

THE COLLARS OF A RIEMANNIAN MANIFOLD AND STABLE ISOSYSTOLIC INEQUALITIES

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We define the p -dimensional collar $\text{Col}_p(M, g)$ of a compact torsion-free Riemannian manifold (M, g) to be the greatest lower bound of the masses of all the p -dimensional currents which represent non-trivial integral homology classes. When the cohomology ring of M satisfies a certain non-degeneracy condition there is an inequality giving a lower bound on the volume of (M, g) in terms of certain p -dimensional collars of (M, g) . This is a version of the stable isosystolic inequality using currents rather than singular homology.

In addition to deriving this version of the stable isosystolic inequality, we show for one class of manifolds that it is a sharp inequality.

THEOREM A. *Let (M, g) be a compact oriented n -dimensional Riemannian manifold with $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$. Then*

$$\text{Vol}(M, g) \geq \text{Col}_1(M, g)\text{Col}_{n-1}(M, g).$$

Furthermore, equality holds if and only if there is a Riemannian submersion of (M, g) onto the circle of length $\text{Col}_1(M, g)$ such that each level hypersurface (i.e. fiber) is a connected minimal submanifold of volume $\text{Col}_{n-1}(M, g)$.

It is interesting to contrast Theorem A with Loewner's inequality [2], [8] which gives a lower bound on the area of a torus in terms of the length of the shortest non-contractible closed curve. In Loewner's theorem equality holds for a class of metrics which differ from one another by a constant multiple. Whereas in Theorem A equality can hold for many very different Riemannian metrics. As an example let $M = S^1 \times S^2$. Certainly the equation $\text{Vol}(M, g) = \text{Col}_1(M, g)\text{Col}_2(M, g)$ will hold for any of the various product metrics g . It also will hold for some non-product metrics. One of the latter can be constructed as follows. Let S^2 be given the canonical constant curvature metric, and let f be a non-trivial orientation preserving isometry of S^2 . Then the group of integers acts as a properly discontinuous group of isometries on the Riemannian product $R \times S^2$ by defining $n(t, x) = (t + n, f^n(x))$ where f^n is the n th iterate of f . The quotient space under this action is diffeomorphic to M . Hence the metric on $R \times S^2$ passes down to a non-product metric on M for which (1) the

projection on the first factor is a Riemannian submersion and (2) the level surfaces of the projection are connected minimal submanifolds. Thus by Theorem A the volume of this metric equals the product of the 1- and 2-dimensional collars.

It is easy to see that the cohomology rings of the complex and quaternionic projective spaces and of the product of spheres satisfy appropriate non-degeneracy conditions, and thus that the stable isosystolic inequalities hold for these spaces. (See Gromov [6], [7].) We show that the cohomology ring of $CP^m \times CP^n$ satisfies a strong non-degeneracy condition. Consequently the following theorem holds.

THEOREM B. *Let $M = CP^m \times CP^n$. Given any positive integers i_1, \dots, i_k such that $i_1 + \dots + i_k = m + n$ there is a constant $C(i_1, \dots, i_k) > 0$ such that*

$$\text{Vol}(M, g) \geq C(i_1, \dots, i_k) \text{Col}_{2i_1}(M, g) \cdots \text{Col}_{2i_k}(M, g)$$

for each Riemannian metric g on M .

There are analogous inequalities for the product of two quaternionic projective spaces and the product of two Cayley planes. This can be proved by modifications in the proof of Theorem B.

We conclude with an inequality bounding the first eigenvalue of the Laplacian of (M, g) from above in terms of some of the collars of (M, g) .

Norms on forms and cohomology. Let (M, g) be a connected compact oriented n -dimensional Riemannian manifold. The canonical volume form of (M, g) will be denoted by v_g . Thus $\text{Vol}(M, g) = \int_M v_g$ is the volume of (M, g) . The pointwise inner product of two p -forms ω and φ on M is the function $\langle \omega, \varphi \rangle$ defined by

$$\omega \wedge * \varphi = \langle \omega, \varphi \rangle v_g$$

where $*$ is the Hodge-star operator on forms. The corresponding pointwise norm of the p -form ω will be denoted by $|\omega|$. The following properties are well known [5, 10].

$$\begin{aligned} *1 &= v_g, \\ **\omega &= (-1)^{(n-p)p} \omega && \text{for all } p\text{-forms } \omega, \\ |*\omega| &= |\omega| && \text{for all } p\text{-forms } \omega, \text{ and,} \\ (1) \quad |\omega \wedge \varphi| &\leq C_1(p, q; n) |\omega| |\varphi| && \text{for all } p\text{-forms } \omega \end{aligned}$$

and q -forms φ where $C_1(p, q; n)$ is a constant depending only on p, q , and n .

REMARK. Always

$$C_1(p, q; n) \leq \binom{p+q}{p}^{1/2},$$

and if either p or q equals $0, 1, n - 1,$ or $n,$ then $C_1(p, q; n) = 1$ because in this case at least one of the two forms is simple. Clearly $C_1(p, q; n) = C_1(q, p; n).$ See [5].

