\mathrm{Sq}^2\mathrm{Sq}^3) = 0$$

such that $\mathrm{Sq}^2\phi_4 + \mathrm{Sq}^1\phi_5 = 0$. Let ψ_5 be a stable tertiary cohomology operation associated with the above relation. We assume that (ϕ_4, ϕ_5) and ψ_5 are chosen to be spin trivial in the sense of Theorem 3.7 of Thomas [13].

Let $\phi_{0,0}$ and ϕ_3 be the Adams basic stable secondary cohomology operations associated with the relations:

$$\phi_{0,0}\colon \mathrm{Sq}^1\mathrm{Sq}^1 = 0 \quad \text{and}$$
$$\phi_3\colon \mathrm{Sq}^2\mathrm{Sq}^2 + \mathrm{Sq}^1(\mathrm{Sq}^2\mathrm{Sq}^1) = 0$$

respectively.

We shall prove the following theorem.

THEOREM 2.1. *Suppose*

$$\mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M), \qquad \mathrm{Sq}^2 H^{n-7}(M; \mathbf{Z}) = \mathrm{Sq}^2 H^{n-7}(M)$$

and $\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$. *Then* $\mathrm{span}(M) \geq 7$ *if and only if* $\delta w_{n-7}(M) = 0$, $w_{n-5}(M) = 0$, $0 \in \phi_4(w_{n-9}(M))$, $0 \in \phi_3(w_{n-7}(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_5(w_{n-9}(M))$.

Suppose $\mathrm{Sq}^1 H^4(M) = 0$, $\phi_{0,0}H^4(M) = 0$ and $H_6(M; \mathbf{Z})$ has no 2-torsion. Then it is easily deduced that $\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$ and $\mathrm{Sq}^3 H^{n-7}(M) = 0$. It follows from Theorem 2.1 the following theorem:

THEOREM 2.2. *Suppose* $\mathrm{Sq}^1 H^4(M) = 0$, $\phi_{0,0}H^4(M) = 0$ *and* $H_6(M; \mathbf{Z})$ *has no 2-torsion. Then* $\mathrm{span}(M) \geq 7$ *if and only if* $\delta w_{n-7}(M) = 0$, $w_{n-5}(M) = 0$, $0 \in \phi_4(w_{n-9}(M))$, $0 \in \phi_3(w_{n-7}(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_5(w_{n-9}(M))$.

If $n = 11 + 16k$ with $k \geq 1$, then according to Wu and 6.6 of [**8**], if $w_4(M) = 0$, $w_{n-9}(M) = w_{n-7}(M) = w_{n-5}(M) = 0$. Hence we have

COROLLARY 2.3. *Suppose either M satisfies the hypothesis of* 2.2 *or M is* 4*-connected* mod 2 *with* $\mathrm{Sq}^2 H^{n-7}(M; \mathbf{Z}) = \mathrm{Sq}^2 H^{n-7}(M)$. *If* $n \equiv 11 \bmod 16 > 11$ *and* $w_4(M) = 0$, *then* $\mathrm{span}(M) \geq 7$ *if and only if* $\chi_2(M) = 0$.

3. Statement of results for the case $k = 8$. Let $B\hat{\mathrm{SO}}_j\langle 8\rangle$ be the classifying space for orientable j-plane bundles ξ satisfying $w_2(\xi) = w_4(\xi) = 0$. We shall consider the modified Postnikov tower for the fibration $B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle \to B\hat{\mathrm{SO}}_n\langle 8\rangle$ through dimension n. The computation is done in [**8**]. We list here the k-invariants for the relevant dimensions only for reference. In particular the k-invariants k_1^2, k_5^2 and k_1^3 for the n-MPT for $B\mathrm{spin}_{n-8} \to B\mathrm{spin}_n$ are the images of k_1^2, k_6^2 and k_1^3, respectively, of the n-MPT for $B\mathrm{spin}_{n-7} \to B\mathrm{spin}_n$ under the map between the towers.

TABLE 2. The n-MPT for $B\mathrm{spin}_{n-8} \to B\mathrm{spin}_n$.

k-invariant	Dimension	Defining relation
k_1^1	$n-7$	$k_1^1 = w_{n-7}$
k_2^1	$n-3$	$k_2^1 = w_{n-3}$
k_1^2	$n-5$	$\mathrm{Sq}^2\mathrm{Sq}^1 k_1^1 = 0$
k_2^2	$n-4$	$(\chi\mathrm{Sq}^4 + w_4)k_1^1 = 0$
k_5^2	n	$(\mathrm{Sq}^4 + w_4)k_2^1 = 0$
k_1^3	$n-4$	$\mathrm{Sq}^2 k_1^2 = 0$

Consider the following stable secondary cohomology operations ϕ_6, ϕ_7 and ζ_7 associated with the following relations in $\mathfrak{A}$ also denoted by the same symbols as for the operations:

$$\phi_6\colon (\mathrm{Sq}^2\mathrm{Sq}^1)\chi\mathrm{Sq}^4 + \mathrm{Sq}^2(\mathrm{Sq}^4\mathrm{Sq}^1) = 0$$
$$\phi_7\colon \chi\mathrm{Sq}^4\chi\mathrm{Sq}^4 + \mathrm{Sq}^2(\mathrm{Sq}^4\mathrm{Sq}^2) + \mathrm{Sq}^1(\mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1) = 0$$
$$\zeta_7\colon (\mathrm{Sq}^2\mathrm{Sq}^1)(\mathrm{Sq}^4\mathrm{Sq}^1) = 0.$$

These operations satisfy the relation

$$\mathrm{Sq}^2\phi_6 + \mathrm{Sq}^1\zeta_7 = 0 \tag{3.1}$$

Let ψ_7 be a stable tertiary cohomology operation associated with relation (3.1). Note that the operations ϕ_6, ϕ_7 and ψ_7 are chosen such that $\phi_4 \circ \mathrm{Sq}^2 \subset \phi_6$, $\phi_5 \circ \mathrm{Sq}^2 \subset \zeta_7$ and $\psi_5 \circ \mathrm{Sq}^2 \subset \psi_7$.

