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A CHARACTERIZATION THEOREM FOR COMPACT UNIONS OF TWO STARSHAPED SETS IN R^3

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Set S in R^d has property P_k if and only if S is a finite union of d-polytopes and for every finite set F in bdry$S$ there exist points $c_1, \ldots, c_k$ (depending on F) such that each point of F is clearly visible via S from at least one c_i, $1 \leq i \leq k$. The following results are established.

(1) Let $S \subseteq R^3$. If S satisfies property P_2, then S is a union of two starshaped sets.

(2) Let $S \subseteq R^d$, $d \geq 3$. If S is a compact union of k starshaped sets, then there exists a sequence $\{S_j\}$ converging to S (relative to the Hausdorff metric) such that each set S_j satisfies property P_k.

When $d = 3$ and $k = 2$, the converse of (2) above holds as well, yielding a characterization theorem for compact unions of two starshaped sets in R^3.

1. Introduction. We begin with some definitions. Let S be a subset of R^d. Hyperplane H is said to *support S locally* at boundary point s of S if and only if $s \in H$ and there is some neighborhood N of s such that $N \cap S$ lies in one of the closed halfspaces determined by H. Point s in S is called a *point of local convexity* of S if and only if there is some neighborhood N of s such that $N \cap S$ is convex. If S fails to be locally convex at q in S, then q is called a *point of local nonconvexity* (lnc point) of S. For points x and y in S, we say x *sees* y via S (x is *visible* from y via S) if and only if the segment $[x, y]$ lies in S. Similarly, x is *clearly visible* from y via S if and only if there is some neighborhood N of x such that y sees via S each point of $N \cap S$. Set S is *locally starshaped* at point x of S if and only if there is some neighborhood N of x such that x sees via S each point of $N \cap S$. Finally, set S is *starshaped* if and only if there is some point p in S such that p sees via S each point of S, and the set of all such points p is called the (convex) *kernel* of S.

A well-known theorem of Krasnosel′skii [3] states that if S is a nonempty compact set in R^d, S is starshaped if and only if every $d + 1$ points of S are visible via S from a common point. Moreover, "points of S" may be replaced by "boundary points of S" to produce a stronger result. In [1], the concept of clear visibility, together with work by Lawrence, Hare, and Kenelly [4], were used to obtain the following

Krasnosel'skii-type theorem for unions of two starshaped sets in the plane: Let S be a compact nonempty set in R^2, and assume that for each finite set F in the boundary of S there exist points c, d (depending on F) such that each point of F is clearly visible via S from at least one of c, d. Then S is a union of two starshaped sets.

In this paper, an analogous result is proved for set S in R^3, where S satisfies the additional hypothesis of being a finite union of polytopes. Furthermore, while not every compact union F of two starshaped sets in R^3 satisfies this hypothesis, F will be the limit (relative to the Hausdorff metric) for a sequence whose members do satisfy it. This in turn leads to a characterization theorem for compact unions of two starshaped sets in R^3.

The following terminology will be used throughout the paper: $\operatorname{Conv} S$, $\operatorname{cl} S$, $\operatorname{int} S$, $\operatorname{rel\,int} S$, $\operatorname{bdry} S$, $\operatorname{rel\,bdry} S$, and $\ker S$ will denote the convex hull, closure, interior, relative interior, boundary, relative boundary, and kernel, respectively, for set S. The distance from point x to point y will be denoted $\operatorname{dist}(x, y)$. For distinct points x and y, $L(x, y)$ will be the line determined by x and y, while $R(x, y)$ will be the ray emanating from x through y. For $x \in S$, A_z will represent $\{x\colon z$ is clearly visible via S from $x\}$. The reader is referred to Valentine [7] and to Lay [5] for a discussion of these concepts and to Nadler [6] for information on the Hausdorff metric.

2. The results. The following definition will be helpful.

DEFINITION 1. Let $S \subseteq R^d$. We say that S has property P_k if and only if S is a finite union of d-polytopes and for every finite set $F \subseteq \operatorname{bdry} S$ there exist points $c_1, \dots, c_k$ (depending on F) such that each point of F is clearly visible via S from at least one c_i, $1 \le i \le k$.