By integrating over M one obtains the global inner product on forms:

$$\langle\langle \omega, \varphi \rangle\rangle = \int_M \langle \omega, \varphi \rangle v_g \quad \text{for } p\text{-forms } \omega \text{ and } \varphi.$$

By Hodge Theory and the de Rham isomorphism, every real cohomology class $\alpha \in H^p(M, R)$ is uniquely represented by a harmonic p -form. We define the inner product of two cohomology classes $\alpha, \beta \in H^p(M, R)$ by the formula

$$\langle\langle \alpha, \beta \rangle\rangle = \langle\langle \omega, \varphi \rangle\rangle$$

where ω and φ are the harmonic p -forms representing α and β respectively. $\| \cdot \|$ will denote the corresponding norms for forms and for cohomology classes.

The comass of a p -form ω at the point x in M is

$$|\omega_x|_0 = \sup \{ \omega_x(X) : X \in \wedge^p T_x M \text{ is a simple } p\text{-vector of length } \leq 1 \},$$

and the comass of ω is

$$\|\omega\|_0 = \sup \{ |\omega_x|_0 : x \in M \}.$$

REMARK. One always has the following inequalities between the pointwise norm and the comass of ω which are valid at every point of M :

$$(2) \quad \begin{aligned} |\omega|_0 &\leq |\omega| \\ |\omega| &\leq C_2(p; n) |\omega|_0 \end{aligned}$$

where $C_2(p; n) \leq \binom{n}{p}$ is a constant depending only on n and $p.$ Furthermore, $C_2(p; n) = 1$ if p equals $0, 1, n - 1,$ or $n,$ and $C_2(p; n) = C_2(n - p; n).$ See [5, 11].

Norms on homology. A p -dimensional current is a continuous linear functional on the vector space of all p -forms endowed with the C^∞ -topology. The mass $|T|$ of a p -dimensional current T is defined by

$$|T| = \sup \{ T(\omega) : \omega \text{ is a } p\text{-form with } \|\omega\|_0 \leq 1 \}.$$

Recall that if T is integration over a submanifold of M , then $|T|$ is the volume of the submanifold, and that if T is a singular chain, then $|T|$ is the mass of T defined in Gromov [7]. The boundary of the p -dimensional current T is the $(p-1)$ -dimensional current ∂T satisfying $\partial T(\omega) = T(d\omega)$ for every $(p-1)$ -form ω . This turns the collection of all currents into a differential complex whose homology is canonically isomorphic to the real homology of M [9]. We define the mass of a real homology class $a \in H_p(M, \mathbb{R})$ by

$$\|a\| = \inf\{|T| : T \in a\}.$$

This definition of mass compares with Gromov's Vol_R as follows. $\|a\| \leq \text{Vol}_R(a)$ at least when a is an integral homology class.

The norms defined in this and the preceding section all depend upon the Riemannian metric g .

Duality. $D(a) \in H^{n-p}(M, \mathbb{R})$ denotes the Poincaré dual class of $a \in H_p(M, \mathbb{R})$. Likewise, $D(\alpha) \in H_{n-p}(M, \mathbb{R})$ denotes the Poincaré dual of $\alpha \in H^p(M, \mathbb{R})$. Thus, if $a \cdot b \in H_{p+q-n}(M, \mathbb{R})$ is the intersection class corresponding to $a \in H_p(M, \mathbb{R})$ and $b \in H_q(M, \mathbb{R})$, then $D(a \cdot b) = D(a) \cup D(b)$ where \cup is the cup product.

LEMMA 1. For every $a \in H_p(M, \mathbb{R})$ and every $(n-p)$ -form ω representing $D(a)$,

$$\|a\| \leq C_2(p; n) \int_M |\omega| v_g.$$

Proof. Define a p -dimensional current T by the formula

$$T(\varphi) = \int_M \omega \wedge \varphi$$

for every p -form φ . Then $T \in a$. It follows from (2) that $|\varphi| \leq C_2(p; n)$ for every p -form φ satisfying $\|\varphi\|_0 \leq 1$. Hence, if $\|\varphi\|_0 \leq 1$,

$$\begin{aligned} T(\varphi) &= \int_M \omega \wedge \varphi = \int_M \langle *\omega, \varphi \rangle v_g \\ &\leq \int_M |*\omega| |\varphi| v_g \leq C_2(p; n) \int_M |\omega| v_g \end{aligned}$$

where we have integrated Schwarz's inequality for the pointwise inner product on forms and used $|*\omega| = |\omega|$. Thus

$$\|a\| \leq |T| \leq C_2(p; n) \int_M |\omega| v_g.$$

LEMMA 2. *If $a \in H_{n-p}(M, R)$ and $b \in H_{n-q}(M, R)$, then*

$$\|D(a)\| \|D(b)\| \geq C_3(p, q; n) \|a \cdot b\|$$

where $C_3(p, q; n) = (C_1(p, q; n)C_2(p + q; n))^{-1}$.

Proof. Let ω and φ be the harmonic forms representing $D(a)$ and $D(b)$ respectively. By applying in succession Schwarz's inequality, (1), and Lemma 1, we have

$$\begin{aligned} \|D(a)\| \|D(b)\| &= \left(\int_M |\omega|^2 v_g \right)^{1/2} \left(\int_M |\varphi|^2 v_g \right)^{1/2} \\ &\geq \int_M |\omega| |\varphi| v_g \geq C_1^{-1}(p, q; n) \int_M |\omega \wedge \varphi| v_g \\ &\geq C_1^{-1}(p, q; n) C_2^{-1}(p + q; n) \|a \cdot b\| \end{aligned}$$

because $\omega \wedge \varphi$ is a form dual to the class $a \cdot b$ and $C_2(n - p - q; n) = C_2(p + q; n)$.