Let U_j be the Thom class of the universal j-plane bundle over $B\hat{\mathrm{S}}0_j\langle 8\rangle$ for $j \geq 4$. Then it is easily seen that

$$U_j(\mathrm{Sq}^2\nu_4, \mathrm{Sq}^3\nu_4, 0) \in (\phi_6, \phi_7, \zeta_7)(U_j)$$

where $\nu_4 \in H^4(B\hat{\mathrm{S}}\mathrm{O}_j\langle 8\rangle) \approx \mathbf{Z}_2$ is a generator. Hence

$$(0,0,0) \in (\phi_6, \phi_7, \zeta_7)(U_j). \tag{3.2}$$

Since $H^7(B\hat{\mathrm{S}}\mathrm{O}_j\langle 8\rangle) \approx \mathbf{Z}_2$ is generated by $\mathrm{Sq}^3\nu_4$, trivially

$$0 \in \psi_7(U_j). \tag{3.3}$$

Let K_4 be the Eilenberg-MacLane space of type $(\mathbf{Z}_2, 4)$. Consider the Massey-Peterson algebra $\mathfrak{A}(K_4)$. Let ι_4 be the fundamental class of K_4. Let $\gamma = (1 \otimes \chi\mathrm{Sq}^4 + \iota_4 \otimes 1)$, $\alpha = (\mathrm{Sq}^2\mathrm{Sq}^1\iota_4 \otimes 1 + \iota_4 \otimes \mathrm{Sq}^2\mathrm{Sq}^1 + \mathrm{Sq}^1\iota_4 \otimes \mathrm{Sq}^2 + 1 \otimes \mathrm{Sq}^4\mathrm{Sq}^2\mathrm{Sq}^1)$, $\theta = (1 \otimes \mathrm{Sq}^4 + \iota_4 \otimes 1)$. Then we have the following relation in $\mathfrak{A}(K_4)$:

$$\tilde{\phi}_7\colon \gamma \cdot \gamma + \mathrm{Sq}^2(\theta \cdot \mathrm{Sq}^2) + \mathrm{Sq}^1\alpha = 0.$$

Let $\tilde{\phi}_7$ be the twisted cohomology operation associated with this relation also denoted by the same symbols as the operation. For $j \geq 4$ let U_j be the Thom class of the universal j-plane bundle over $B\mathrm{spin}_j$. Then $\tilde{\phi}_7$ is defined on (U_j, w_4). Since $H^7(B\mathrm{spin}_j)$ is generated by $w_7 = \mathrm{Sq}^1 w_6$, trivially

$$0 \in \tilde{\phi}_7(U_j, w_4). \tag{3.4}$$

We shall prove the following theorems.

THEOREM 3.1. *Suppose either* $\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M)$ *and* $\mathrm{Sq}^4 H^{n-8}(M) = \mathrm{Sq}^2 H^4(M) = \mathrm{Sq}^1 H^4(M) = 0$ *or* $\mathrm{Sq}^2 H^{n-7}(M) = 0$ *and* $\mathrm{Sq}^1 H^{n-5}(M) + \mathrm{Sq}^2 H^{n-6}(M) \subset \chi\mathrm{Sq}^4 H^{n-8}(M)$. *If* $w_4(M) = 0$ *and* $\mathrm{Indet}^{n-4}(\psi_7, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$, *then* $\mathrm{span}(M) \geq 8$ *if and only if* $w_{n-7}(M) = 0$, $0 \in \phi_6(w_{n-11}(M))$, $0 \in \phi_7(w_{n-11}(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_7(w_{n-11}(M))$.

THEOREM 3.2. *Suppose either* $\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M)$, $(\mathrm{Sq}^4 + w_4(M)\cdot)H^{n-8}(M) = 0$ *and* $\mathrm{Sq}^2 H^4(M) = \mathrm{Sq}^1 H^4(M) = 0$ *or* $\mathrm{Sq}^2 H^{n-7}(M) = 0$ *and*

$$\mathrm{Sq}^2 H^{n-6}(M) + \mathrm{Sq}^1 H^{n-5}(M) \subset \left(\chi\mathrm{Sq}^4 + w_4(M)\cdot\right)H^{n-8}(M).$$

If $\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$ *then* $\mathrm{span}(M) \geq 8$ *if and only if* $w_{n-7}(M) = 0$, $0 \in \phi_4(w_{n-9}(M))$, $0 \in \tilde{\phi}_7(w_{n-11}(M), w_4(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_5(w_{n-9}(M))$.

By the choice of operations (ϕ_6, ζ_7) and ψ_7 we can deduce that the indeterminacy $\mathrm{Indet}^{n-4}(\psi_7, M)$ is given by the image of the stable operation ϕ_3 on classes $\mathrm{Sq}^1 x + y$, where $x \in H^{n-8}(M)$ and $y \in H^{n-7}(M)$ satisfy $\mathrm{Sq}^2\mathrm{Sq}^1 x + \mathrm{Sq}^2 y = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 y = 0$. Note that the map from the tower of operations for $(\psi_7, (\phi_6, \zeta_7))$ on N-dimensional classes to the tower of operations for $(\psi_5, (\phi_4, \phi_5))$ on $N + 2$ dimensional classes is induced by the commutative square:

$$\begin{array}{ccc} K_N & \xrightarrow{\mathrm{Sq}^2} & K_{N+2} \\ (\chi\mathrm{Sq}^4, \mathrm{Sq}^4\mathrm{Sq}^1)\downarrow & & \downarrow(\mathrm{Sq}^2\mathrm{Sq}^1, \mathrm{Sq}^2\mathrm{Sq}^3) \\ K_{N+4}\times K_{N+5} & \xrightarrow{(\mathrm{Sq}^1\iota_{N+4}+\iota_{N+5},\,0)} & K_{N+5}\times K_{N+7} \end{array}$$

where K_j is the Eilenberg-Maclane space of type $(\mathbf{Z}_2, j)$ and ι_j is its fundamental class.