Several lemmas will be needed to prove Theorem 1. The first of these is a variation of [2, Lemma 2].

LEMMA 1. *Let $S \subseteq R^d$, $z \in S$, and assume that S is locally starshaped at z. If $p \in \operatorname{conv} A_z$ and $p \neq z$, then there exists some point $p' \in [p, z)$ such that $p' \in A_z$.*

Proof. As in [2, Lemma 2], use Carathéodory's theorem to select a set of $d + 1$ or fewer points $p_1, \dots, p_k$ in A_z with $p \in \operatorname{conv}\{p_1, \dots, p_k\}$. Say $p = \Sigma\{\lambda_i p_i\colon 1 \le i \le k\}$, where $0 \le \lambda_i \le 1$ and $\Sigma\{\lambda_i\colon 1 \le i \le k\} = 1$. Observe that for any $0 \le \mu \le 1$, point $\mu z + (1 - \mu)p$ on $[z, p]$ is a convex conbination of the points $\mu z + (1 - \mu)p_i$, $1 \le i \le k$. Also $\mu z + (1 - \mu)p_i \in [z, p_i]$, $1 \le i \le k$. By the definition of locally starshaped, together with the definition of clear visibility, we may choose a spherical neighborhood N of z, $p \notin N$, such that z and each p_i see via S every point of $N \cap S$. We may choose μ_0, $0 < \mu_0 < 1$ and μ_0 sufficiently near 1 that each point $\mu_0 z + (1 - \mu_0)p_i = p_i'$ belongs to N. Define

$$p' = \Sigma\{\lambda_i p_i' : 1 \leq i \leq k\}$$
$$= \mu_0 z + (1 - \mu_0)p \in \operatorname{conv}\{p_1', \ldots, p_k'\} \cap (z, p) \cap N.$$

We will show that p' satisfies the lemma. For $x \in N \cap S$, $[x, z] \subseteq N \cap S$, p_1 sees $[x, z]$ via S, and hence $\operatorname{conv}\{p_1', x, z\} \subseteq N \cap S$. By an easy induction, $\operatorname{conv}\{p_k', \ldots, p_1', x, z\} \subseteq N \cap S$. Since $p' \in \operatorname{conv}\{p_k', \ldots, p_1'\}$, $[p', x] \subseteq S$. We conclude that p' sees via S each point of $N \cap S$, $p' \in A_z$, and Lemma 1 is established.

LEMMA 2. *Let S be a closed set in R^d. Let P be a plane in R^d, B a component of $P \sim S$, with S locally starshaped at $z \in \operatorname{bdry} B$. Assume that line L in plane P supports B locally at z and that $B \cap M$ is in the open halfplane L_1 determined by L for an appropriate neighborhood M of z. Then* $(\operatorname{conv} A_z) \cap P \subseteq \operatorname{cl} L_2$, *where L_2 is the opposite open halfplane determined by L.*

Proof. Suppose on the contrary that there is some point $p \in (\operatorname{conv} A_z) \cap P \cap L_1$, to obtain a contradiction. Then $p \neq z$, so by Lemma 1 there exist point $p' \in [p, z)$ and convex neighborhood N of z such that p' sees via S each point of $N \cap S$. For convenience of notation, assume that $N \subseteq M \subseteq P$.

By a simple geometric argument, we may choose a point $b \in B \cap N$ such that $R(p', b)$ meets $N \cap L$ at some point w. Since $B \cap N \subseteq B \cap M \subseteq L_1$, $w \notin B$, so $(b, w]$ meets $\operatorname{bdry} B$ at a point c. We have $c \in [b, w] \subseteq N$ and $c \in \operatorname{bdry} B \subseteq S$, so $c \in N \cap S$. Therefore, by our choice of p', $[p', c] \subseteq S$. Hence $b \in [p', c] \subseteq S$, impossible since $b \in B \subseteq P \sim S$. We have a contradiction, our supposition is false, and $(\operatorname{conv} A_z) \cap P \subseteq \operatorname{cl} L_2$. Thus Lemma 2 is proved.