COROLLARY 3. *For every $a \in H_{n-p}(M, R)$*

$$(\text{Vol}(M, g))^{1/2} \|D(a)\| \geq C_4(p; n) \|a\|$$

where $C_4(p; n) = C_2^{-1}(p; n)$. *If equality holds then the pointwise norm of the harmonic form representing $D(a)$ is constant.*

Proof. Let ω be the harmonic form representing $D(a)$. Then by Lemma 1 and Schwarz's inequality,

$$\begin{aligned} \|a\| &\leq C_2(p; n) \int_M |\omega| v_g \\ &\leq C_2(p; n) \left(\int_M |\omega|^2 v_g \right)^{1/2} \left(\int_M 1^2 v_g \right)^{1/2} \\ &= C_2(p; n) \|D(a)\| (\text{Vol}(M, g))^{1/2}. \end{aligned}$$

If equality holds then $|\omega|$ is constant by Schwarz's inequality.

The collars of (M, g) . From now on suppose M is torsion free. Thus the integral homology $H_*(M, Z)$ and the integral cohomology $H^*(M, Z)$ may be identified with subsets of $H_*(M, R)$ and $H^*(M, R)$ respectively which form lattices of these real vector spaces.

The p -dimensional collar $\text{Col}_p(M, g)$ of (M, g) is defined by

$$\text{Col}_p(M, g) = \inf\{\|a\|: a \in H_p(M, Z), a \neq 0\}.$$

REMARK. $\text{Col}_n(M, g) = \text{Vol}(M, g)$.

Let $\det(H^p(M, Z))$ denote the determinant of the lattice $H^p(M, Z)$ in the inner product space $H^p(M, R)$. Recall the theorem of Minkowski which states

$$\det(H^p(M, Z)) \geq u(b_p)\lambda^{b_p}$$

where $\lambda = \min\{\|\alpha\|: \alpha \in H^p(M, Z), \alpha \neq 0\}$ and where $u(b_p)$ is a positive constant related to the Hermite constant depending only on the p th Betti number b_p of M . We will have need of the following inequality [4].

LEMMA 4. $\det(H^p(M, Z)) \geq u(b_p)\lambda_1 \cdots \lambda_{b_p}$ where $\lambda_1 \leq \cdots \leq \lambda_{b_p}$ are the successive minimums of the lattice $H^p(M, Z)$.

For notational convenience we set $h(k) = (u(k))^{1/k}$.

We say M satisfies the dual lattice condition in degree p if for all $\alpha \in H^p(M, R)$, $(\alpha \cup \beta) [M] \in Z$ for every $\beta \in H^{n-p}(M, Z)$ implies $\alpha \in H^p(M, Z)$. This condition is important because of the following lemma.

LEMMA 5. (Berger [2].) *If M satisfies the dual lattice condition in degree p , then*

$$\det(H^p(M, Z)) \det(H^{n-p}(M, Z)) = 1.$$

This condition is satisfied in all degrees by compact orientable surfaces, the product of spheres, the complex and quaternionic projective spaces, and the Cayley plane. Details may be found in [2]. Similar computations to those in [2] show that $CP^m \times CP^n$, $HP^m \times HP^n$, and $OP^2 \times OP^2$ also satisfy the dual lattice condition in all degrees.

PROPOSITION 6. *Suppose M satisfies the dual lattice condition in degree p , then*

$$\text{Vol}(M, g) \geq C_4^2(p; n)h^2(b_p)\text{Col}_p(M, g)\text{Col}_{n-p}(M, g).$$

Proof. Let $\alpha \in H^p(M, Z)$ and $\beta \in H^{n-p}(M, Z)$ be the minimum lattice points. Then $D(\alpha) \in H_{n-p}(M, Z)$ and $D(\beta) \in H_p(M, Z)$ are both non-zero. Hence by Corollary 3 and the definition of the collars of

(M, g) ,

$$\begin{aligned} (\text{Vol}(M, g))^{1/2} \|\alpha\| &\geq C_4(p; n) \|D(\alpha)\| \\ &\geq C_4(p; n) \text{Col}_{n-p}(M, g) \end{aligned}$$

and

$$\begin{aligned} (\text{Vol}(M, g))^{1/2} \|\beta\| &\geq C_4(n-p; n) \|D(\beta)\| \\ &\geq C_4(n-p; n) \text{Col}_p(M, g). \end{aligned}$$

By Lemmas 4 and 5,

$$\begin{aligned} 1 &= \det(H^p(M, Z)) \det(H^{n-p}(M, Z)) \\ &\geq u(b_p) \|\alpha\|^{b_p} u(b_{n-p}) \|\beta\|^{b_{n-p}} \end{aligned}$$

from which follows the inequality

$$1 \geq h^2(b_p) \|\alpha\| \|\beta\|$$

by taking b_p roots since $b_p = b_{n-p}$. Therefore

$$\begin{aligned} \text{Vol}(M, g) &\geq \text{Vol}(M, g) h^2(b_p) \|\alpha\| \|\beta\| \\ &\geq h^2(b_p) C_4(p; n) C_4(n-p; n) \text{Col}_p(M, g) \text{Col}_{n-p}(M, g) \\ &\geq h^2(b_p) C_4^2(p; n) \text{Col}_p(M, g) \text{Col}_{n-p}(M, g) \end{aligned}$$

since $C_4(n-p; n) = C_4(p; n)$.