Now the map from the n-MPT for $B\mathrm{spin}_{n-8} \to B\mathrm{spin}_n$ to the n-MPT for $B\mathrm{spin}_{n-7} \to B\mathrm{spin}_n$ is induced by the commutative square:

$$\begin{array}{ccc}
B\mathrm{spin}_n & \overline{\overline{\qquad\qquad}} & B\mathrm{spin}_n \\
{\scriptstyle (w_{n-7}, w_{n-3})}\downarrow & & \downarrow{\scriptstyle (w_{n-7}, w_{n-5}, w_{n-3})} \\
K_{n-7} \times K_{n-3} & \xrightarrow{(\delta\iota_{n-7}, \mathrm{Sq}^2\iota_{n-7}, \iota_{n-3})} & K^*_{n-6} \times K_{n-5} \times K_{n-3}
\end{array}$$

where K^*_{n-6} is the Eilenberg-Maclane space of type $(\mathbf{Z}, n-6)$. In view of the fact that (k_1^2, k_2^2) is independent of w_{n-3} (see [**15**, §4]) and k_1^3 is independent of k_5^2 we can deduce that $\mathrm{Indet}^{n-4}(k_1^3, M)$ is the image of the stable operations ϕ_3^* on classes δx, where $x \in H^{n-8}(M)$ satisfies $\mathrm{Sq}^2\mathrm{Sq}^1 x = 0$ and $(\chi\mathrm{Sq}^4 + w_4(M)\cdot)x = 0$ and ϕ_3^* is a stable cohomology operation associated with the relation $\mathrm{Sq}^2\mathrm{Sq}^2 = 0$ on integral classes.

Suppose M satisfies the following conditions:

(A) $H_6(M; \mathbf{Z})$ has no 2-torsion and

(B) $H_7(M; \mathbf{Z})$ has no free parts and its 2-torsion elements are all of order 2.

Then $\mathrm{Sq}^1 H^{n-8}(M) = H^{n-7}(M)$. If further

$$\mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M) \text{ and } \big(\chi\mathrm{Sq}^4 + w_4(M)\cdot\big)H^{n-8}(M) = 0,$$

then $\mathrm{Indet}^{n-4}(\psi_7, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$. Hence we have by Theorem 3.1 the following immediate corollary:

COROLLARY 3.3. *Suppose M satisfies both condition* (A) *and* (B). *If* $w_4(M) = 0$, $\mathrm{Sq}^1 H^4(M) \simeq \mathrm{Sq}^2 H^4(M) \simeq 0$ *and* $\chi\mathrm{Sq}^4 H^{n-8}(M) = 0$ *then*

(i) $\phi_4(w_{n-9}(M)) = \phi_6(w_{n-11}(M))$;

(ii) *if* $\phi_{0,0}H^4(M) = 0$ *then* $\psi_7(w_{n-11}(M)) = \psi_5(w_{n-9}(M))$;

(iii) $\mathrm{span}(M) \geq 8$ *if and only if* $w_{n-7}(M) = 0$, $0 \in \phi_6(w_{n-11}(M))$, $0 \in \phi_7(w_{n-11}(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_7(w_{n-11}(M))$.

Similarly from Theorem 3.2 we have

COROLLARY 3.4. *Suppose M satisfies conditions* A *and* B. *Assume* $\mathrm{Sq}^1 H^4(M) = 0$, $\phi_{0,0}H^4(M) = 0$, $\mathrm{Sq}^2 H^4(M) = 0$ *and*

$$\big(\chi\mathrm{Sq}^4 + w_4(M)\cdot\big)H^{n-8}(M) = 0.$$

Then $\mathrm{span}(M) \geq 8$ *if and only if* $w_{n-7}(M) = 0$, $0 \in \phi_4(w_{n-9}(M))$, $0 \in \tilde{\phi}_7(w_{n-11}(M), w_4(M))$, $\chi_2(M) = 0$ *and* $0 \in \psi_5(w_{n-9}(M))$.

Suppose $n = 11 + 16k$ with $k \geq 1$, $w_4(M) = w_8(M) = 0$. Then by Wu $w_{n-5}(M) = w_{n-7}(M) = w_{n-9}(M) = 0$ and $w_{n-11}(M) = v_{8k}^2$, where $v_j \in H^j(M)$ is the jth Wu-class of M. Since $w_8(M) = 0$, utilizing the Adém relations for $\mathrm{Sq}^8\mathrm{Sq}^{8k-4}$, $\mathrm{Sq}^4\mathrm{Sq}^{8k-2}$ and $\mathrm{Sq}^2\mathrm{Sq}^{8k-1}$ we can show that $\mathrm{Sq}^4 v_{8k} = 0$. Therefore ϕ_7 is defined on v_{8k}. By a Cartan formula for ϕ_7, (see [7])

$$\begin{aligned}\phi_7(w_{n-11}(M)) = \phi_7(v_{8k}^2) &= \phi_7(v_{8k}) \cdot v_{8k} + v_{8k} \cdot \phi_7(v_{8k}) \\ &= \{0\} \quad \text{modulo indeterminacy of } \phi_7.\end{aligned}$$

If $n = 19 + 32k$ with $k \geq 0$ and $w_4(M) = w_8(M) = 0$, then $w_{n-5}(M) = w_{n-7}(M) = w_{n-9}(M) = w_{n-11}(M) = 0$. Thus we have from Corollary 3.3 the following:

THEOREM 3.5. *Suppose* $n \equiv 11 \bmod 16 > 11$ *or* $n \equiv 19 \bmod 32$ *and* $w_4(M) = w_8(M) = 0$. *Then*

(a) *Suppose* $\mathrm{Sq}^1 H^4(M) \simeq \mathrm{Sq}^2 H^4(M) \simeq \mathrm{Sq}^4 H^{n-8}(M) \simeq 0$. *Then if* M *satisfies* (A) *and* (B), $\operatorname{span}(M) \geq 8$ *if and only if* $\chi_2(M) = 0$.

(b) *Suppose* M *is* 4*-connected* mod 2 *and* $\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M)$. *Then* $\operatorname{span}(M) \geq 8$ *if and only if* $\chi_2(M) = 0$.

4. Proof of Theorem 2.1. Let w_{n-9} be the $(n-9)$th mod 2 universal Stiefel-Whitney class considered as in $H^{n-9}(B\mathrm{spin}_{n-7})$. Then according to [**13**] we have

PROPOSITION 4.1 (*E. Thomas*).

(a) $(0,0) \in (\phi_4, \phi_5)(w_{n-9}) \subset H^{n-5}(B\mathrm{spin}_{n-7}) \oplus H^{n-4}(B\mathrm{spin}_{n-7})$

(b) $0 \in \psi_5(w_{n-9}) \subset H^{n-4}(B\mathrm{spin}_{n-7})$.

