

ON VALUED, COMPLETE FIELDS AND THEIR AUTOMORPHISMS

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The following theorem is proved:

THEOREM. *Let K be a valued, complete field, and assume that the valuation topology admits countable neighbourhood bases. All the automorphisms of K are continuous if and only if K is not algebraically closed.*

Introduction. In this paper we study the following problem: Which valued, complete fields have the property that all their automorphisms are continuous? In the case of Archimedean valuations the answer is well known: There are only two complete fields, \mathbf{R} and \mathbf{C} . $\text{Aut}(\mathbf{R}) = \{\text{Id}\}$; \mathbf{C} has exactly two continuous automorphisms, but $\text{card Aut}(\mathbf{C}) = 2^{(2^{\aleph_0})}$, hence very many $\tau \in \text{Aut}(\mathbf{C})$ are discontinuous. Henceforth we shall consider exclusively fields K with non-Archimedean valuations ϕ in the sense of Krull. We assume that the valuation topology admits countable neighbourhood bases. Suppose now that (K, ϕ) is complete. Our main result is that all the automorphisms of K are continuous if and only if K is not algebraically closed. Thus there is a perfect analogy to the Archimedean case.

If the valuation ϕ has rank 1, Hensel's lemma is available and a rather short proof can be given (see §4). The case where ϕ has infinite rank requires a new method; our proof in §5 will be based on a lemma on solvability of certain infinite systems of equations. The technique of this lemma can be applied to other problems on complete fields; two illustrations for this are given in §6.

We should like to mention that the paper was motivated by studies in the theory of orthomodular spaces. These are, by definition, vector spaces E endowed with a hermitian form Ψ such that the lattice $L = \{U \subseteq E: U = (U^\perp)^\perp\}$ of all orthogonally closed subspaces of (E, Ψ) satisfies the orthomodular law: $U \leq V \Rightarrow V = U \vee (V \wedge U^\perp)$ for all $U, V \in L$. Classical examples are the Hilbert spaces over \mathbf{R} or \mathbf{C} . In recent years numerous non-classical, infinite dimensional orthomodular spaces have been discovered. All these new spaces are constructed over certain valued,

complete fields; the valuations in question have infinite rank. Investigations on the particular properties of these spaces E lead, via the fundamental theorem of projective geometry, to maps $f: E \rightarrow E$ which are semilinear, i.e. linear up to some fixed automorphism τ of the base field K . It is then crucial to know that τ and τ^{-1} are continuous. We refer to [2] for a comprehensive account on orthomodular spaces.

1. Notations. Let K be a field and $\phi: K \rightarrow \Gamma \cup \{\infty\}$ a Krull valuation on K , i.e. Γ is an ordered abelian group, written additively, and for all $x, y \in K$ we have (i) $\phi(x) = \infty \Leftrightarrow x = 0$, (ii) $\phi(xy) = \phi(x) + \phi(y)$, (iii) $\phi(x + y) \geq \min\{\phi(x), \phi(y)\}$. We assume that $\Gamma \neq \{0\}$ and that ϕ is onto. The valuation topology $\mathcal{T} = \mathcal{T}(\phi)$ on K is defined by taking $\{U_\gamma(a): \gamma \in \Gamma\}$, where $U_\gamma(a) = \{x \in K: \phi(a - x) \geq \gamma\}$, as a neighbourhood basis of $a \in K$. Notions as “continuity” etc. always refer to this topology. Throughout the paper we make the assumption that $\mathcal{T}(\phi)$ admits a countable zero-neighbourhood basis, or, equivalently, that Γ contains a countable cofinal subset. Thus by “valued field” we mean a field with a Krull valuation satisfying this countability condition.

A subgroup $\Delta \leq \Gamma$ is called isolated if $\delta \in \Delta$, $\gamma \in \Gamma$, $0 \leq \gamma \leq \delta$ implies $\gamma \in \Delta$. We set $\mathcal{D} = \mathcal{D}(\Gamma) = \{\Delta \leq \Gamma: \Delta \text{ is an isolated subgroup}\}$. Every $\Delta \in \mathcal{D}$ gives rise to a valuation $\Phi_\Delta: K \rightarrow (\Gamma/\Delta) \cup \{\infty\}$. The valuation ring corresponding to Φ_Δ is $A_\Delta = \{x \in K: \phi(x) \geq \delta \text{ for some } \delta \in \Delta\}$. We let J_Δ denote the maximal ideal of A_Δ ; $\hat{K}_\Delta = A_\Delta/J_\Delta$ is the residue field of A_Δ . For details see [1, §7].