Proof of Theorem A. Let (M, g) be a compact oriented n -dimensional Riemannian manifold with $H_1(M, Z) \cong Z$. Since M is oriented, $H_{n-1}(M, Z)$ is torsion-free. Thus it is evident that M satisfies the dual lattice condition in degree one. Thus by Proposition 6

$$\text{Vol}(M, g) \geq \text{Col}_1(M, g) \text{Col}_{n-1}(M, g)$$

because $h(1) = C_4(1; n) = 1$.

Suppose $f: M \rightarrow R/LZ$ is a Riemannian submersion onto the circle of length L such that every level surface (or fiber) is a connected minimal hypersurface of (M, g) . Let $\omega = f^*(dt)$ where dt is the canonical volume form on the circle R/LZ . Then $|\omega| = 1$ because f is a Riemannian submersion. Thus $*\omega = *\omega/|\omega|$ restricts to the volume form on each fiber of f . Hence a simple calculation shows that $d(*\omega) = \eta \wedge *\omega$ where η is the 1-form defined by $\eta(X) = -g(H, X)$, H being the mean curvature vector of the fibers of f . Consequently, $d(*\omega) = 0$ because the fibers are

minimal submanifolds. Hence all the fibers have the same volume V . Because f is a Riemannian submersion, we may calculate $\text{Vol}(M, g)$ by first integrating over the fibers and then over the circle R/LZ . Therefore $\text{Vol}(M, g) = LV$.

Let $a \in H_1(M, Z) \cong Z$ be a generator. Then, because the fibers of f are connected, f_*a generates $H_1(R/LZ, Z)$. Thus if s is a closed curve representing a ,

$$\left| \int_s \omega \right| = \left| \int_{f \circ s} dt \right| = L.$$

If T is any current in the homology class a , then \int_s and T are homologous, i.e.

$$\int_s - T = \partial S$$

for some 2-dimensional current S . Thus

$$\int_s \omega - T(\omega) = \partial S(\omega) = S(d\omega) = S(0) = 0.$$

Therefore, because $\|\omega\|_0 = 1$ (since $|\omega|_0 = |\omega| = 1$)

$$|T| \geq |T(\omega)| = \left| \int_s \omega \right| = L.$$

Consequently, $\text{Col}_1(M, g) \geq L$.

Let $b \in H_{n-1}(M, Z) \cong Z$ be a generator and T a current in the homology class b . Since the fibers of f also represent b , integration over any fiber W is homologous to T . Therefore, since $d(*\omega) = 0$,

$$T(*\omega) = \int_W *\omega = V.$$

But $|\omega|_0 = |*\omega| = |\omega| = 1$. Hence $|T| \geq |T(*\omega)| = V$. Consequently, $\text{Col}_{n-1}(M, g) \geq V$. Therefore

$$\text{Col}_1(M, g)\text{Col}_{n-1}(M, g) \geq LV = \text{Vol}(M, g).$$

Since we have already proved the reverse inequality we conclude $\text{Col}_1(M, g) = L$, $\text{Col}_{n-1}(M, g) = V$, and $\text{Vol}(M, g) = \text{Col}_1(M, g)\text{Col}_{n-1}(M, g)$.

Conversely, suppose $\text{Vol}(M, g) = \text{Col}_1(M, g)\text{Col}_{n-1}(M, g)$. Let ω be the harmonic 1-form representing a generator $\alpha \in H^1(M, Z)$. Then $|\omega|$ is constant by Corollary 3 because by the proof of Proposition 6, if equality

holds in Proposition 6, then it holds in Corollary 3. Therefore $\varphi = \omega/|\omega|$ is a harmonic 1-form with $|\varphi| = 1$. Since $H_1(M, Z) \cong Z$, the periods of φ are all integral multiples of a fixed number $L > 0$. (In fact $L = |\omega|^{-1}$.) Thus we may define a map $f: M \rightarrow R/LZ$ such that $f^*(dt) = \varphi$ by first fixing a point x_0 in M and then defining $f(x)$ to be the value modulo L of the integral of φ along any path joining x_0 to x . (See [2].) f is a Riemannian submersion because $|f^*(dt)| = |\varphi| = 1$. Consequently $*\varphi$ restricts to the volume form on the fibers of f . Therefore, because $d(*\varphi) = 0$ since φ is harmonic, the fibers of f are all minimal submanifolds of M . Furthermore, they must be connected since each fiber represents the generator $D(\alpha)$ of $H_{n-1}(M, Z)$. Finally the first part of the proof shows that $L = \text{Col}_1(M, g)$ and that $\text{Col}_{n-1}(M, g)$ is the volume of a fiber.