Proof. Part (a) is proved in [**13**, Proposition 4.2] and we shall not present it here. Let j: $B\mathrm{spin}_{n-9} \to B\mathrm{spin}_{n-7}$ be the inclusion map. Since ψ_5 is spin trivial $0 \in \psi_5(w_{n-9}) \subset H^{n-4}(B\mathrm{spin}_{n-9})$. Now $\mathrm{Indet}^{n-4}(\psi_5, B\mathrm{spin}_{n-9})$ is determined by a stable cohomology operation Θ defined on cohomology vectors

$$(x, y) \in H^{n-7}(B\mathrm{spin}_{n-9}) \times H^{n-5}(B\mathrm{spin}_{n-9})$$

satisfying $\mathrm{Sq}^2 x = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 x + \mathrm{Sq}^1 y = 0$. Since j^* is an epimorphism in dimensions $\leq n-4$, there are classes $x' \in H^{n-7}(B\mathrm{spin}_{n-7})$ and $y' \in H^{n-5}(B\mathrm{spin}_{n-7})$ such that $j^*x' = x$ and $j^*y' = y$. Since j^* is an isomorphism in dimension $n-5$, $\mathrm{Sq}^2 x = 0$ implies that $\mathrm{Sq}^2 x' = 0$.

Since $\operatorname{Ker} j^*$ in dimension $n-4$ is generated by $w_{n-8}\cdot w_4$, $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^1 y' = \alpha w_{n-8}\cdot w_4$ for some $\alpha \in \mathbf{Z}_2$. Hence

$$\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^1(y' + \alpha w_{n-9}\cdot w_4) = 0.$$

Thus Θ is defined on $(x', y' + \alpha w_{n-9}\cdot w_4)$. Now Θ is defined on $(0, w_{n-9}\cdot w_4) \in H^{n-7}(B\mathrm{spin}_{n-9}) \times H^{n-5}(B\mathrm{spin}_{n-9})$. $\Theta(0, w_{n-9}\cdot w_4) = \phi_{0,0}(w_{n-9}\cdot w_4) \subset H^{n-4}(B\mathrm{spin}_{n-9})$ modulo indeterminacy of Θ. But

$$\phi_{0,0}(w_{n-9}\cdot w_4) = \phi_{0,0}(w_{n-9})\cdot w_4 + w_{n-9}\cdot \phi_{0,0}(w_4) = 0,$$

since $\phi_{0,0}$ is spin-trivial and $H^5(B\mathrm{spin}_{n-9}) \approx 0$. Thus $\Theta(0, w_{n-9}\cdot w_4) = \{0\} \subset H^{n-4}(B\mathrm{spin}_{n-9})$. Thus

$$\begin{aligned}\Theta(x, y) &= \Theta(x, y) + \Theta(0, \alpha w_{n-9}\cdot w_4) = \Theta(x, y + \alpha w_{n-9}\cdot w_4)\\ &= \Theta(j^*x', j^*(y' + \alpha w_{n-9}\cdot w_4)) = j^*\Theta(x', y' + \alpha w_{n-9}\cdot w_4)\end{aligned}$$

since j^* is an epimorphism in dimensions $\leq n-4$. Thus

$$\mathrm{Indet}^{n-4}(\psi_5, B\mathrm{spin}_{n-9}) = j^*\mathrm{Indet}^{n-4}(\psi_5, B\mathrm{spin}_{n-7}).$$

This clarifies an argument in [**13**, Proposition 4.2]. Since $w_{n-8}\cdot w_4 = \mathrm{Sq}^1(w_{n-9}\cdot w_4)$ we conclude that $0 \in \psi_5(w_{n-9}) \subset H^{n-4}(B\mathrm{spin}_{n-7})$.

Similarly, since ϕ_3 is spin-trivial, we have

PROPOSITION 4.2. $0 \in \phi_3(w_{n-7}) \subset H^{n-4}(B\mathrm{spin}_{n-7})$.

Let the n-MPT for $\pi\colon B\mathrm{spin}_{n-k} \to B\mathrm{spin}_n$ for $k = 7$ or 8 be indicated by the following diagram:

$$\begin{array}{ccccc} & & & & B\mathrm{spin}_{n-k} \\ & q_2 \swarrow & & \swarrow q_1 & \downarrow \pi \\ E_2 & \xrightarrow{p_2} & E_1 & \xrightarrow{p_1} & B\mathrm{spin}_n \end{array}$$

when $k = 8$, it is understood this tower is induced over $B\hat{\mathrm{S}}\mathrm{O}_n\langle 8\rangle$. We shall still use the same symbols for the second and third stage of the tower over $B\hat{\mathrm{S}}\mathrm{O}_n\langle 8\rangle$ when no confusion need arise.

Recall the definition of a generating class [**12**]. Then we have

PROPOSITION 4.3 (*E. Thomas*).

(a) The class w_{n-9} in $H^{n-9}(B\mathrm{spin}_n)$ is a generating class for the pair $(k_1^2, 0)$ in $H^{n-5}(E_1) \oplus H^{n-4}(E_1)$ relative to the pair (ϕ_4, ϕ_5).

(b) The class $p_1^*(w_{n-9})$ is a generating class for k_1^3, relative to the operation ψ_5.

(c) The class w_{n-7} in $H^{n-7}(B\mathrm{spin}_n)$ is a generating class for k_2^2 in $H^{n-4}(E_1)$ relative to the operation ϕ_3.

Now by inspection of the k-invariants for the n-MPT for π and the connectivity condition on M, together with Proposition 4.1, 4.2, 4.3, the generating class theorem of **[12]** and the admissible class theorem **[8]** we have:

THEOREM 4.4. *Let η be an n-plane bundle over M. Suppose*

$$\mathrm{Sq}^2 H^{n-7}(M;\mathbf{Z}) = \mathrm{Sq}^2 H^{n-7}(M), \qquad \mathrm{Sq}^1 H^{n-5}(M) \subset \mathrm{Sq}^2 H^{n-6}(M)$$

and $\mathrm{Indet}^{n-4}(\psi_5, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$.