We denote by ω the set of all finite ordinals.

2. The main theorem. Our purpose is to prove

THEOREM 1. *Let the valued field (K, ϕ) be complete. Then the following conditions are equivalent:*

- (a) *All automorphisms of K are continuous.*
- (b) *K is not algebraically closed.*

The proof will be divided into several steps. We begin with the easy part.

Proof of the implication (a) \Rightarrow (b). Suppose (K, ϕ) is complete and algebraically closed. Let P be the prime field. Then $\text{tr deg}(K|P) > \aleph_0$. It is easy to construct a transcendence basis $B = \{t_\mu: \mu \in M\}$ for K over P such that $\{\phi(t_\mu): \mu \in M\}$ has no upper bound in Γ . Then (by virtue of the general assumption on Γ) there is a sequence $(t_i)_{i \in \omega}$, $t_i \in B$, with

$\phi(t_i) \rightarrow \infty$ as $i \rightarrow \infty$. Since $\text{card } M > \aleph_0$ there exists a $\gamma \in \Gamma$ with $\{\mu \in M: \phi(t_\mu) \leq \gamma\}$ infinite, so we can define a permutation $\sigma: B \rightarrow B$ such that $\phi(\sigma(t_i)) \leq \gamma$ for all $i \in \omega$. σ induces an automorphism τ_0 of $P(t_\mu)_{\mu \in M}$ which is not continuous, and τ_0 extends to an automorphism $\tau: K \rightarrow K$, since K is algebraically closed. Hence (a) implies (b).

3. On the rank of ϕ . The set $\mathcal{D} = \mathcal{D}(\Gamma)$ is totally ordered by the inclusion \subseteq . The order type of (\mathcal{D}, \subseteq) is called the rank of the valuation ϕ and denoted by $\text{rg}(\phi)$. The cofinality of \mathcal{D} , denoted by $\text{cf}(\mathcal{D})$, is the least ordinal which is similar to a cofinal subset of (\mathcal{D}, \subseteq) . There are two possibilities.

1st case. $\text{cf}(\mathcal{D}) = 1$. This means that \mathcal{D} has a largest element, say Δ_0 . Then the valuation $\phi_{\Delta_0}: K \rightarrow (\Gamma/\Delta_0) \cup \{\infty\}$ has rank 1, because Δ_0 is maximal. Furthermore, ϕ_{Δ_0} defines the same topology on K as ϕ . Thus, in the proof of Theorem 1, we may replace ϕ by ϕ_{Δ_0} , i.e. we may suppose that $\text{rg}(\phi) = 1$.

2nd case. $\text{cf}(\mathcal{D}) = \omega$. This means that all $\phi_\Delta (\Delta \in \mathcal{D})$ have infinite rank.

We now continue the proof of Theorem 1. We have to show that the existence of a discontinuous $\tau \in \text{Aut}(K)$ implies that K is algebraically closed. The above two cases must be treated differently.

4. The first case: $\text{rg}(\phi) = 1$. We need the following

LEMMA 1. *Let (K, ϕ) be a valued field and suppose that $\tau \in \text{Aut}(K)$ is discontinuous. Then*

(i) *There is a sequence $(v_i)_{i \in \omega}$ in K such that $v_i \rightarrow 0$ and $\tau(v_i) \rightarrow \infty$ as $i \rightarrow \infty$.*

(ii) *Given $a, b \in K$ and $\gamma \in \Gamma$ there exists a $c \in K$ with $\phi(b - c) \geq \gamma$ and $\phi(a - \tau(c)) \geq \gamma$.*

Proof. (i) follows by routine arguments. To show (ii) let $(v_i)_{i \in \omega}$ be as in (i). Put $u_i = b + (\tau^{-1}(a) - b) \cdot v_i \cdot (1 + v_i)^{-1}$. Then $u_i \rightarrow b$ and $\tau(u_i) \rightarrow a$, so we can take $c = u_j$ with j sufficiently large.