Fully non-degenerate bilinear maps. Let E, F , and W be three finite-dimensional real vector spaces, and let $B: E \times F \rightarrow W$ be a bilinear map. Choose bases e_1, \dots, e_l of E and f_1, \dots, f_k of F . Thus $l = \dim(E)$ and $k = \dim(F)$. We will assume $l \leq k$. Form the $l \times k$ matrix $A = (B(e_i, f_j))$ whose entries are elements of W . The set of $l \times l$ submatrices of A formed by deleting $k - l$ columns can be indexed by the set Π consisting of all subsets π of $\{1, 2, 3, \dots, k\}$ of cardinality l so that the submatrix A_π is obtained by deleting the j th column if and only if $j \notin \pi$. There are $\binom{k}{l}$ such matrices.

If we consider the entries of A_π to be in the real symmetric algebra over W , we can form the determinant $\det(A_\pi)$ which is a homogeneous element of degree l in the symmetric algebra.

Suppose the set $\{\det(A_\pi): \pi \in \Pi\}$ is linearly independent over R in the symmetric algebra. It is a simple matter to show that this condition is independent of the choice of bases of E and F . For first suppose e'_1, \dots, e'_l is a second basis of E related to e_1, \dots, e_l in one of the following three ways:

Case 1.

$$e'_i = \begin{cases} e_i & i \neq i_0 \\ ce_{i_0} & i = i_0. \end{cases}$$

Case 2.

$$e'_i = \begin{cases} e_i & i \neq i_0 \\ e_{i_0} + ce_{j_0} & i = i_0. \end{cases}$$

Case 3.

$$e'_i = \begin{cases} e_i & i \neq i_0, j_0 \\ e_{j_0} & i = i_0 \\ e_{i_0} & i = j_0. \end{cases}$$

Thus, if we let A' be the matrix $(B(e'_i, f_j))$ and A'_π be the submatrix corresponding to $\pi \in \Pi$, then, by the properties of the determinant, we have:

Case 1. $\det(A'_\pi) = c \det(A_\pi)$

Case 2. $\det(A'_\pi) = \det(A_\pi)$

Case 3. $\det(A'_\pi) = -\det(A_\pi)$.

Clearly $\{\det(A_\pi) : \pi \in \Pi\}$ is linearly independent in every case.

Now suppose f'_1, \dots, f'_k is a second basis of E related to f_1, \dots, f_k in one of the following three ways:

Case 1.

$$f'_i = \begin{cases} f_i & i \neq i_0 \\ cf_{i_0} & i = i_0. \end{cases}$$

Case 2.

$$f'_i = \begin{cases} f_i & i \neq i_0 \\ f_{i_0} + cf_{j_0} & i = i_0. \end{cases}$$

Case 3.

$$f'_i = \begin{cases} f_i & i \neq i_0, j_0 \\ f_{j_0} & i = i_0 \\ f_{i_0} & i = j_0. \end{cases}$$

Thus, if we let A' be the matrix $(B(e_i, f'_j))$ and A'_π be the submatrix corresponding to $\pi \in \Pi$, then, by the properties of determinants, we have:

Case 1.

$$\det(A'_\pi) = \begin{cases} c \det(A_\pi) & \text{if } i_0 \in \pi \\ \det(A_\pi) & \text{if } i_0 \notin \pi. \end{cases}$$

Case 2.

$$\det(A'_\pi) = \begin{cases} \det(A_\pi) & \text{if either } i_0 \notin \pi \text{ or} \\ & \text{both } i_0, j_0 \in \pi \\ \det(A_\pi) + c \det(A_{\pi'}) & \text{if } i_0 \in \pi, j_0 \notin \pi \\ & \text{and } \pi' = \pi \cup \{j_0\} - \{i_0\}. \end{cases}$$

Case 3.

$$\det(A'_\pi) = \begin{cases} \det(A_\pi) & \text{if } i_0, j_0 \notin \pi \\ -\det(A_\pi) & \text{if } i_0, j_0 \in \pi \\ \det(A_{\pi'}) & \text{if } i_0 \in \pi, j_0 \notin \pi \\ & \text{and } \pi' = \pi \cup \{j_0\} - \{i_0\} \\ \det(A_{\pi'}) & \text{if } i_0 \notin \pi, j_0 \in \pi \\ & \text{and } \pi' = \pi \cup \{i_0\} - \{j_0\} \end{cases}$$

Once again, in every case, $\{\det(A'_\pi) : \pi \in \Pi\}$ is linearly independent.

Since any two pairs of bases for E and F are related to each other by a sequence of the elementary changes of bases just described, we see that $\{\det(A_\pi) : \pi \in \Pi\}$ is linearly independent for every pair of bases for E and F if it is for one such pair. We will say B is fully non-degenerate if the set $\{\det(A_\pi) : \pi \in \Pi\}$ is linearly independent in the symmetric algebra over W .

Suppose B is fully non-degenerate. Then for any bases e_1, \dots, e_l of E and f_1, \dots, f_k of F and for every $\pi \in \Pi$, $\det(A_\pi) \neq 0$. Recall that $\det(A_\pi)$ is the sum of terms where each term is ± 1 times a product of l entries chosen from A_π in such a way that an entry has been chosen from each row and column exactly once. Since $\det(A_\pi) \neq 0$, one of these terms must be non-zero. Thus we are able to choose l non-zero entries from A_π such that any entry has been chosen from each row and column exactly once. Therefore, doing this for each π , we can choose $\binom{k}{l}l$ non-zero entries from A so that an entry has been chosen exactly $\binom{k}{l}$ -times from each row and exactly $\binom{k-1}{l-1}$ -times from each column. Some entries may have been chosen more than once.