LEMMA 3. *Let S be a compact set in R^3, and assume that S is a finite union of polytopes. Let P be a plane in R^3, with b a bounded component of $P \sim S$. For z a point of local convexity of* $\operatorname{cl} B$, *z in edge $e \subseteq$* $\operatorname{rel\,bdry\,cl} B$, *there exists a plane H such that the following are true*:

(1) *$H \cap P$ is a line containing e.*

(2) *The two open halfspaces determined by H can be denoted H_1 and H_2 in such a way that for N any neighborhood of z such that* $(\operatorname{cl} B) \cap N$ *is convex, $B \cap N$ lies in H_1 while $A_z \subseteq$* $\operatorname{cl} H_2$.

Proof. Notice that S is locally starshaped at each of its points and that bdry$B$ is a closed polygonal curve in P. Let J be a plane, $J \neq P$, such that J contains edge e of bdry$B$. If N is any neighborhood of z such that $(\mathrm{cl}\, B) \cap N$ is convex, then J supports $(\mathrm{cl}\, B) \cap N$ at e, and $B \cap N$ necessarily lies in one of the open halfspaces J_1 determined by J. If $A_z \subseteq \mathrm{cl}\, J_2$, then J satisfies the lemma. Otherwise, $A_z \cap J_1 \neq \varnothing$.

For convenience of notation, let P_1 and P_2 denote distinct open halfspaces in R^3 determined by plane P, let $L = P \cap J$, and label the halfplanes in P determined by L so that $B \cap N \subseteq L_1 \equiv J_1 \cap P$. (See Figure 1.) Observe that convA_z is necessarily disjoint from one of $J_1 \cap P_1$ or $J_1 \cap P_2$, for otherwise $(\mathrm{conv} A_z) \cap J_1 \cap P \equiv (\mathrm{conv} A_z) \cap L_1 \cap P \neq \varnothing$, contradicting Lemma 2. Thus we may assume that $(\mathrm{conv} A_z) \cap J_1 \cap P_2 = \varnothing$, and since $(\mathrm{conv} A_z) \cap L_1 = \varnothing$, $(\mathrm{conv} A_z) \cap J_1 \subseteq P_1$.