For the basic properties of henselian fields we refer to [3, Chap. F] or [1, §16].

LEMMA 2. *Suppose the valued field (K, ϕ) is henselian. Let L be the algebraic closure of K and ψ the unique valuation on L which extends ϕ . If K has a discontinuous automorphism τ , then K is topologically dense in (L, ψ) .*

Proof. Let Γ and Γ' be the value groups of ϕ and ψ respectively. Γ is cofinal in Γ' . Let $A = \{x \in K: \phi(x) \geq 0\}$ be the valuation ring of ϕ , \hat{K} its residue field and $\hat{\cdot}: A \rightarrow \hat{K}$ the canonical epimorphism.

(1) We assert: given $f(x) = \sum_{j=0}^r a_j X^j \in K[X]$ (where $a_r = 1, r \geq 1$) and $\gamma \in \Gamma$ there exist $a'_0, \dots, a'_r \in K$ such that

(i) $\phi(a'_j - a_j) \geq \gamma$ for $0 \leq j \leq r-1, a'_r = 1$.

(ii) $f^*(X) = \sum_{j=0}^r a'_j X^j$ decomposes into linear factors in $K[X]$.

We show this by induction on r . The case $r = 1$ is trivial, so let $r \geq 2$. Using Lemma 1, (ii) we find $c_0, \dots, c_{r-1} \in K$ such that $\phi(a_j - \tau(c_j)) \geq \gamma$ for $0 \leq j \leq r-1$, and $\phi(1 - c_1) > 0, \phi(0 - c_j) > 0$ for $j = 0, 2, \dots, r-1$. Let $h(X) = \sum_{j=0}^r c_j X^j$ where $c_r = 1$. Then $h(X) \in A[X]$, $h(X)$ is monic, and $\hat{h}(X) = \sum_{j=0}^r \hat{c}_j X^j = X + X^r \in \hat{K}[X]$ has a simple root in \hat{K} . It follows that $h(X)$ has a root in K , hence also $h^\tau(X) = \sum_{j=0}^r \tau(c_j) X^j$ has a root in K . Thus $h^\tau(X) = (X - t)g(X)$ where $t \in K$ and $g(X) = \sum_{j=0}^{r-1} d_j X^j \in K[X], d_{r-1} = 1$. By the induction hypothesis, applied to $g(X)$ and $\varepsilon = \max\{\gamma, \gamma - \phi(t)\}$, there are $d'_0, \dots, d'_{r-1} \in K$ with $\phi(d'_j - d_j) \geq \varepsilon$ for $0 \leq j \leq r-2, d'_{r-1} = 1$, such that

$$g^*(X) = \sum_{j=0}^{r-1} d'_j X^j$$

is a product of linear factors in $K[X]$. We determine a'_0, \dots, a'_r so that $\sum_{j=0}^r a'_j X^j = (X - t)g^*(X)$. Clearly (ii) holds, and (i) is readily checked.

(2) Now let $w \in L$ and $\delta \in \Gamma'$ be given. Let

$$f(X) = \text{irr}(w, K) = \sum_{j=0}^r a_j X^j \in K[X]$$

and choose $\gamma \in \Gamma$ such that $\gamma \geq \max\{r \cdot \delta - j \cdot \psi(w): 0 \leq j \leq r-1\}$. By (1) there are $a'_0, \dots, a'_r \in K$ with $\phi(a'_j - a_j) \geq \gamma$ for $0 \leq j \leq r-1, a'_r = 1$, such that

$$f^*(X) = \sum_{j=0}^r a'_j X^j = \prod_{k=1}^r (X - t_k)$$

where $t_1, \dots, t_r \in K$. Then

$$\prod_{k=1}^r (w - t_k) = f^*(w) = f^*(w) - f(w) = \sum_{j=0}^{r-1} (a'_j - a_j) w^j.$$

It follows that

$$\begin{aligned} \sum_{k=1}^r \psi(w - t_k) &\geq \min\{\psi((a'_j - a_j)w^j): 0 \leq j \leq r - 1\} \\ &\geq \min\{\gamma + j \cdot \psi(w): 0 \leq j \leq r - 1\} \geq r \cdot \delta. \end{aligned}$$

Hence for some $k \in \{1, \dots, r\}$ we have $\psi(w - t_k) \geq \delta$ and $t_k \in K$. Hence K is dense in L .