LEMMA 7. Suppose $\cup : H^p(M, R) \times H^q(M, R) \rightarrow H^{p+q}(M, R)$ with $p + q \leq n$ is fully non-degenerate. Then

$$\begin{aligned} & (\det(H^p(M, Z)))^{(1/b_p)} (\det(H^q(M, Z)))^{(1/b_q)} \\ & \geq h(b_p)h(b_q)C_3(p, q; n)\text{Col}_{n-p-q}(M, g). \end{aligned}$$

Proof. Let $b_p = l$, $b_q = k$, and assume $l \leq k$. Let $\alpha_1, \dots, \alpha_l$ and β_1, \dots, β_k be bases of $H^p(M, R)$ and $H^q(M, R)$ respectively which are the successive minimums of the lattices $H^p(M, Z)$ and $H^q(M, Z)$. Since \cup is fully non-degenerate, we can choose $\binom{k}{l}l$ pairs (α_i, β_j) with $\alpha_i \cup \beta_j \neq 0$ so that each α_i is a member of a pair exactly $\binom{k}{l}$ -times and

the hypothesis that the cup products are fully non-degenerate implies that the cup product form $H^h(M) \times \cdots \times H^{i_k}(M) \rightarrow H^n(M)$ is non-degenerate in the sense of §7.4 of [7]. Thus Gromov’s result 7.4.C gives a constant C and non-zero homology classes h_j in $H_{i_j}(M, \mathbb{Z})$ such that $\text{Vol}_R(h_1) \cdots \text{Vol}_R(h_k) \leq C \text{Vol}(M, g)$. Since $\text{Col}_{i_j}(M, g) \leq \text{Vol}_R(h_j)$ Proposition 8 follows. However we will prove Theorem B by showing the cohomology ring of $CP^m \times CP^n$ satisfies the hypothesis of Proposition 8.

Proof of Theorem B. Let $M = CP^m \times CP^n$ with $m \geq n$. $H^*(M, R)$ is generated as an algebra by two elements $x, y \in H^2(M, R)$ satisfying $xy = yx$, $x^{m+1} = 0$ and $y^{n+1} = 0$. Thus $H^{2p}(M, R)$ has a basis consisting of

$$\begin{aligned} &x^p, x^{p-1}y, x^{p-2}y^2, \dots, y^p && \text{if } p < n, \\ &x^p, x^{p-1}y, x^{p-2}y^2, \dots, x^{p-n}y^n && \text{if } n \leq p \leq m, \text{ or} \\ &x^m y^{p-m}, x^{m-1}y^{p-m-1}, \dots, x^{p-n}y^n && \text{if } p > m. \end{aligned}$$

Thus if e_1, \dots, e_l is this basis for $H^{2p}(M, R)$ and f_1, \dots, f_k is this basis for $H^{2q}(M, R)$ where we assume $l = b_{2p} \leq k = b_{2q}$ and $p + q \leq m + n$, then $e_i \cup f_j = e_{i'} \cup f_{j'}$ whenever $i + j = i' + j'$. Hence, if we let $z_{i+j} = e_i \cup f_j$, then the matrix A of the bilinear map $\cup: H^{2p}(M, R) \times H^{2q}(M, R) \rightarrow H^{2p+2q}(M, R)$ is of the form

$$\begin{pmatrix} z_2 & z_3 & z_4 & \cdots & z_{k+1} \\ z_3 & z_4 & z_5 & \cdots & z_{k+2} \\ z_4 & z_5 & z_6 & \cdots & z_{k+3} \\ \vdots & \vdots & \vdots & & \vdots \\ z_{l+1} & z_{l+2} & z_{l+3} & \cdots & z_{l+k} \end{pmatrix}.$$

The z_i are distinct elements of the basis for $H^{2p+2q}(M, R)$ except that the first few z_i ’s and the last few, namely z_2, \dots, z_l and z_{k+2}, \dots, z_{l+k} , may possibly be zero when $p + q$ is large enough. In any case, $z_{l+1}, z_{l+2}, \dots, z_{l+k}$ are linearly independent elements of $H^{2p+2q}(M, R)$.

Again recall that the determinant of a square matrix is the sum of terms where each term is ± 1 times a product of entries chosen from the matrix so that an entry is chosen once from each row and each column. For $\pi \in \Pi$, we break down the determinant of A_π as follows:

$$\det(A_\pi) = D_\pi + E_\pi$$

where D_π is the sum of the terms involving only z_{l+1}, \dots, z_{k+1} , and E_π is the sum of the terms that contain at least one of $z_2, \dots, z_l, z_{k+2}, \dots, z_{l+k}$. We will prove the set $\{\det(A_\pi): \pi \in \Pi\}$ to be linearly independent by

first proving that $\{\det(A_\pi): \pi \in \Pi\}$ is linearly independent if $\{D_\pi: \pi \in \Pi\}$ is linearly independent, and then by proving that $\{D_\pi: \pi \in \Pi\}$ is linearly independent.

