PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 2, 1987

FOUR DIMENSIONAL HOMOGENEOUS ALGEBRAS

L. G. SWEET AND J. A. MACDOUGALL

An algebra is homogeneous if the automorphism group acts transitively on the one dimensional subspaces of the algebra. The purpose of this paper is to determine all homogeneous algebras of dimension 4. It continues previous work of the authors in which all homogeneous algebras of dimensions 2 and 3 were described. Our main result is the proof that the field must be $GF(2)$ and the algebras are of a type previously described by Kostrikin. There are 5 non-isomorphic algebras of dimension 4; a description of each is given and the automorphism group is calculated in each case.

All algebras considered are finite dimensional and not necessarily associative. By $\mathrm{Aut}(A)$ we denote the group of algebra automorphisms of the algebra A. Thus, an algebra A is homogeneous if $\mathrm{Aut}(A)$ acts transitively on the one dimensional subspaces of A. A general discussion of homogeneous algebras may be found in [**8**] along with references to related literature. Djokovic [**2**] has classified all homogeneous algebras over the field of real numbers and has found that the only examples exist in dimensions 3, 6 and 7. Sweet [**9**] has shown that non-trivial examples cannot exist over any algebraically closed field. Homogeneous algebras of dimension 2 were studied in [**8**] where it was shown the field must be $GF(2)$. The authors have also previously classified dimension 3 homogeneous algebras [**6**] where it was found that either the algebra is a truncated quaternion algebra or else the field must be $GF(2)$. The purpose of this paper is to determine the structure and automorphism group of all homogeneous algebras of dimension 4.

Kostrikin has shown, in [**5**], how to construct homogeneous algebras over the field $GF(2)$ in every dimension.

DEFINITION. *Let $K = GF(2^n)$ and let μ be any fixed element in K. Let $\circ\colon K \times K \to K$ be the map defined by $x \circ y = \mu(xy)^{2^{n-1}}$. Then $A(n, \mu)$ denotes the algebra over $GF(2)$ obtained by replacing the usual multiplication in K by the map $\circ$. We call $A(n, \mu)$ a Kostrikin Algebra.*

These algebras are shown to be homogeneous by Kostrikin and are obviously commutative. We can now state the main result of the paper. It is summarized in the following theorem.

THEOREM. *Let A be a non-trivial four dimensional homogeneous algebra over a field K. Then $K = GF(2)$ and A is a Kostrikin algebra.*

There are actually 5 non-isomorphic algebras of this type and these, along with their automorphism groups, are described in more detail in §III of the paper. Section I contains general results about homogeneous algebras of arbitrary dimension which will be useful later. In §II, we deal with four separate cases depending on whether the dimension of the subalgebra generated by a single element is $1, 2, 3$ or 4. In cases $1, 2$ and 4, it is shown that the field must be $GF(2)$ and in case 3 no homogeneous algebra exists.

I. General results. Assume A is a homogeneous algebra. We say that A is non-trivial if $A^2 \neq 0$ and $\dim A > 1$. If a is any nonzero element of A then $\langle a \rangle$ denotes the subalgebra generated by a. From [8] we know that $\langle a \rangle$ is also a homogeneous algebra and $\langle a \rangle$ does not have any non-trivial subalgebras. As in [8], L_a denotes the linear map on A defined by left multiplication by some fixed $a \in A$ and L_a is usually represented by a matrix relative to some basis for A. If a, b are any non-zero elements of A then there exists $\alpha \in \operatorname{Aut}(A)$ such that $\alpha(a) = \lambda b$ for some nonzero scalar λ. Hence $\alpha L_a \alpha^{-1} = \lambda L_b$ and we say that L_a and L_b are projectively similar. We denote by $E_r(L_a)$ the rth elementary symmetric function of the eigenvalues of L_a, that is, the sum of the principal $r \times r$ sub-determinants of L_a. In particular, E_1 is the trace and E_n is the determinant. From [8], we know that $E_1(L_a) = 0$ for all $a \in A$. Finally we say that A is a quasi-division algebra if the nonzero elements of A form a quasi-group under multiplication.

THEOREM 1. *Let A be a commutative or anti-commutative homogeneous algebra over an infinite field K. If L_a is nilpotent for some nonzero $a \in A$ then $A^2 = 0$.*

Proof. Since L_a is nilpotent for some nonzero $a \in A$, homogeneity implies that L_x is nilpotent for all $x \in A$. If a and b are any nonzero elements of A then L_a and L_b are projectively similar nilpotent matrices. But projectively similar nilpotent matrices are in fact similar and so A is a left special nil algebra as defined in [9]. But A is commutative or anti-commutative so A is also a right special nil algebra. Since K is infinite, Theorem 2 of [9] implies that $A^2 = 0$.