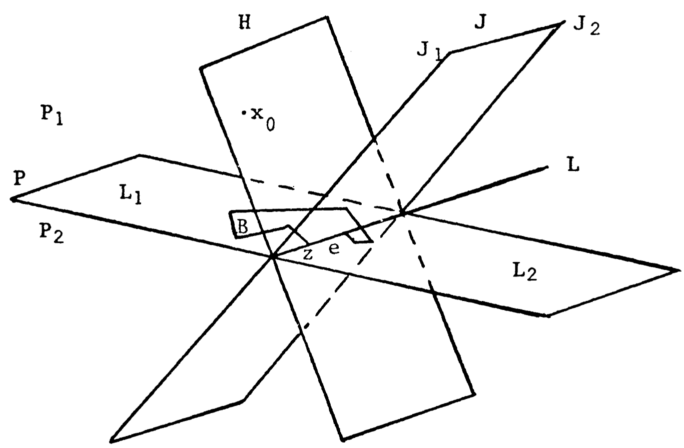

FIGURE 1

Examine the points of $A_z \cap J_1 \subseteq P_1$. For $x \in A_z \cap J_1$, x sees via S a nondegenerate segment s_z at z contained in edge e, thus generating a planar set $T_x \equiv \mathrm{conv}(s_x \cup \{x\})$. Since none of the T_x sets lie in P, each determines with $\mathrm{cl}\, L_1$ an angle of positive measure $m(x)$. Define $m \equiv \mathrm{glb}\{m(x)\colon x \in A_z \cap J_1\}$. Since S is a finite union of polytopes, the T_x sets lie in a finite union of polytopes, each meeting edge e in a nondegenerate segment at z, each contained in $P_1 \cup L$. This forces m to be positive. Using a standard argument, select sequence $\{x_i\}$ in $A_z \cap J_1$ so that $\{m(x_1)\}$ converges to m. Some subsequence of $\{x_1\}$ also converges, say to x_0. Moreover, the angle determined by $\mathrm{conv}(e \cup \{x_0)\}$ and $\mathrm{cl}\, L_1$ has measure m, and $x_0 \in (\mathrm{cl}\, A_z) \cap J_1 \subseteq P_1$. Let H be the plane determined by $\mathrm{conv}(e \cup \{x_0\})$. Of course $H \cap P = L$. Furthermore, for an appropriate labeling of halfspaces determined by H, $L_1 \subseteq H_1$ so $B \cap N \subseteq H_1$.

It remains to show that $A_z \subseteq \operatorname{cl} H_2$. Suppose on the contrary that $y \in A_z \cap H_1$. If $y \in P_1$, then the angle m chosen above would not be minimal. If $y \in P$, then $y \in A_z \cap P \cap L_1$, contradicting Lemma 2. If $y \in P_2$, then since $y \in P_2 \cap H_1$ and $x_0 \in P_1 \cap H$, $[y, x_0]$ would meet $P \cap H_1 = L_1$. Moreover, since $x_0 \in \operatorname{cl} A_z$, there would be a point $x_0' \in A_z$ sufficiently near x_0 that $[y, x_0']$ would meet $P \cap H_1 = L_1$ also, say at point w. Then $w \in (\operatorname{conv} A_z) \cap P \cap L_1$, again contradicting Lemma 2. We conclude that $A_z \cap H_1 = \varnothing$, and $A_z \subseteq \operatorname{cl} H_2$, finishing the proof of Lemma 3.

The final lemma follows immediately from [**4**, Theorem 1].

LEMMA 4 (*Lawrence, Hare, Kenelly Lemma*). *Let S be a closed set in R^d. Assume that every finite set F in* bdry$S$ *may be partitioned into two sets F_1 and F_2 such that each point of F_i is clearly visible from a common point of S. Then* bdry$S$ *may be partitioned into two sets S_1 and S_2 such that for every finite set F in* bdry$S$, *each point of $F \cap S_i$ is clearly visible from a common point of S, $i = 1, 2$.*

We are ready to prove the following theorem.

THEOREM 1. *Let $S \subseteq R^3$. If S satisfies property P_2, then S is a union of two starshaped sets.*

Proof. Using Lemma 4, select a partition S_1, S_2 for bdry$S$ such that for every finite set F in bdry$S$, each point of $F \cap S_i$ is clearly visible via S from a common point. For $i = 1, 2$, define $\mathscr{T}_i = \{\operatorname{cl} A_z\colon z \in S_i\}$. Then each $\mathscr{T}_i$ is a collection of compact subsets of S. Moreover, by our choice of S_1 and S_2, each $\mathscr{T}_i$ has the finite intersection property. Hence $\bigcap\{T\colon T$ in $\mathscr{T}_i\} \neq \varnothing$, and we may select points c and d with $c \in \bigcap\{T\colon T$ in $\mathscr{T}_1\}$ and $d \in \bigcap\{T\colon T$ in $\mathscr{T}_2\}$. Observe that for $z \in \operatorname{bdry} S = S_1 \cup S_2$, one of c or d, say c, belongs to $\operatorname{cl} A_z$. Then $[c, z] \subseteq S$. We conclude that each boundary point of S sees via S either c or d.