Suppose $\{D_\pi: \pi \in \Pi\}$ is linearly independent, and let

$$\sum a_\pi \det(A_\pi) = 0.$$

Thus

$$\sum a_\pi D_\pi + \sum a_\pi E_\pi = 0.$$

But $\sum a_\pi D_\pi$ is contained in the subspace of the symmetric algebra over $H^{2p+2q}(M, R)$ spanned by simple homogeneous elements of degree l involving only z_{l+1}, \dots, z_{k+1} while $\sum a_\pi E_\pi$ is contained in the subspace spanned by simple homogeneous elements of degree l each of which contains at least one of $z_2, \dots, z_l, z_{k+2}, \dots, z_{l+k}$. Since these two subspaces have only zero in common.

$$\sum a_\pi D_\pi = -\sum a_\pi E_\pi = 0.$$

Hence, $a_\pi = 0$ for all $\pi \in \Pi$ by the linear independence of the D_π . Therefore $\{\det(A_\pi): \pi \in \Pi\}$ is linearly independent.

We next prove $\{D_\pi: \pi \in \Pi\}$ is linearly independent by double induction on l and k .

Clearly, $\{D_\pi: \pi \in \Pi\}$ is linearly independent if $l = 1$ and $k \geq 1$ since z_2, \dots, z_{k+1} is linearly independent. Also, if $l = k$, $\{D_\pi: \pi \in \Pi\} = \{(z_{l+1})^l\}$ is linearly independent because $z_{l+1} \neq 0$.

Now suppose that $1 < l < k$ and that we have linear independence for matrices of dimensions $(l - 1) \times (k - 1)$ and $l \times (k - 1)$. Decompose Π into two disjoint subsets

$$\Pi = \Pi_0 \cup \Pi_1$$

where Π_0 is the set of all subsets of $\{1, 2, \dots, k - 1\}$ of cardinality l , and Π_1 is the set of all subsets of $\{1, 2, \dots, k\}$ of cardinality l which contain k . Π_1 is in one-to-one correspondence with the set Π' of all subsets of $\{1, 2, \dots, k - 1\}$ of cardinality $l - 1$ where $\pi \in \Pi_1$ corresponds to $\pi' \in \Pi'$ if and only if $\pi = \pi' \cup \{k\}$.

Let A^0 be the $l \times (k - 1)$ matrix

$$\begin{pmatrix} z_2 & z_3 & \cdots & z_k \\ z_3 & z_4 & \cdots & z_{k+1} \\ \vdots & \vdots & & \vdots \\ z_{l+1} & z_{l+2} & \cdots & z_{k+l-1} \end{pmatrix}$$

obtained from A by deleting the last column. Then, for $\pi \in \Pi_0$ let

$$\det(A_\pi^0) = D_\pi^0 + E_\pi^0$$

where D_π^0 is the sum of the terms involving only z_{l+1}, \dots, z_k . By induction $\{D_\pi^0: \pi \in \Pi_0\}$ is linearly independent. Observe also that if $\pi \in \Pi_0$, then

$$D_\pi = D_\pi^0 + F_\pi$$

where D_π^0 is as above and F_π is the sum of the terms in D_π which contain z_{k+1} .

Let A^1 be the $(l - 1) \times (k - 1)$ matrix

$$\begin{pmatrix} z_3 & z_4 & \cdots & z_k \\ z_4 & z_5 & \cdots & z_{k+1} \\ \vdots & \vdots & \ddots & \vdots \\ z_{l+1} & z_{l+2} & \cdots & z_{k+l-1} \end{pmatrix}$$

obtained from A by deleting the last column and the first row. For $\pi' \in \Pi'$, let

$$\det(A_{\pi'}^1) = D'_{\pi'} + E'_{\pi'}$$

where $D'_{\pi'}$ is the sum of the terms involving only z_{l+1}, \dots, z_k and $E'_{\pi'}$ are the other terms. By induction, $\{D'_{\pi'}: \pi' \in \Pi'\}$ is linearly independent. Observe that if $\pi \in \Pi_1$ corresponds to $\pi' \in \Pi'$, then

$$D_\pi = z_{k+1}(D'_{\pi'} + F'_{\pi'})$$

where the terms in $F'_{\pi'}$ must contain z_{k+1} . If we set $D_\pi^1 = (D'_{\pi'} + F'_{\pi'})$, then $D_\pi = z_{k+1}D_\pi^1$ and the set $\{D_\pi^1: \pi \in \Pi_1\}$ is linearly independent. The linear independence of the D_π^1 follows easily from the linear independence of the $D'_{\pi'}$ and the fact that the $D'_{\pi'}$ lie in the subspace of the symmetric algebra over $H^{2p+2q}(M, R)$ spanned by simple homogeneous elements of degree $l - 1$ involving only z_{l+1}, \dots, z_k while the $F'_{\pi'}$ lie in the subspace spanned by the simple homogeneous elements that involve z_{k+1} . In short, the $D'_{\pi'}$ and the $F'_{\pi'}$ lie in two different subspaces which have only zero in common.