COROLLARY 1. *Let (K, ϕ) be a complete field, $\text{rg}(\phi) = 1$. If K admits a discontinuous automorphism, then K is algebraically closed.*

Proof. (K, ϕ) is a henselian field ([3], p. 198) and the assertion follows immediately from the lemma, because K is topologically closed in L .

Thus we have the implication (b) \Rightarrow (a) in Theorem 1 in case $\text{rg}(\phi) = 1$.

COROLLARY 2. *Let (K, ϕ) be a henselian field which has a discontinuous automorphism. Then K is separably algebraically closed. Furthermore, the value group Γ of ϕ is divisible and the residue field \hat{K}_Δ is algebraically closed for all $\Delta \in \mathcal{D}$.*

Proof. With the notations of Lemma 2, let $K_s = \{w \in L: w \text{ is separable over } K\}$, $\psi' = \psi|_{K_s}$, and let $G = \text{Gal}(K_s|K)$. Since ψ' is the unique extension of ϕ to K_s , $\psi' \circ \sigma = \psi'$ for every $\sigma \in G$. This implies that every $\sigma \in G$ is continuous. K is the fixed field of G , consequently $K = \bigcap_{\sigma \in G} \text{Ker}(\sigma - \text{Id}_{K_s})$ is topologically closed in K_s . By Lemma 2, K is dense in L , so dense in K_s . We conclude that $K = K_s$. The last two statements follow from the fact that (L, ψ) is an immediate extension of (K, ϕ) , because K is dense in L .

5. The second case: $\text{cf}(\mathcal{D}) = \omega$.

LEMMA 3. *Suppose the valued field (K, ϕ) is complete and $\text{cf}(\mathcal{D}) = \omega$. Let $(v_i)_{i \in \omega}$ be a sequence in K with $v_i \rightarrow 0$ as $i \rightarrow \infty$. Let $q(X, Z) = \sum_{k,l} a_{kl} X^k Z^l$ be a polynomial in $K[X, Z]$ with $q(0, 0) = 0$. Then there exist elements $c_0, c_1, \dots, c_m, \dots$ in K such that*

$$(1) \quad c_m = q(c_{m+1}, v_m) = \sum_{k,l} a_{kl} c_{m+1}^k v_m^l \quad \text{for all } m \in \omega.$$

Proof. Let S be the subring of K generated by the coefficients a_{kl} of $q(X, Z)$. We note that $\{\phi(s) : s \in S\}$ has a lower bound in Γ , by virtue of the hypothesis $\text{cf}(\mathcal{D}) = \omega$. Let Y_i ($i \in \omega$) be new variables and form the polynomial ring $R = S[Y_i]_{i \in \omega}$. For each $n \in \omega$ we define recursively $f_n^{(0)}, \dots, f_n^{(n)} \in R$ by

$$(2) \quad \begin{cases} f_n^{(0)} = q(0, Y_n) \\ f_n^{(j)} = q(f_n^{(j-1)}, Y_{n-j}) \quad \text{for } j = 1, \dots, n. \end{cases}$$

Thus $f_n^{(j)}$ is a polynomial in $Y_n, Y_{n-1}, \dots, Y_{n-j}$. We put $g^{(0)} = f_0^{(0)}$, $g^{(n)} = f_n^{(n)} - f_{n-1}^{(n-1)}$ for $n \geq 1$; then

$$(3) \quad f_n^{(n)} = \sum_{i=0}^n g^{(i)} \quad \text{for all } n \in \omega.$$

We claim that every $g^{(n)} \in R$ has the form

$$(4) \quad g^{(n)} = Y_n \cdot h^{(n)}, \quad \text{where } h^{(n)} \in R.$$

In fact, for $n = 0$ this is true because $q(0, 0) = 0$. Let $n \geq 1$. Substituting $Y_n = 0$ in (2) we see by induction on j that $f_n^{(j)}|_{Y_n=0} = f_{n-1}^{(j-1)}$ for $j = 1, \dots, n$. For $j = n$ this means that $g^{(n)}$ vanishes at $Y_n = 0$, as claimed.