We will show that each point of S sees via S either c or d. Portions of the argument will resemble the proof of [**1**, Theorem 1]. Let $x \in S$ and suppose on the contrary that neither c nor d sees x, to reach a contradiction. Certainly $x \notin \{c, d\}$, and by a previous observation. $x \in \operatorname{int} S$. As in [**1**, Theorem 1], choose the segment at x in $S \cap L(c, x)$ having maximal length, and let p and q denote its endpoints, with the order of

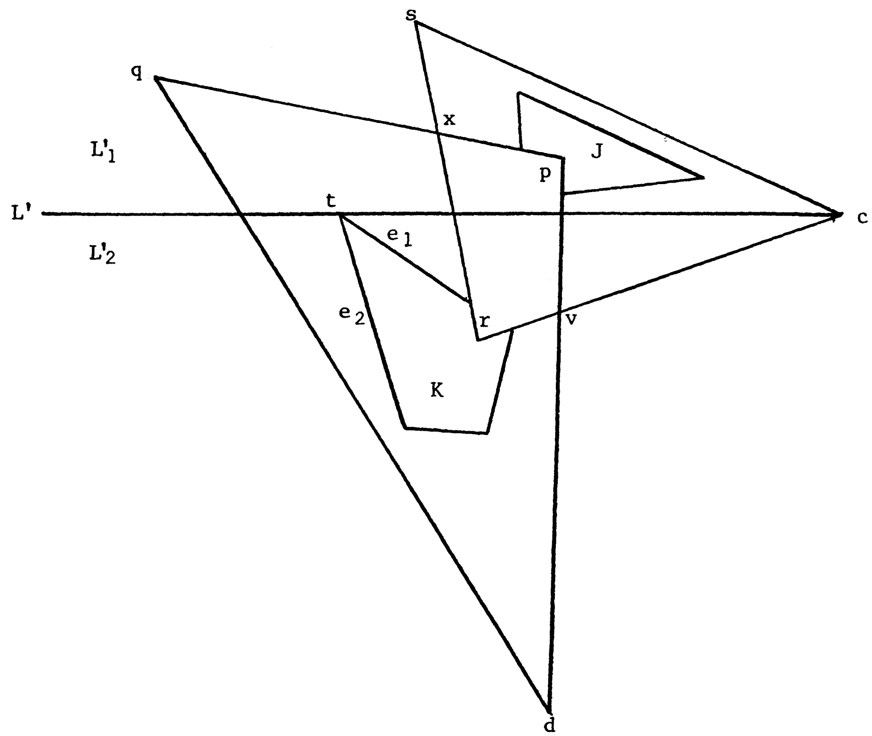

FIGURE 2

the points $c < p < x < q$. Then $p, q \in \operatorname{bdry} S$, neither is seen by c, so d sees via S both p and q. Notice that $d \notin L(c, x)$ since d cannot see x. Similarly, choose a segment at x in $S \cap L(d, x)$ having maximal length, and let r and s denote its endpoints, $d < r < x < s$. Then point c sees via S both r and s. (See Figure 2.)

Since points c, d, x are not collinear, they determine a plane P in R^3. In the next part of our proof, we restrict our attention to P. Since $[d, x] \not\subseteq S$, there is a segment in $(d, r) \sim S$, and this segment lies in a bounded component K of $P \sim S$, $K \subseteq \operatorname{rel\,int\,conv}\{d, p, q\}$. Likewise, there is a segment in $(c, p) \sim S$ belonging to a bounded component J of $P \sim S$, $J \subseteq \operatorname{rel\,int\,conv}\{c, s, r\}$. Letting $L(c, r) \cap L(d, p) = \{v\}$, it is not hard to show that J and K lie in opposite open halfplanes of P determined by $L(v, x)$.

For future reference, observe that for any line U from c meeting K, $d \notin U$, d cannot see via S all points of $\operatorname{bdry} K$ on the opposite side of U from d, so c sees via S some of these points. Thus if line U' from c supports $\operatorname{conv} K$, by a convergence argument, c sees via S some point of $U' \cap (\operatorname{bdry} K)$. We will use this observation in the next part of the proof.