(a) *Suppose $w_4(\eta) \neq w_4(M)$. Then η admits 7-linearly independent cross sections if and only if $\delta w_{n-7}(\eta) = 0$, $w_{n-5}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \phi_3(w_{n-7}(\eta))$ and $0 \in \psi_5(w_{n-9}(\eta))$.*

(b) *Suppose $w_4(\eta) = w_4(M)$. Then η admits 7-linearly independent cross sections if and only if $\delta w_{n-7}(\eta) = 0$, $w_{n-5}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \phi_3(w_{n-7}(\eta))$, $0 \in \psi_5(w_{n-9}(\eta))$ and $\Gamma(U(\eta)) = 0$.*

We now specialise to the case when η is the tangent bundle of M τ. Let $g\colon M \times M \to T(\tau)$ be the map that collapses the complement of a tubular neighbourhood of the diagonal in $M \times M$ to a point. Let $U = g^*(U(\tau))$. Then we have the following decomposition of Milnor and Wu:

$$U \bmod 2 = \sum_{2i<n} \sum_k \alpha_i^k \otimes \beta_{n-i}^k + \sum_{2i<n} \sum_k \beta_{n-i}^k \otimes \alpha_i^k$$

where $\alpha_i^k \in H^i(M)$, $\beta_{n-i}^k \in H^{n-i}(M)$ and $\alpha_i^k \cup \beta_{n-i}^j = \delta_{kj}\mu$ where $\mu \in H^n(M)$ is a generator and δ_{kj} is the Kronecker function.

Then we have

LEMMA 4.5. *Let* $A = \sum_{2i<n}\sum_k \alpha_i^k \otimes \beta_{n-i}^k$.

(i) $\underline{U} \bmod 2 = A + t^*A$ *where* $t^*\colon H^*(M \times M) \to H^*(M \times M)$ *is the homomorphism induced by the map that interchanges the factors.*

(ii) $A \cup t^*A = \chi_2(M)\mu \otimes \mu$.

(iii) $\mathrm{Sq}^{n-3} A = \mathrm{Sq}^{n-3}\mathrm{Sq}^2 A = (\mathrm{Sq}^{n-3}\mathrm{Sq}^3 + \mathrm{Sq}^{n-1}\mathrm{Sq}^1)A = 0$.

Proof. (i) and (ii) are due to Thomas **[15]** (iii) is a consequence of the connectivity condition on M.

Then according to Mahowald **[4]** we have

THEOREM 4.6 (*Mahowald-Thomas*). *Γ is defined on A and so on t^*A. In particular modulo zero indeterminacy,*

$$\Gamma(U(\tau)) = \chi_2(M)U(\tau)\mu.$$

4.7. *Proof of Theorem* 2.1. This now follows immediately from Theorem 4.4 (b) and Theorem 4.6.

5. Proof of Theorem 3.1 and Theorem 3.2. Consider the fibration π: $B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle \to B\hat{\mathrm{S}}\mathrm{O}_n\langle 8\rangle$. Let U_j be the Thom class of the universal j-plane bundle over $B\hat{\mathrm{S}}\mathrm{O}_j\langle 8\rangle$ for $j \geq 4$. Then from §3 we know that $(0,0,0) \in (\phi_6, \phi_7, \zeta_7)U_j$. Let l: $B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle \to B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle$ be the obvious inclusion. Then w_{n-11} in $H^*(B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle)$ is the reduction mod 2 of the Euler class of the universal $(n-11)$-plane bundle over $B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle$. Hence

$$(5.1) \qquad (0,0,0) \in (\phi_6, \phi_7, \zeta_7)(w_{n-11}) \subset H^{n-5}(B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle).$$

Now it can be easily checked that l^*: $H^*(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \to H^*(B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle)$ is an epimorphism in dimension $\leq n-4$ for $n > 43$, while for $n = 19$ and 27 l^* is monomorphic in dimension $\leq n-4$. This is readily derived by considering the Leray-Serre spectral sequence for $B\hat{\mathrm{S}}\mathrm{O}_j\langle 8\rangle$. Also for $n > 27$, $\ker l^*$ in dimension $n-4$ is generated by $\{w_{n-10}\mathrm{Sq}^2 \nu_4, w_{n-8}\cdot\nu_4\}$, where $\nu_4 \in H^4(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \approx \mathbf{Z}_2$ is a generator, while in dimension $n-5$ $\ker l^*$ is generated by $\{w_{n-9}\cdot\nu_4\}$.

Now for $n \leq 43$, $\mathrm{Indet}^{n-5,n-4,n-4}((\phi_6, \phi_7, \zeta_7), B\hat{\mathrm{S}}\mathrm{O}_{n-11}\langle 8\rangle)$ on Cok l^* is contained in Coker l^*. Therefore in view of (5.1), $(0,0,0) \in l^*(\phi_6, \phi_7, \zeta_7)(w_{n-11})$ modulo $l^*\ \mathrm{Indet}^{n-5,n-4,n-4}((\phi_6, \phi_7, \zeta_7), B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle)$ for $n \geq 19$. Thus by naturality there exist classes $U_1 \in H^{n-5}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle)$, $U_2, U_3 \in H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle)$ such that $(U_1, U_2, U_3) \in (\phi_6, \phi_7, \zeta_7)w_{n-11}$ and $l^*U_1 = 0$, $l^*U_2 = 0$ and $l^*U_3 = 0$.

Since l^* is injective for $n = 19$, 27, in dimension $n-4$, $n-5$, $(0,0,0) \in (\phi_6, \phi_7, \zeta_7)(w_{n-11}) \subset H^{n-5}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle)$ for $n = 19$ and 27. For $n > 27$, $\mathrm{Sq}^3(w_{n-9}\nu_4) \neq 0$ for $w_{n-9}\nu_4 \in H^{n-5}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle)$. But by (3.1), $\mathrm{Sq}^3 U_1 = 0$, since $\mathrm{Sq}^3\phi_6 = 0$. Thus $U_1 = 0$. Since $\mathrm{Sq}^2(w_{n-10}\nu_4) = w_{n-10}\mathrm{Sq}^2\nu_4$ and $\mathrm{Sq}^1(w_{n-9}\nu_4) = w_{n-8}\nu_4$, we may assume that $U_2 = 0$. Now $U_3 \neq w_{n-10}\mathrm{Sq}^2\nu_4$ since by (3.1) $\mathrm{Sq}^1 U_3 = \mathrm{Sq}^2 U_1 = 0$ but $\mathrm{Sq}^1(w_{n-10}\mathrm{Sq}^2\nu_4) = w_{n-10}\mathrm{Sq}^3\nu_4 \neq 0$. Now $U_3 \neq w_{n-8}\nu_4$ for $\mathrm{Sq}^2 U_3 = \mathrm{Sq}^2\zeta_7(w_{n-11}) = \alpha\,\mathrm{Sq}^8\mathrm{Sq}^1 w_{n-11}$ for some $\alpha \in \mathbf{Z}_2$ and $\mathrm{Sq}^2(w_{n-8}\nu_4) = w_{n-8}\mathrm{Sq}^2\nu_4 \neq w_8 w_{n-10}$. Thus $U_3 = 0$.