We define the homomorphism $\sigma: R \rightarrow R$ by $\sigma|_S = \text{Id}_S$, $\sigma(Y_i) = Y_{i+1}$ ($i \in \omega$). From (2) it follows easily that $\sigma(f_n^{(j)}) = f_{n+1}^{(j)}$ for $j = 0, \dots, n$, in particular

$$(5) \quad \sigma(f_n^{(n)}) = f_{n+1}^{(n)} \quad \text{for all } n \in \omega.$$

For $n, m \in \omega$ we set $g_m^{(n)} = \sigma^m(g^{(n)})$, where $\sigma^0 = \text{Id}_R$. By (4) we have

$$(6) \quad g_m^{(n)} = Y_{n+m} \cdot h_m^{(n)}, \quad \text{where } h_m^{(n)} = \sigma^m(h^{(n)}) \in R.$$

Now we consider the homomorphism $\eta: R \rightarrow K$ defined by $\eta|_S = \text{Id}_S$, $\eta(Y_i) = v_i$ ($i \in \omega$). We put $d_m^{(n)} = \eta(g_m^{(n)})$; then, by (6),

$$(7) \quad d_m^{(n)} = v_{n+m} \cdot \eta(h_m^{(n)}).$$

We claim that for all $m \in \omega$ we have

$$(8) \quad d_m^{(n)} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

In fact, ϕ is bounded from below on S and on $\{v_i : i \in \omega\}$, therefore (since $\text{cf}(\mathcal{D}) = \omega$) also on $\eta(R)$ = the subring generated by S and the v_i 's. Now $v_{n+m} \rightarrow 0$ as $n \rightarrow \infty$, so (8) follows from (7).

By (8) and by completeness, the series $\sum_{i=0}^\infty d_m^{(i)}$ converges. We set, for $m \in \omega$, $c_m = \sum_{i=0}^\infty d_m^{(i)}$ and verify that these elements satisfy (1). Let $m \in \omega$. For any $n \in \omega$ we have

$$\begin{aligned} \sum_{i=0}^{n+1} g^{(i)} &= f_{n+1}^{(n+1)} = q(f_{n+1}^{(n)}, Y_0) \quad \text{by (2), (3)} \\ &= q(\sigma(f_n^{(n)}), Y_0) \quad \text{by (5)} \\ &= q\left(\sum_{i=0}^n \sigma(g^{(i)}), Y_0\right) \quad \text{by (3)}. \end{aligned}$$

Applying σ^m we get

$$\sum_{i=0}^{n+1} g_m^{(i)} = q\left(\sum_{i=0}^n g_{m+1}^{(i)}, Y_m\right)$$

and applying η we get

$$\sum_{i=0}^{n+1} d_m^{(i)} = q\left(\sum_{i=0}^n d_{m+1}^{(i)}, v_m\right)$$

Now we let $n \rightarrow \infty$ and obtain (1). The proof is complete.

We now show how Lemma 3 can be used to derive consequences from the existence of a discontinuous $\tau \in \text{Aut}(K)$.

LEMMA 4. *Let (K, ϕ) be complete, $\text{cf}(\mathcal{D}) = \omega$. If K admits a discontinuous automorphism τ , then every residue field \hat{K}_Δ ($\Delta \in \mathcal{D}$) is algebraically closed.*