THEOREM 2. *Let A be a homogeneous quasi-division algebra over a field K. If a is any nonzero element of A then L_a has precisely one eigenvalue in K and the corresponding eigenspace is one dimensional.*

Proof. If $A = \langle a \rangle$ then this is the result of Theorem 8 of **[8]**. Assume $A \neq \langle a \rangle$ and suppose L_a has an eigenvector $b \notin \langle a \rangle$. Then $ab = \lambda b$ for some nonzero scalar λ but $\langle b \rangle$ is also a quasi-division algebra so $xb = \lambda b$ has a unique solution $x \in \langle b \rangle$. This implies that $a \in \langle b \rangle$. But from Theorem 3 of **[8]** we know that $\langle a \rangle \cap \langle b \rangle = \{0\}$ and we have a contradiction.

The only known examples of homogeneous algbras over an infinite field have the property that $x^2 = 0$ for every x in the algebra (see **[2]** and **[6]**). Thus the following theorem is of interest.

THEOREM 3. *Let A be a nontrivial homogeneous algebra over a field K. If a is a nonzero element of A such that $a^2 = 0$ then L_a has no nonzero eigenvalues in K.*

Proof. Since $a^2 = 0$ for some nonzero element a in A, homogeneity implies that $x^2 = 0$ for every $x \in A$ and hence A is anticommutative. Also, clearly A is not a quasi-division algebra and so the results of Shult **[7]** and Gross **[3]** imply that K must be infinite. Let a be any nonzero element in A and suppose that L_a has a non-zero eigenvalue $\lambda \in K$ with corresponding eigenvector b. Then with respect to a basis $\{a, b, \dots\}$ L_a and L_b are $n \times n$ matrices as follows:

$$L_a = \begin{bmatrix} 0 & 0 & \alpha_{13} & \cdots & \alpha_{1n} \\ 0 & \lambda & \alpha_{23} & & \alpha_{2n} \\ 0 & 0 & \alpha_{33} & & \alpha_{3n} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & \alpha_{n3} & & \alpha_{nn} \end{bmatrix}, \quad L_b = \begin{bmatrix} 0 & 0 & \beta_{13} & \cdots & \beta_{1n} \\ -\lambda & 0 & \beta_{23} & & \beta_{2n} \\ 0 & 0 & \beta_{33} & & \beta_{3n} \\ \cdot & \cdot & \cdot & & \\ \cdot & \cdot & \cdot & & \\ 0 & 0 & \beta_{n3} & & \beta_{nn} \end{bmatrix}.$$

We proceed by showing L_b to be nilpotent, which contradicts Theorem 1. Let $t \in K$ be a variable and let B be the $(n-2) \times (n-2)$ block of $L_a + tL_b$ obtained by deleting the first two rows and columns. We will show that $E_k(L_b) = 0$ for $k = 1, \dots, n-1$. Clearly $E_{n-1}(L_b) = 0$, so that by Theorem 1(ii) of [7], $E_{n-1}(L_a + tL_b) = 0$. But

$$0 = E_{n-1}(L_a + tL_b) = \lambda E_{n-2}(B).$$

Consequently $E_{n-2}(B) = 0$ for all $t \in K$. But this is a polynomial of degree $n-2$ in t whose coefficients must be identically 0. The coefficient of t^{n-2} is just $E_{n-2}(L_b)$, so we have $E_{n-2}(L_b) = 0$ and by similarity $E_{n-2}(L_a + tL_b) = 0$. We find that

$$0 = E_{n-2}(L_a + tL_b) = \lambda E_{n-3}(B) + E_{n-2}(B) = \lambda E_{n-3}(B).$$

Therefore $E_{n-3}(B) = 0$ for all t and as before, by examining the coefficient of the highest power of t in this polynomial, we find

$$E_{n-3}(L_b) = 0.$$

This argument may be repeated to show that $E_{n-4}(L_b) = \cdots = E_1(L_b) = 0$ and the proof is complete.

THEOREM 4. *Let A be a homogeneous algebra over a field K. If there exists an $a \in A$ such that* $\dim\langle a\rangle$ *is a prime or* 4 *then* $K = GF(2)$.