We have thus proved.

PROPOSITION 5.2. *Let the operation* $(\phi_6, \phi_7, \zeta_7)$ *be given as in* §3. *Then*

$$(0,0,0) \in (\phi_6, \phi_7, \zeta_7)(w_{n-11}) \subset H^{n-5}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle) \oplus H^{n-4}(B\hat{\mathrm{S}}\mathrm{O}_{n-8}\langle 8\rangle).$$

$\operatorname{Ker} l^*$ is generated by $\{w_{n-10}, w_{n-9}\}$ as a $\mathfrak{A}(H^*(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle))$-module in dimensions $\leq n+1$. Now $\operatorname{Indet}^{n-4}(\psi_7, B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle)$ is determined by a cohomology operation Ω on cohomology vectors $(x, y) \in H^{n-8}(B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle) \times H^{n-7}(B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle)$ satisfying $\mathrm{Sq}^2\mathrm{Sq}^1 x + \mathrm{Sq}^2 y = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 y = 0$. Since l^* is onto in dimensions $\leq n-8$ and onto in dimension $n-7$ if $n > 43$ and since for $n \leq 43$, $\mathrm{Sq}^2(\operatorname{Cok} l^*)_{n-7} \subseteq (\operatorname{Cok} l_*)_{n-5}$ and is non-zero in $(\operatorname{Cok} l_*)_{n-5}$, there exist classes $x' \in H^{n-8}(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle)$ and $y' \in H^{n-7}(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle)$ such that $l^*x' = x$ and $l^*y' = y$. Therefore $l^*(\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^2 y') = 0$ and $l^*(\mathrm{Sq}^2\mathrm{Sq}^1 y') = 0$. For $n = 19$ and 27, l^* is a monomorphism in dimension $n-4$ and $n-5$ and so $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^2 y' = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 y' = 0$.

Now consider the case $n \geq 35$. $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^2 y' = \alpha w_{n-9} \cdot \nu_4$ for some $\alpha \in \mathbf{Z}_2$. Let $K\colon B\hat{\mathrm{SO}}_j\langle 8\rangle \to B\mathrm{spin}_j$ be the obvious map. Then by considering for $p \leq n-4$,

$$\begin{aligned} H^p(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle) = K^*H^p(B\mathrm{spin}_{n-8}) &+ \nu_4 \cdot K^*H^{p-4}(B\mathrm{spin}_{n-8}) \\ &+ \mathrm{Sq}^2\nu_4 \cdot K^*H^{p-6}(B\mathrm{spin}_{n-8}) \\ &+ \mathrm{Sq}^3\nu_4 \cdot K^*H^{p-7}(B\mathrm{spin}_{n-8}) \end{aligned}$$

modulo $\{(\mathrm{Sq}^I \nu_4)^j K^*H^*(B\mathrm{spin}_{n-8})$; I is an admissible sequence of excess < 4, length $l(I) \geq 2$ and $j \geq 1$ or $l(I) \leq 1$ and $j \geq 2\}$ and the fact that l^* is an isomorphism in dimensions $\leq n-11$ we can show that

$$w_{n-9} \cdot \nu_4 \notin \left(\operatorname{Im}\mathrm{Sq}^2\mathrm{Sq}^1 + \operatorname{Im}\mathrm{Sq}^2\right) \cap \operatorname{Ker} l^* \cap H^{n-5}(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle).$$

Thus $\alpha = 0$ and $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^2 y' = 0$. Similarly it can be shown that $\mathrm{Sq}^2\mathrm{Sq}^1 x' + \mathrm{Sq}^2 y' = 0$ and $l^*(\mathrm{Sq}^2\mathrm{Sq}^1 y') = 0$ implies that $\mathrm{Sq}^2\mathrm{Sq}^1 y' = 0$. Hence Ω is defined on (x', y'). Thus $\Omega(x, y) = \Omega(l^*x', l^*y') = l^*\Omega(x', y')$ modulo $\operatorname{Indet}^{n-4}(\Omega, B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle)$. In view of this and the fact that ψ_7 is trivial on the Thom class of the universal $(n-11)$-plane bundle over $B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle$,

$$l^*\psi_7(w_{n-11}) \subseteq \mathrm{Sq}^2 H^{n-6}(B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle) + \mathrm{Sq}^1 H^{n-7}(B\hat{\mathrm{SO}}_{n-11}\langle 8\rangle)$$

modulo $l^* \operatorname{Indet}^{n-4}(\psi_7, B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle)$. Since for $n \leq 43$ $\mathrm{Sq}^2(\operatorname{Cok} l^*)_{n-6} + \mathrm{Sq}^1(\operatorname{Cok} l^*)_{n-5}$ is non-trivial in $(\operatorname{Cok} l^*)_{n-4}$ and since l^* is an epimorphism in dimensions $\leq n-4$ for $n > 43$, we can conclude that $l^*\psi_7(w_{n-11}) \subset l^* \operatorname{Indet}^{n-4}(\psi_7, B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle)$. Since $\operatorname{Ker} l^*$ is generated by

$$\begin{aligned} \{w_{n-10}\mathrm{Sq}^2\nu_4, w_{n-8}\nu_4\} \subset \mathrm{Sq}^2 H^{n-6}&(B\hat{SO}_{n-8}\langle 8\rangle) \\ &+ \mathrm{Sq}^1 H^{n-5}(B\hat{SO}_{n-8}\langle 8\rangle), \end{aligned}$$

we deduce that $0 \in \psi_7(w_{n-11}) \subset H^{n-4}(B\hat{\mathrm{SO}}_{n-8}\langle 8\rangle)$.