Proof. Fix $\Delta \in \mathcal{D}$ and let $\hat{\cdot} : A_\Delta \rightarrow \hat{K}_\Delta$ be the canonical map. Let $\hat{p}(X) = \hat{a}_0 + \hat{a}_1 X + \dots + \hat{a}_{r-1} X^{r-1} + X^r \in \hat{K}_\Delta[X]$; we must find a root of $\hat{p}(X)$ in \hat{K}_Δ . By Lemma 1, (i) there is a sequence $(v_i)_{i \in \omega}$ in K with $v_i \rightarrow 0$ and $\tau(v_i) \rightarrow \infty$ as $i \rightarrow \infty$. We apply Lemma 3 to this sequence (v_i) and the polynomial

$$\begin{aligned} q(X, Z) &= \tau^{-1}(a_0)Z^r + \tau^{-1}(a_1)XZ^{r-1} \\ &\quad + \dots + \tau^{-1}(a_{r-1})X^{r-1}Z + X^r. \end{aligned}$$

We get elements $c_m \in K$ ($m \in \omega$) such that $c_m = q(c_{m+1}, v_m)$ for all m . We put $d_m = \tau(c_m)$, $w_m = \tau(v_m)$; then $d_m = q^\tau(d_{m+1}, w_m)$, that is

$$(9) \quad d_m = a_0 w_m^r + a_1 d_{m+1} w_m^{r-1} + \dots + a_{r-1} d_{m+1}^{r-1} w_m + d_{m+1}^r$$

for all $m \in \omega$. Choose a $\Delta' \in \mathcal{D}$ such that $\Delta \leq \Delta'$ and $d_0, w_0 \in A_{\Delta'}$; this is possible since $\text{cf}(\mathcal{D}) = \omega$. Note that $a_0, \dots, a_{r-1} \in A_{\Delta'}$. Now $w_i \rightarrow \infty$ as $i \rightarrow \infty$, hence we have $w_0, \dots, w_{n-1} \in A_{\Delta'}$, $w_n \notin A_{\Delta'}$ for some $n \in \omega$.

We claim that $d_0, \dots, d_n \in A_{\Delta}$. In fact, $d_0 \in A_{\Delta}$ by the choice of Δ' , and if $d_j \in A_{\Delta}$, $j < n$, then equation (9) with $m = j$ shows that d_{j+1} is integral over A_{Δ} , hence $d_{j+1} \in A_{\Delta}$ because A_{Δ} is integrally closed.

Now consider equation (9) with $m = n$. We divide it by w_n^r and get

$$(10) \quad \frac{d_n}{w_n^r} = a_0 + a_1 t + \dots + a_{r-1} t^{r-1} + t^r, \quad \text{where } t = \frac{d_{n+1}}{w_n}.$$

We have $d_n \in A_{\Delta}$ and $w_n \notin A_{\Delta}$, so $d_n/w_n^r \in J_{\Delta} \subseteq J_{\Delta} \subseteq A_{\Delta}$. It follows that $t \in A_{\Delta}$, and reducing (10) modulo J_{Δ} we get $0 = \hat{a}_0 + \dots + \hat{a}_{r-1} \hat{t}^{r-1} + \hat{t}^r$, i.e. $\hat{t} \in \hat{K}_{\Delta}$ is a root of $\hat{p}(X)$.

LEMMA 5. *Suppose (K, ϕ) is complete, $\text{cf}(\mathcal{D}) = \omega$. If every residue field \hat{K}_{Δ} ($\Delta \in \mathcal{D}$) is algebraically closed, then K is algebraically closed.*

Proof. We fix in \mathcal{D} a chain $\Delta_0 \leq \Delta_1 \leq \dots \leq \Delta_n \leq \dots$ with $\bigcup_{n=0}^{\infty} \Delta_n = \Gamma$. We write A_n, J_n, \hat{K}_n instead of $A_{\Delta_n}, J_{\Delta_n}, \hat{K}_{\Delta_n}$, and we let $\pi_n: A_n \rightarrow \hat{K}_n$ be the canonical map. Let be given $p(X) = \sum_{j=0}^r a_j X^j \in K[X]$. We may assume that $a_0, \dots, a_r \in A_0$, $a_r = 1$. Put

$$\hat{p}_n(X) = \sum_{j=0}^r \pi_n(a_j) X^j \in \hat{K}_n[X].$$

We construct recursively a sequence $(t_n)_{n \in \omega}$ in K with

- (i) $t_n \in A_0 \subseteq A_n$, and $\pi_n(t_n) \in \hat{K}_n$ is a root of $\hat{p}_n(X)$ (for all $n \in \omega$).
- (ii) $\pi_{n-1}(t_{n-1}) = \pi_{n-1}(t_n)$ for all $n \geq 1$.