Now, suppose

$$\sum_{\pi \in \Pi_0} a_\pi D_\pi + \sum_{\pi \in \Pi_1} a_\pi D_\pi = 0.$$

Then

$$\sum_{\pi \in \Pi_0} a_\pi D_\pi^0 + \sum_{\pi \in \Pi_0} a_\pi F_\pi + \sum_{\pi \in \Pi_1} a_\pi z_{k+1} D_\pi^1 = 0.$$

But the first term of the last equation lies in the subspace spanned by terms involving only z_{l+1}, \dots, z_k whereas the last two terms lie in the subspace spanned by terms which all involve z_{k+1} . Since these two subspaces have only zero in common,

$$\sum_{\pi \in \Pi_0} a_\pi D_\pi^0 = 0 = \sum_{\pi \in \Pi_0} a_\pi F_\pi + \sum_{\pi \in \Pi_1} a_\pi z_{k+1} D_\pi^1.$$

Since the D_π^0 are linearly independent, $a_\pi = 0$ for all $\pi \in \Pi_0$. Therefore

$$\sum_{\pi \in \Pi_1} a_\pi z_{k+1} D_\pi^1 = 0.$$

Therefore

$$\sum_{\pi \in \Pi_1} a_\pi D_\pi^1 = 0.$$

Therefore $a_\pi = 0$ for all $\pi \in \Pi_1$ since the D_π^1 are linearly independent. This completes the induction step proving that $\{D_\pi: \pi \in \Pi\}$ is linearly independent.

Therefore \cup is fully non-degenerate. Hence Theorem B follows from Proposition 8.

Example. Proposition 8 does not apply to the product of more than two projective spaces. For example, let $M = CP^2 \times CP^2 \times CP^2$, and let $x, y, z \in H^2(M, R)$ be generators of $H^*(M, R)$ which satisfy $x^3 = y^3 = z^3 = 0$ and commute. With respect to the basis $\{x^2, xy, y^2, xz, z^2, yz\}$ for $H^4(M, R)$ and the basis $\{x^2y, x^2z, xy^2, y^2z, xz^2, yz^2, xyz\}$ for $H^6(M, R)$ the bilinear map $\cup: H^4(M, R) \times H^6(M, R) \rightarrow H^{10}(M, R)$ has the matrix

$$\begin{pmatrix} 0 & 0 & 0 & x^2y^2z & 0 & x^2yz^2 & 0 \\ 0 & 0 & 0 & 0 & x^2yz^2 & xy^2z^2 & x^2y^2z \\ 0 & x^2y^2z & 0 & 0 & xy^2z^2 & 0 & 0 \\ 0 & 0 & x^2y^2z & xy^2z^2 & 0 & 0 & x^2yz^2 \\ x^2yz^2 & 0 & xy^2z^2 & 0 & 0 & 0 & 0 \\ x^2y^2z & x^2y^2z & 0 & 0 & 0 & 0 & xy^2z^2 \end{pmatrix}.$$

By direct calculation the submatrix formed by deleting the last column has determinant equal to zero. This shows that \cup is not fully non-degenerate.

On the other hand, it seems safe to conjecture that the product of more than two complex projective spaces satisfies stable isosystolic inequalities. To prove this one would need to show that the cohomology

ring of the product of several complex projective spaces satisfies the appropriate non-degeneracy conditions in the hypothesis of Gromov's version of the stable isosystolic inequality.

The First Eigenvalue.

PROPOSITION 9. *Let $\lambda_1(M, g)$ denote the first eigenvalue of the Laplacian of (M, g) . Suppose M satisfies the dual lattice condition in degree 1, then*

$$\lambda_1(M, g) \leq \frac{4\pi^2(\text{Vol}(M, g))^2}{(h(b_1))^8 \text{Col}_1^4(M, g) \text{Col}_{n-1}^2(M, g)}.$$

Proof. Set $\mu_p = \inf\{\|\alpha\|: \theta \in H^p(M, Z), \alpha \neq 0\}$. Berger [3] proves $\lambda_1 \leq 4\pi^2 c^{-2} \mu_1^4$ where c is the $(n - 1)$ -dimensional carcan of (M, g) which he defines as the infimum of the volumes of all compact oriented $(n - 1)$ -dimensional submanifolds of M whose fundamental class is not null-homologous in M . Since integration over every such submanifold defines an $(n - 1)$ -dimensional current whose mass equals the volume of the submanifold, $\text{Col}_{n-1}(M, g) \leq c$. Thus, by Lemma 4,

$$\lambda_1(M, g) \text{Col}_{n-1}^2(M, g) \leq 4\pi^2 \mu_1^4 \leq 4\pi^2 \frac{(\det(H^1(M, Z)))^{4b_1^{-1}}}{(h(b_1))^4}.$$

Now by Corollary 3 and Lemma 4,

$$\begin{aligned} \text{Col}_1(M, g) &\leq \mu_{n-1}(\text{Vol}(M, g))^{1/2} \\ &\leq \frac{(\det(H^{n-1}(M, Z)))^{b_{n-1}^{-1}}}{h(b_{n-1})} (\text{Vol}(M, g))^{1/2}. \end{aligned}$$

Raising the last inequality to the fourth power and multiplying it with the previous inequality gives

$$\begin{aligned} &\lambda_1(M, g) \text{Col}_1^4(M, g) \text{Col}_{n-1}^2(M, g) \\ &\leq 4\pi^2 \frac{(\det(H^1(M, Z)))^{4b_1^{-1}}}{(h(b_1))^4} \cdot \frac{(\det(H^{n-1}(M, Z)))^{4b_{n-1}^{-1}}}{(h(b_{n-1}))^4} (\text{Vol}(M, g))^2 \end{aligned}$$

which by Lemma 5 is equivalent to the inequality to be proved since $b_1 = b_{n-1}$.

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