Define line L' and associated point t as follows: Clearly $L(c,v) \cap J = \varnothing$. In case $L(c,v) \cap K \neq \varnothing$, let L_1 denote the open halfplane of P determined by $L(c,v)$ and containing J. Let L' be the line from c supporting $\operatorname{conv} K$ at a point of L_1. Using our previous observation, $L' \cap (\operatorname{bdry}\operatorname{conv} K)$ contains some point t of $\operatorname{bdry} K$ such that $[c,t] \subseteq S$. In case $L(c,v) \cap K = \varnothing$, rotate $L(c,v)$ about c toward d until $\operatorname{bdry} K$ is met. Let L' be the corresponding rotated line. Again using our observation, there is some $t \in L' \cap (\operatorname{bdry}\operatorname{conv} K) \cap (\operatorname{bdry} K)$ with $[c,t] \subseteq S$. Of course, in each case t may be chosen to be the furthest point from c having the required property. Moreover, $[c,t] \cap J = \varnothing$, and we may label the open halfplanes of P determined by L' so that $J \subseteq L'_1$. Then $K \cup \{d\}$ lies in the opposite halfplane L'_2.

Since S is a finite union of polytopes, $\operatorname{bdry} K$ is necessarily a simple closed polygonal curve in plane P. By our choice of t, clearly t is a point of local convexity of $\operatorname{cl} K$. Also, t must be a vertex of $\operatorname{bdry} K$, so $\operatorname{bdry} K$ contains two edges e_1 and e_2 at t. Moreover, for an appropriate labeling of these edges, $e_1 \subseteq \operatorname{cl} L'_2$, $e_2 \subseteq L'_2 \cup \{t\}$, and for any neighborhood N of t with $(\operatorname{cl} K) \cap N$ convex, $K \cap N$ and c lie in the same open halfplane of P determined by $L(e_2)$.

Using Lemma 3, select a plane H such that $H \cap P$ is a line containing e_2, $K \cap N \subseteq H_1$, and $A_t \subseteq \operatorname{cl} H_2$. Similarly, select plane M for e_1 so that $K \cap N \subseteq M_1$ and $A_t \subseteq \operatorname{cl} M_2$. Recall that by our choice of c and d, at least one of these points lies in $\operatorname{cl} A_t \subseteq \operatorname{cl} H_2 \cap \operatorname{cl} M_2$. Since c and $K \cap N$ are in the same open halfplane of P determined by $L(e_2)$, $c \in H_1$. This forces d to belong to $\operatorname{cl} H_2 \cap \operatorname{cl} M_2 \cap P$. However, clearly $\operatorname{cl} H_2 \cap \operatorname{cl} M_2 \cap P \subseteq \operatorname{cl} L'_1$, while $d \in L'_2$. We have a contradiction, our supposition is false, and every point of S must see via S either c or d. Hence S is a union of two starshaped sets, and Theorem 1 is established.

THEOREM 2. *For $k \geq 1$ and $d \geq 1$, let $\mathscr{F}(k,d)$ denote the family of all compact unions of k (or fewer) starshaped sets in R^d, $\mathscr{C}(k,d)$ the subfamily of $\mathscr{F}(k,d)$ whose members are finite unions of d-polytopes. Then $\mathscr{C}(k,d)$ is dense in $\mathscr{F}(k,d)$, relative to the Hausdorff metric. Moreover, $\mathscr{F}(k,d)$ is closed, relative to the Hausdorff metric.*

Proof. In the proof, h will denote the Hausdorff metric on compact subsets of R^d. That is, if $(A)_\delta = \{x\colon \operatorname{dist}(x,A) < \delta\}$, then for A and B compact in R^d, $h(A,B) = \inf\{\delta\colon A \subseteq (B)_\delta \text{ and } B \subseteq (A)_\delta, \delta > 0\}$.

To see that $\mathscr{C}(k,d)$ is dense in $\mathscr{F}(k,d)$, let $S \in \mathscr{F}(k,d)$. For an arbitrary $\delta > 0$, we must find some C in $\mathscr{C}(k,d)$ for which $h(S,C) < \delta$. Assume that each point of S is visible via S from one of $s_1,\ldots,s_k$. Form an open cover for S, using interiors of d-simplices whose diameters are at most $\delta/2$. Using the compactness of S, reduce to a finite subcover, say $\{\operatorname{int} P_j\colon 1 \le j \le m\}$, where P_j is a d-simplex. For $1 \le i \le k$, define $C_i = \bigcup\{\operatorname{conv}(s_i \cup P_j)\colon s_i$ sees via S some point of P_j, $1 \le j \le m\}$. Certainly set $C \equiv C_1 \cup \cdots \cup C_k$ is a union of k starshaped sets as well as a finite union of d-polytopes. Thus $C \in \mathscr{C}(k, d)$.