Proof. As noted above, we know that $\langle a\rangle$ is a homogeneous algebra with no proper subalgebras. Since $\dim\langle a\rangle$ is a prime or 4 it follows from the corollary of Theorem 3 of Artamonov [1] that $\mathrm{Aut}\langle a\rangle$ is finite. But since $\langle a\rangle$ is homogeneous this implies that K is finite. Hence, according to Schult [7], the field $K = GF(2)$.

II. Homogeneous algebras of dimension four. Let A be a homogeneous algebra of dim 4 over a field K and let a be a nonzero element of A. We consider four cases, depending on $\dim\langle a\rangle$. In each case it will be shown that $K = GF(2)$. In §III we investigate homogeneous algebras of dim 4 over $K = GF(2)$.

Case 1. $\dim\langle a\rangle = 1$.

If $\dim\langle a\rangle = 1$ then $a^2 = \lambda a$ for some $\lambda \in K$ and there are two possibilities.

THEOREM 5. *Let A be a homogeneous algebra over a field K. If* $\dim A = 4$ *and $a^2 = \lambda a$ for some nonzero element $a \in A$ and nonzero scalar λ then* $K = GF(2)$.

Proof. Homogeneity implies that $x^2 = \lambda_x x$ for all nonzero $x \in A$ where λ_x is a nonzero scalar which may depend on x. The result follows directly from Theorem 7 of [8].

THEOREM 6. *Let A be a homogeneous algebra over a field K. If* $\dim A = 4$ *and $a^2 = 0$ for some nonzero $a \in A$ then $A^2 = 0$.*

Proof. As noted in Theorem 3, we may assume that $x^2 = 0$ for every $x \in A$. A is anti-commutative, and K is infinite. It follows from Theorem 3, from $a^2 = 0$, from $E_1(L_a) = 0$ and from $\dim A = 4$ that the only possible rational canonical forms for L_a are the following

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

Type 1. We may assume that the basis which produced this form for L_a is $\{a, b, c, d\}$. But then

$$L_b = \begin{bmatrix} 0 & 0 & \beta_1 & \beta_5 \\ 0 & 0 & \beta_2 & \beta_6 \\ 0 & 0 & \beta_3 & \beta_7 \\ 0 & 0 & \beta_4 & -\beta_3 \end{bmatrix}$$

and

$$\beta_4 L_a - L_b = \begin{bmatrix} 0 & 0 & -\beta_1 & -\beta_5 \\ 0 & 0 & -\beta_2 & -\beta_6 \\ 0 & 0 & -\beta_3 & -\beta_7 + \beta_4 \\ 0 & 0 & 0 & \beta_3 \end{bmatrix}.$$

Theorem 3 implies that $\beta_3 = 0$. But then $L_{\beta_4 a - b} = \beta_4 L_a - L_b$ is nilpotent and $A^2 = 0$ by Theorem 1.

Type 2. We may assume that the basis which produced this form for L_a is $\{b, a, c, d\}$. But then $ba = -a$ which contradicts Theorem 3 so this type does not occur.

Type 3. We may assume that the basis which produced this form for L_a is $\{a, b, c, d\}$. But then

$$L_a = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_1 \\ 0 & 1 & 0 & \alpha_2 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad L_b = \begin{bmatrix} 0 & 0 & \beta_1 & \beta_5 \\ 0 & 0 & \beta_2 & \beta_6 \\ -1 & 0 & \beta_3 & \beta_7 \\ 0 & 0 & \beta_4 & -\beta_3 \end{bmatrix}.$$

We may assume $\alpha_1 \neq 0$ since otherwise L_a is similar to Type 1 or L_a is nilpotent. Hence rank $L_a = 3$. Again consider $\beta_4 L_a - L_b$

$$\beta_4 L_a - L_b = \begin{bmatrix} 0 & 0 & -\beta_1 & -\beta_5 \\ 0 & 0 & -\beta_2 & -\beta_6 + \beta_4\alpha_1 \\ 1 & \beta_4 & -\beta_3 & -\beta_7 + \beta_4\alpha_2 \\ 0 & 0 & 0 & \beta_3 \end{bmatrix}.$$

As before, Theorem 3 implies that $\beta_3 = 0$. But then $E_3(\beta_4 L_a - L_b) = 0$ whereas $E_3(L_a) \neq 0$, and therefore $\beta_4 L_a - b$ is not projectively similar to L_a. Thus, this type does not occur.