Suppose t_0, \dots, t_{n-1} have already been constructed satisfying (i), (ii). Since \hat{K}_n is algebraically closed, there are $w_1, \dots, w_r \in A_n$ such that $\hat{p}_n(X) = \prod_{k=1}^r (X - \pi_n(w_k))$. Then $p(w_k) \in J_n \subseteq J_0 \subseteq A_0$, i.e. $a_0 + \dots + a_{r-1} w_k^{r-1} + w_k^r \in A_0$, which implies that $w_k \in A_0$ for all $k = 1, \dots, r$. The coefficients of $p(X) - \prod_{k=1}^r (X - w_k)$ lie in J_n , hence in J_{n-1} , consequently $\hat{p}_{n-1}(X) = \prod_{k=1}^r (X - \pi_{n-1}(w_k))$. Therefore $\pi_{n-1}(t_{n-1})$ must be equal to $\pi_{n-1}(w_k)$ for some $k \in \{1, \dots, r\}$. We put $t_n = w_k$; clearly (i) and (ii) are satisfied.

Now by (ii) we have $t_n - t_{n-1} \in J_{n-1}$ for $n \geq 1$, which shows that $(t_n)_{n \in \omega}$ is a Cauchy sequence. It follows from (i) that $t = \lim_{n \rightarrow \infty} t_n \in K$ is a root of $p(X)$. This proves the lemma.

Combining Lemmas 4 and 5 we get the implication (b) \Rightarrow (a) in Theorem 1 in the case where $\text{cf}(\mathcal{D}) = \omega$. The proof of Theorem 1 is complete.

6. Applications. We give two applications of the technique of §5. Theorem 2 below shows that all valued, complete fields have some purely algebraic features in common.

THEOREM 2. *Let (K, ϕ) be a valued, complete field and F a proper subfield of K . Then K is not a purely transcendental extension of F .*

Proof. Suppose, indirectly, that K were purely transcendental over F . It is sufficient to consider the case where $\text{tr deg}(K|F) = 1$, thus $K = F(t)$ with t transcendental over F . Of course K is not algebraically closed. Again (cf. §3) we may assume that either $\text{rg}(\phi) = 1$ or $\text{cf}(\mathcal{D}) = \omega$.

1. Suppose $\text{rg}(\phi) = 1$. Replacing t by $c \cdot t$ or $c \cdot t^{-1}$ ($0 \neq c \in F$) if necessary we may assume that $\phi(t) > 0$. Define $\tau \in \text{Aut}(K)$ by $\tau|_F = \text{Id}_F$, $\tau(t) = t^{-1}$. Then $t^i \rightarrow 0$ as $i \rightarrow \infty$, but $\tau(t^i) = t^{-i} \not\rightarrow 0$, which contradicts Theorem 1.

2. Suppose $\text{cf}(\mathcal{D}) = \omega$. It is clear that $\{\phi(v) : 0 \neq v \in F\} \subseteq \Gamma$ has no upper bound in Γ , for otherwise there would exist a $\Delta \in \mathcal{D}$ with $\phi(t) \in \Delta$ and $\phi(v) \in \Delta$ for all $0 \neq v \in F$, so $\phi(x) \in \Delta$ for all $0 \neq x \in F(t) = K$, which is impossible. It follows that there is a sequence $(v_i)_{i \in \omega}$, $0 \neq v_i \in F$, with $v_i \rightarrow 0$ as $i \rightarrow \infty$. Let $q(X, Z) = X^2 + t \cdot Z \in K[X, Z]$. We use Lemma 3 to find elements $c_m \in K$ ($m \in \omega$) such that $c_m = q(c_{m+1}, v_m)$ for all $m \in \omega$. Write $c_m = f_m(t)/g_m(t)$, where $f_m(t), g_m(t) \in F[t]$, relatively prime. We get

$$\frac{f_m(t) - v_m \cdot t \cdot g_m(t)}{g_m(t)} = \frac{f_{m+1}^2(t)}{g_{m+1}^2(t)} \quad \text{for all } m.$$