Clearly $S \subseteq C$, so $S \subseteq (C)_\delta$. To see that $C \subseteq (S)_\delta$, let $x \in C \sim S$. Then $x \in \operatorname{conv}(s_i \cup P_j)$ for some i and j. Moreover, for an appropriate i and j, there is some $y' \in P_j \cap S$ with $[s_i, y'] \subseteq S$. If x, s_i, y' are collinear, then since $x \notin S$, x must belong to P_j, and $\operatorname{dist}(x, y') \le \delta/2$. Thus $x \in (S)_\delta$. If x, s_i, y are not collinear, assume $x \in [s_i, y]$ where $y \in P_j$, and let x' be the point of $[s_i, y']$ such that $[x, x']$ and $[y, y']$ are parallel. Then $x' \in S$ and $\operatorname{dist}(x, x') \le \operatorname{dist}(y, y') \le \delta/2$. Again $x \in (S)_\delta$. We conclude that $C \subseteq (S)_\delta$, $h(S, C) < \delta$, and $\mathscr{C}(k, d)$ is indeed dense in $\mathscr{F}(k, d)$.

Finally, to see that $\mathscr{F}(k, d)$ is closed, let $\{S_i\}$ be a sequence in $\mathscr{F}(k, d)$ converging to the compact set S_0, to show that $S_0 \in \mathscr{F}(k, d)$ also. For convenience of notation, for $i \ge 1$, let S_i be a union of k starshaped sets whose compact kernels are $A_{i1}, A_{i2}, \ldots, A_{ik}$, respectively. Then by standard results concerning the Hausdorff metric [6], $\{A_{i1}\colon i \ge 1\}$ has a subsequence $\{A'_{i1}\}$ converging to some compact convex set A_1. Pass to the associated subsequence $\{S'_i\}$ of $\{S_i\}$, and repeat the argument for corresponding kernels $\{A'_{i2}\}$. By an obvious induction, in k steps we obtain subsequences $\{A_{i1}^{(k)}\}, \{A_{i2}^{(k)}\}, \ldots, \{A_{ik}^{(k)}\}$ converging to compact convex sets $A_1, \ldots, A_k$, respectively. It is a routine matter to show that S_0 is a union of k or fewer compact starshaped sets having kernels $A_1, \ldots, A_k$.

THEOREM 3. *Let S be a compact union of k starshaped sets in R^d, $k \ge 1$, $d \ge 3$. Then there is a sequence $\{S_j\}$ converging to S (relative to the Hausdorff metric) such that each S_j satisfies property P_k. That is, using the notation of Theorem 2, sets having property P_k are dense in $\mathscr{F}(k, d)$.*

Proof. As in the proof of Theorem 2, h will denote the Hausdorff metric on compact subsets of R^d. For any $\delta > 0$, we must find some C having property P_k for which $h(S, C) < \delta$.

Assume that each point of S is visible via S from one of the distinct points $s_1, \ldots, s_k$. Form an open cover for S using spheres of radius $\delta/4$, centered at points of S. Reduce to a finite subcover, and choose the center of each sphere. Say these centers are the points $t_1, \ldots, t_m$. Partition

$\{t_1,\ldots,t_m\}$ into k subsets $V_1,\ldots,V_k$ such that the following is true: If $t \in V_i$, then s_i is a point of $\{s_1,\ldots,s_k\}$ closest to t with $[s_i,t] \subseteq S$. Define $T_i = \bigcup\{[s_i,t]\colon t \in V_i\}$. Observe that $s_i \notin T_j$ for $i \neq j$: Otherwise, $s_i \in (s_j,t]$ for some $t \in V_j$, $[s_i,t] \subseteq (s_j,t] \subseteq S$, and s_i would be closer to t than s_j is to t, impossible by the definition of V_j.