Hence

PROPOSITION 5.3. *Let ψ_7 be the stable tertiary cohomology operation given as in §3. Then* $0 \in \psi_7(w_{n-11}) \subset H^{n-4}(B\hat{SO}_{n-8}\langle 8\rangle)$.

We can now use the generating class theorem to realize the k-invariants for liftings. Now in $H^*(B\hat{SO}_n\langle 8\rangle)$, $\chi Sq^4 w_{n-11} = w_{n-7}$, $Sq^4 Sq^1 w_{n-11} = 0$, $Sq^4 Sq^2 w_{n-11} = 0$ and $Sq^4 Sq^2 Sq^1 w_{n-11} = 0$. Π^*: $H^*(B\hat{SO}_n\langle 8\rangle) \to H^*(B\hat{SO}_{n-8}\langle 8\rangle)$ is an epimorphism in dimension $\leq n - 4$ for $n \geq 35$. Now for $n = 27$ or 19, Π^* is an epimorphism in dimensions $n-8$, $n-6$, $n-5$ and $n-4$ and q_1^* is an epimorphism in dimensions $\leq n$. A computational check shows that

$$\{(Sq^2 x, Sq^2 Sq^1 x) : x \in H^{n-7}(B\hat{SO}_{n-8}\langle 8\rangle)\}$$
$$\subset \Pi^*\{(Sq^2 x, Sq^2 Sq^1 x) : x \in H^{n-7}(B\hat{SO}_n\langle 8\rangle)\}.$$

Thus by the generating class theorem, Proposition 5.2 and Proposition 5.3 we have

PROPOSITION 5.4.

(a) *The class* $w_{n-11} \in H^{n-11}(B\hat{SO}_n\langle 8\rangle)$ *is a generating class for* $(k_1^2, k_2^2, 0)$ *in* $H^{n-5}(E_1) \oplus H^{n-4}(E_1) \oplus H^{n-4}(E_1)$ *relative to the operation* $(\phi_6, \phi_7, \zeta_7)$.

(b) *The class* $p_1^*(w_{n-11})$ *in* $H^{n-11}(E_1)$ *is a generating class for* $k_1^3 \in H^{n-4}(E_2)$ *relative to the operation* ψ_7.

Consider now π: $B\mathrm{spin}_{n-8} \to B\mathrm{spin}_n$ for $n \geq 19$. Then π^* is an epimorphism in dimension $\leq n$. Then by the admissible class theorem we have

$$U(E_1) \cdot \{k_6^2\} \in \Gamma(U(E_1))$$

where E_1 is the first stage of the modified Postnikov tower for π and $U(E_1)$ is the Thom class of the n-plane bundle over E_1 induced from the universal n-plane bundle over $B\mathrm{spin}_n$ by p_1. Therefore by naturality we have

PROPOSITION 5.5. $U(E_1) \cdot \{k_6^2\} \in \Gamma(U(E_1)) \subset H^{2n}(T(E_1))$ *where* $T(E_1)$ *is the Thom space of the n-plane bundle over* E_1 *induced from the universal n-plane bundle over* $B\hat{SO}_n\langle 8\rangle$ *by* p_1 *and* $U(E_1)$ *the corresponding Thom class.*

We also have the following proposition since $\tilde{\phi}_7$ is in the terminology of [**12**], of bundle type 2.

PROPOSITION 5.6.
(a) $0 \in \tilde{\phi}_7(w_{n-11}, w_4) \subset H^{n-4}(B\mathrm{spin}_{n-8})$
(b) *The class* $w_{n-11} \in H^{n-11}(B\mathrm{spin}_n)$ *is a generating class for* k_2^2 *in* $H^{n-4}(E_1)$ *relative to the operation* $\tilde{\phi}_7$ *with twisting given by* w_4.

Proof. (a) is a consequence of (3.4). Since $(\chi\,\mathrm{Sq}^4 + w_4)w_{n-11} = w_{n-7} \in H^{n-7}(B\mathrm{spin}_n)$, $(\theta \cdot \mathrm{Sq}^2)(w_{n-11}) = 0$, $\alpha(w_{n-11}) = 0$ and π^*: $H^*(B\mathrm{spin}_n) \to H^*(B\mathrm{spin}_{n-8})$ is an epimorphism in dimension $\leq n$. (b) follows.

Suppose *either* $\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M)$ and $\mathrm{Sq}^4 H^{n-8}(M) = \mathrm{Sq}^2 H^4(M) = \mathrm{Sq}^1 H^4(M) = 0$ *or* $\mathrm{Sq}^2 H^{n-7}(M) = 0$ and $\mathrm{Sq}^1 H^{n-5}(M) + \mathrm{Sq}^2 H^{n-6}(M) \subset \chi\mathrm{Sq}^4 H^{n-8}(M)$.

Suppose $\mathrm{Indet}^{n-4}(\psi_7, M) = \mathrm{Indet}^{n-4}(k_1^3, M)$. Then we have with the above assumption

THEOREM 5.7. *Let* η *be an n-plane bundle over M such that* $w_4(\eta) = 0$.
(a) *Suppose* $w_4(M) \neq 0$. *Then* $\mathrm{span}(\eta) \geq 8$ *if and only if* $w_{n-7}(\eta) = 0$, $0 \in \phi_6(w_{n-11}(\eta))$, $0 \in \phi_7(w_{n-11}(\eta))$ *and* $0 \in \psi_7(w_{n-11}(\eta))$.
(b) *Suppose* $w_4(M) = 0$. *Then* $\mathrm{span}(\eta) \geq 8$ *if and only if* $w_{n-7}(\eta) = 0$, $0 \in \phi_6(w_{n-11}(\eta))$, $0 \in \phi_7(w_{n-11}(\eta))$, $\Gamma(U(\eta)) = 0$ *and* $0 \in \psi_7(w_{n-11}(\eta))$.

Proof. Since $w_4(M) \neq 0$. $\mathrm{Sq}^4 H^{n-4}(M) = H^n(M)$. Trivially $0 \in k_6^2(\eta)$. Part (a) then follows from Propositions 5.2, 5.3, 5.4 and the generating class theorem.

Part (b) is similar and it follows from Propositions 5.2, 5.3, 5.4, 5.5 the generating class theorem and the admissible class theorem. We leave the details to the reader.

Similarly we have a variation of Theorem 5.7.