Case 2. $\dim\langle a\rangle = 2$.

THEOREM 7. *Let A be a homogeneous algebra over a field K. If $\dim A = 4$ and $\dim\langle a\rangle = 2$ for some $a \in A$ then $K = GF(2)$.*

Proof. Since $\langle a\rangle$ is a nontrivial homogeneous algebra of dim 2, Theorem 9 of [**8**] implies that $K = GF(2)$.

Case 3. $\dim\langle a\rangle = 3$.

THEOREM 8. *Let A be a homogeneous algebra over a field K. If $\dim A = 4$ then $\dim\langle a\rangle \neq 3$ for any $a \in A$.*

Proof. Suppose there exists an $a \in A$ such that $\dim\langle a\rangle = 3$. Then homogeneity implies that $\dim\langle x\rangle = 3$ for every nonzero $x \in A$. Fix a nonzero $a \in A$ and choose any nonzero $b \notin \langle a\rangle$. Then $\dim\langle b\rangle = 3$ and so $\dim(\langle a\rangle \cap \langle b\rangle) \geq 1$. But Theorem 3 of [**8**] says that $\langle a\rangle \cap \langle b\rangle = \{0\}$ and we have a contradiction.

Case 4. $\dim\langle a\rangle = 4$.

THEOREM 9. *Let A be a homogeneous algebra over a field K. If $\dim A = 4$ and $\langle a\rangle = A$ for some $a \in A$ then $K = GF(2)$.*

Proof. This is simply a special case of Theorem 4.

III. Homogeneous algebras of dim 4 **over** $GF(2)$. Now assume that A is a homogeneous algebra over $GF(2)$ of dimension 4. By direct (but tedious) computation using methods similar to those of §II, the authors have shown that there are exactly 5 non-isomorphic algebras of this type, all of which are Kostrikin algebras. This work has been superseded by a result of Ivanov. In [**4**], the following general theorem is proved which applies in any dimension.

THEOREM 10 (*Ivanov*). *If A is a homogeneous algebra over $GF(2)$, then A is a Kostrikin algebra.*

The question of when two Kostrikin algebras of the same dimension are isomorphic was answered by Gross in [**3**].

THEOREM 11 (*Gross*). *The algebras $A(n,\mu)$ and $A(n,\lambda)$ are isomorphic if and only if there is an automorphism T of $GF(2^n)$ such that $T(\lambda)=\mu$.*

Using this result, the authors in **[10]** derived the following formula, which in the case $n=4$ shows that there exist 5 nontrivial algebras.

THEOREM 12. *The number of non-isomorphic Kostrikin algebras of dimension n is given by*

$$N_n = \frac{1}{n}\sum_{d\mid n}\phi(d)2^{n/d},$$

We proceed to determine the multiplication table of a representative of each of the isomorphism classes. As explained in the proof of Theorem 12, the automorphism group of $GF(2^n)$ is generated by the squaring map, so by Theorem 11, $A(n,\mu)$ and $A(n,\lambda)$ will be non-isomorphic if and only if λ and μ belong to different orbits of $GF(2^n)$. We construct $GF(16)$ by extending $GF(2)$ by α which is a root of the irreducible x^4+x+1. Then the orbits of $GF(16)$ are:

I $\{0\}$,
II $\{1\}$,
III $\{\alpha^2+\alpha, \alpha^2+\alpha+1\}$,
IVa $\{\alpha, \alpha^2, \alpha+1, \alpha^2+1\}$,
IVb $\{\alpha^3, \alpha^3+\alpha^2, \alpha^3+\alpha^2+\alpha+1, \alpha^3+\alpha\}$,
IVc $\{\alpha^3+1, \alpha^3+\alpha^2+1, \alpha^3+\alpha^2+\alpha, \alpha^3+\alpha+1\}$.

By choosing μ from each of the orbits in turn and substituting into the definition

$$x\circ y = \mu(xy)^{2^3}$$

we obtain the six different cases (orbit I giving the trivial algebra). In each case we use the basis $\{1,\alpha,\alpha^2,\alpha^3\}$.