In case the sets $T_1,\ldots,T_k$ are pairwise disjoint, let $T_i' = T_i$, $1 \le i \le k$, and define T to be their union. Otherwise, suppose T_1 meets $T_2 \cup \cdots \cup T_k$. Then for some point in V_1, call it t_1 (for convenience of notation), $(s_1,t_1]$ meets $T_2 \cup \cdots \cup T_k$. Using the facts that each T_i set is a finite union of edges at s_i, $s_1 \notin T_2 \cup \cdots \cup T_k$, and $d \ge 3$, it is not hard to show that there exists an edge $[s_1,t_1']$ not collinear with $[s_1,t_1]$ such that $[s_1,t_1']$ is disjoint from $T_2 \cup \cdots \cup T_k$ and $\operatorname{dist}(t_1,t_1') < \delta/4$. Thus $h([s_1,t_1],[s_1,t_1'])$ $< \delta/4$, also. Repeating the procedure for each edge of T_1, in finitely many steps we obtain a new set T_1' starshaped at s_1 such that T_1' is disjoint from $T_2 \cup \cdots \cup T_k$ and $h(T_1,T_1') < \delta/4$.

Continuing the process for $T_2,\ldots,T_k$, by an obvious induction we obtain pairwise disjoint starshaped sets $T_1', T_2',\ldots,T_k'$ with $h(T_i,T_i') <$ $\delta/4$, $1 \le i \le k$. Define $T = T_1' \cup \cdots \cup T_k'$. Standard arguments reveal that

$$h(S,T_1 \cup \cdots \cup T_k) < \frac{\delta}{4}, \quad h(T_1 \cup \cdots \cup T_k, T) < \frac{\delta}{4},$$

and hence $h(S,T) < \delta/2$.

Finally, we extend the sets $T_1',\ldots,T_k'$ to finite unions of d-polytopes. define $m = \min\{h(T_i',T_j')\colon i \neq j\}$. Using techniques from Theorem 2, select set $C \equiv C_1 \cup \cdots \cup C_k$ in $\mathscr{C}(k,d)$ with $h(T_i,C_i) < \min\{\delta/2, m/2\}$ and with $s_i \in \ker C_i$, $1 \le i \le k$. Since $h(T_i,C_i) < m/2$, certainly the C_i sets must be pairwise disjoint. Therefore, each boundary point of C is clearly visible from some s_i, $1 \le i \le k$, and C has property P_k. Moreover,

$$h(S,C) \le h(S,T) + h(T,C) < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$$

Theorem 3 is established.

It is interesting to observe that while Theorem 3 holds when $d \ge 3$, it fails in the plane, as the following easy example reveals.

EXAMPLE 1. Let S be the set in Figure 3. Then S is a union of two starshaped sets with kernels $\{c\}$, $\{d\}$, respectively. However, sets sufficiently close to S fail to satisfy the clear visibility condition required for property P_2.

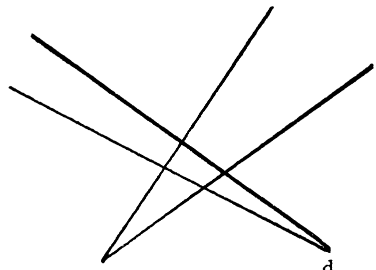

FIGURE 3

Finally, the characterization theorem for unions of two starshaped sets in R^3 is an easy consequence of our previous results.

COROLLARY 1. *Let $S \subseteq R^3$. Then S is a compact union of two starshaped sets if and only if there is a sequence $\{S_j\}$ converging to S (relative to the Hausdorff metric) such that each set S_j satisfies property P_2.*

Proof. The necessity follows immediately from Theorem 3. For the sufficiency, Theorem 1 implies that each set S_j is a compact union of two starshaped sets in R^3. By Theorem 2, their limit S is a compact union of two starshaped sets as well.

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