By unique factorization in $F[t]$ we conclude that $g_m(t) = g_{m+1}^2(t)$ for all m , therefore $g_m(t) = 1$. We are left with

$$f_m(t) - v_m \cdot t = f_{m+1}^2(t) \quad \text{for all } m \in \omega.$$

Since $0 \neq v_m \in F$ we must have $\text{deg } f_m \neq 0$ for all m , but then $\text{deg } f_m = 2 \cdot \text{deg } f_{m+1} > \text{deg } f_{m+1}$ for all m , a contradiction. This proves the theorem.

Our second application is suggested by the theory of orthomodular spaces. As mentioned in the introduction, all known non-classical examples of such spaces are constructed over certain valued, complete fields. One may ask if some of these ground fields can be embedded algebraically into \mathbf{R} ; such an embedding would yield a quick answer to several open problems. However, we now show

THEOREM 3. *A (non-Archimedeanly) valued, complete field (K, ϕ) cannot be algebraically isomorphic to a subfield of the reals \mathbf{R} .*

Proof. Suppose there were a monomorphism $\rho: K \rightarrow \mathbf{R}$. We first consider the case where $\text{cf}(\mathcal{D}) = \omega$.

(1) We construct a sequence $(v_i)_{i \in \omega}$ in K with

$$v_i \rightarrow 0 \quad \text{as } i \rightarrow \infty, \text{ and } \rho(v_i) > 1 \quad \text{for all } i \in \omega.$$

We choose $\Delta_i \in \mathcal{D}$ ($i \in \omega$) such that $\Delta_0 \not\subseteq \cdots \not\subseteq \Delta_1 \not\subseteq \cdots, \bigcup_{i=0}^{\infty} \Delta_i = \Gamma$. For every $i \in \omega$ we pick a $b_i \in A_{\Delta_{i+1}} \setminus A_{\Delta_i}$ with $\rho(b_i) > 0$, then we choose integers $r_i \in \mathbf{Z}$ so that $\rho(b_i)^{r_i} > \rho(b_{i+1})$. Put $v_i = b_i^{r_i}/b_{i+1}$. Clearly $\rho(v_i) > 1$, and by choice of the b_i 's we have $v_i \in J_{\Delta_{i+1}}$, hence $v_i \rightarrow 0$.

(2) Let $q(X, Z) = X^2 + Z$. We use Lemma 3 to obtain elements $c_m \in K$ ($m \in \omega$) such that $c_m = q(c_{m+1}, v_m)$ for all m . Put $d_m = \rho(c_m)$, $w_m = \rho(v_m)$; then

$$(11) \quad d_m = d_{m+1}^2 + w_m \quad \text{for all } m \in \omega.$$

But $d_m, w_m \in \mathbf{R}$ and $w_m > 1$ by construction, so it is obvious that (11) leads to a contradiction.

It remains to consider the case where $\text{rg}(\phi) = 1$. We shall prove, more generally, that no henselian field can be embedded into \mathbf{R} . In fact, let (K, ϕ) be henselian. Let A be the valuation ring of ϕ , J its maximal ideal and $\hat{K} = A/J$. If $\text{char } \hat{K} \neq 2$, then $1 + J \subseteq K^2$, hence $\rho(1 + J) \subseteq \mathbf{R}^2$. But this is impossible, because $\rho(1 + J) = 1 + \rho(J)$ is a coset of a non-zero additive subgroup of \mathbf{R} and therefore it must contain negative numbers. Now suppose $\text{char } \hat{K} = 2$. Take any $a \in J$, $a \neq 0$. Then $f(X) = X^2 - X - n \cdot a$ has a root in K for every $n \in \mathbf{Z}$. Choose $n \in \mathbf{Z}$ such that $n \cdot \rho(a) < -1/4$; then $f^\rho(X) = X^2 - X - n \cdot \rho(a)$ is irreducible over \mathbf{R} , a contradiction. This completes the proof.

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Received July 9, 1984. The author was partially supported by DIUC (No. 30/83) and by FNC (No. 0649).

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