THEOREM 5.8. *Let* η *be an n-plane bundle over M. Suppose either*

$$\mathrm{Sq}^2\mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M), \qquad \left(\mathrm{Sq}^4 + w_4(\eta)\right)H^{n-8}(M) = 0$$

and

$$\mathrm{Sq}^2 H^4(M) = \mathrm{Sq}^1 H^4(M) = 0 \quad or \quad \mathrm{Sq}^2 H^{n-7}(M) = 0$$

and

$$\mathrm{Sq}^2 H^{n-6}(M) + \mathrm{Sq}^1 H^{n-5}(M) \subset \left(\chi\,\mathrm{Sq}^4 + w_4(\eta)\right)H^{n-8}.$$

Assume $\operatorname{Indet}^{n-4}(\psi_5, M) = \operatorname{Indet}^{n-4}(k_1^3, M)$. *Then*

(a) *If* $w_4(\eta) \neq w_4(M)$, *then* $\operatorname{span}(\eta) \geq 8$ *if and only if* $w_{n-7}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \tilde{\phi}_7(w_{n-11}(\eta), w_4(\eta))$ *and* $0 \in \psi_5(w_{n-9}(\eta))$.

(b) *If* $w_4(\eta) = w_4(M)$, *then* $\operatorname{span}(\eta) \geq 8$ *if and only if* $w_{n-7}(\eta) = 0$, $0 \in \phi_4(w_{n-9}(\eta))$, $0 \in \tilde{\phi}_7(w_{n-11}(\eta), w_4(\eta))$, $\Gamma(U(\eta)) = 0$ *and* $0 \in \psi_5(w_{n-9}(\eta))$.

The proof is similar to that of 5.7 using Proposition 5.6. We leave the details to the reader.

5.9. *Proof of Theorem* 3.1 *and Theorem* 3.2. Theorem 3.1 is now a consequence of Theorem 5.7(b) by taking η to be the tangent bundle of M and the fact that $\Gamma(U(\tau)) = \chi_2(M)U(\tau)\mu$ (cf. Theorem 4.6). Similarly Theorem 3.2 follows from Theorem 5.8(b) and Theorem 4.6.

6. Applications.

6.1. Let $M = S^{7+8k} \times QP^{2l+1}$ for $k \geq 0, l \geq 0$. Then we have

THEOREM. $\operatorname{Span}(M) \geq 8$.

Proof. Plainly $H^{n-5}(M) \simeq H^{n-6}(M) \simeq 0$, $w_4(M) = 0$ and $\chi_2(M) = 0$. For $k \geq 1$, $H^{n-7}(M) \simeq 0$ and so $\operatorname{Indet}^{n-4}(\psi_7, M) = \operatorname{Indet}^{n-4}(k_1^3, M) \simeq 0$. For $k = 0$, $\delta H^{n-8}(M) \simeq 0$ and so $\operatorname{Indet}^{n-4}(k_1^3, M) \simeq 0$. For $k = 0$, $H^{n-7}(M) \simeq H^{n-7}(QP^{2l+1})$ and so by naturality $\operatorname{Indet}^{n-4}(\psi_7, M) \simeq 0$. Now $w_{n-11}(M) = 0$ if $k \geq 1$ and $w_{n-11}(M) = 1 \times U^{2l}$ if $k = 0$ where $U \in H^4(QP^{2l+1}) \simeq \mathbf{Z}_2$ is a generator. Trivially $w_{n-7}(M) = 0$. Thus if $k \geq 1$ $\operatorname{span}(M) \geq 8$. Now for $k = 0$ $w_{n-11}(M) = j^*(U^{2l})$, where $j\colon M \to QP^{2l+1}$ is the projection. For dimensional reasons $(\phi_6, \phi_7, \zeta_7)(U^{2l}) = (0,0,0)$ and $\psi_7(U^{2l}) = 0$. Therefore by naturality $0 \in \psi_7(w_{n-11}(M))$. It follows from Theorem 3.1 that $\operatorname{span}(M) \geq 8$. This completes the proof.

Similarly we have

THEOREM 6.2. $\operatorname{Span}(S^{3+8k} \times QP^{2l+1} \times QP^{2j+1}) \geq 8$ *for* $k \geq 1$, $l, j \geq 0$.

We now give an application to immersion of M into euclidean spaces. Suppose $w_4(M) = 0$ and $\dim M = n \equiv 11 \bmod 16 > 11$. Then following Massey one readily deduces that $\bar{w}_{n-j}(M) = 0$ for $j = 0, 1, 2, \ldots, 10, 11$.

Let ν be the stable normal bundle of M. For stable bundle we can ignore the secondary obstruction in the top dimension given by Table 1 and Table 2. Thus by Theorem 4.4 we have

THEOREM 6.3. *Suppose* $\mathrm{Sq}^1 H^4(M) = 0$, $\phi_{0,0}H^4(M) = 0$ *and* $H_6(M; \mathbf{Z})$ *has no 2-torsion. Then if* $w_4(M) = 0$ *and* $n \equiv 11 \bmod 16 > 11$, *$M$ immerses in* $\mathbf{R}^{2n-7}$.

Consequently we have by Theorem 5.8:

THEOREM 6.4. *Suppose M is 4-connected* mod 2 *and* $\dim M = n \equiv 11 \bmod 16 > 11$. *Then M immerses in* $\mathbf{R}^{2n-7}$ *if* $\mathrm{Sq}^2 H^{n-7}(M; \mathbf{Z}) = \mathrm{Sq}^2 H^{n-7}(M)$ *and immerses in* $\mathbf{R}^{2n-8}$ *if* $\mathrm{Sq}^2 \mathrm{Sq}^1 H^{n-8}(M) = \mathrm{Sq}^2 H^{n-7}(M)$.

Similarly we have by Theorem 5.7:

THEOREM 6.5. *Suppose M satisfies conditions* A *and* B *of* §3, $w_4(M) = 0$ *and* $n \equiv 11 \bmod 16 > 11$. *If* $\mathrm{Sq}^1 H^4(M) = \mathrm{Sq}^2 H^4(M) = 0$ *and if either* $\chi\mathrm{Sq}^4 H^{n-8}(M) = 0$ *or* $\phi_3 H^4(M) = 0$ *and* $\mathrm{Sq}^2 H^{n-7}(M) = 0$ *then M immerses in* $\mathbf{R}^{2n-8}$.

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