Case II. ($\mu=1$)

	1	α	α^2	α^3
1	1	α^2+1	α	$\alpha^3+\alpha$
α		α	$\alpha^3+\alpha$	α^2
α^2			α^2	$\alpha^2+\alpha+1$
α^3				α^3

Case III. ($\mu = \alpha^2 + \alpha$)

	1	α	α^2	α^3
1	$\alpha^2 + \alpha$	$\alpha^3 + \alpha^2 + 1$	$\alpha^3 + \alpha^2$	$\alpha^3 + 1$
α		$\alpha^3 + \alpha^2$	$\alpha^3 + 1$	$\alpha^3 + \alpha + 1$
α^2			$\alpha^3 + \alpha + 1$	1
α^3				$\alpha^2 + 1$

Case IVa. ($\mu = \alpha$)

	1	α	α^2	α^3
1	α	$\alpha^3 + \alpha$	α^2	$\alpha^2 + \alpha + 1$
α		α^2	$\alpha^2 + \alpha + 1$	α^3
α^2			α^3	$\alpha^3 + \alpha^2 + \alpha$
α^3				$\alpha + 1$

Case IVb. ($\mu = \alpha^3$)

	1	α	α^2	α^3
1	α^3	$\alpha^3 + \alpha^2 + \alpha$	$\alpha + 1$	$\alpha^3 + \alpha^2 + \alpha + 1$
α		$\alpha + 1$	$\alpha^3 + \alpha^2 + \alpha + 1$	$\alpha^2 + \alpha$
α^2			$\alpha^2 + \alpha$	$\alpha^3 + \alpha + 1$
α^3				$\alpha^3 + \alpha^2$

Case IVc. ($\mu = \alpha^3 + 1$)

	1	α	α^2	α^3
1	$\alpha^3 + 1$	$\alpha^3 + \alpha + 1$	1	$\alpha^2 + 1$
α		1	$\alpha^2 + 1$	α
α^2			α	$\alpha^3 + \alpha$
α^3				α^2

The algebra of Case II has the property that $x^2 = x$ for every $x \in A$, i.e. A has 1-dimensional homogeneous subalgebras. The algebra of Case III has 2-dimensional homogeneous subalgebras. The algebras of Cases IVa, IVb, and IVc enjoy the property that each is generated by any nonzero element.

We conclude by describing the automorphism group for each algebra. In **[10]**, the authors have determined $\mathrm{Aut}(A)$ for any Kostrikin algebra. $\mathrm{Aut}(A)$ has 2 generators: (1) $T_\gamma(x) = \gamma x$, where γ is a generator of the multiplicative group of K; (2) S^m, where S is the squaring map and m is the smallest non-negative integer for which $S^m(\mu) = \mu$. Readers are referred to **[10]** for further details.

Case II. Aut(A) is of order 60 and the generators T_α and S satisfy the relations $T_\alpha^{15} = 1 = S^4$ and $S^{-1}T_\alpha S = T_\alpha^8$.

Case III. Aut(A) is of order 30 and the generators T_α and S^2 satisfy the relations $T_\alpha^{15} = 1 = (S^2)^2$ and $(S^2)^{-1}T_\alpha S^2 = T_\alpha^4$.

Cases IVa, IVb, IVc. Aut(A) is cyclic of order 15 and is generated by T_α.

Note. The authors wish to thank the referee for suggestions which shortened and improved §II.

REFERENCES

[1] V. A. Artamonov, *On algebras without proper subalgebras*, Math. USSR Sbornik, **33** (1977), 375–401.

[2] D. Z. Djokovic, *Real homogeneous algebras*, Proc. Amer. Math. Soc., **41** (1973), 457–62.

[3] F. Gross, *Finite automorphic algebras over GF*(2), Proc. Amer. Math. Soc., **31** (1972), 10–4.

[4] D. N. Ivanov, *On homogeneous algebras over GF*(2), Vestnik Moskov. Univ. Matematika, **37** (1982), 69–72.

[5] A. I. Kostrikin, *On homogeneous algebras*, Izvestiya Akad. Nauk USSR, **29** (1965), 471–84.

[6] J. A. MacDougall and L. G. Sweet, *Three dimensional homogeneous algebras*, Pacific J. Math., **74** (1978), 153–62.

[7] E. E. Shult, *On finite automorphic algebras*, Illinois J. Math., **13** (1969), 625–53.

[8] L. G. Sweet, *On homogeneous algebras*, Pacific J. Math., **59** (1975), 585–94.

[9] ______, *On the triviality of homogeneous algebras over an algebraically closed field*, Proc. Amer. Math. Soc., **48** (1975), 321–4.

[10] L. G. Sweet and J. A. MacDougall, *Algebras with transitive automorphism groups*, Canad. Math. Bull., **29** (1986), 224–226.

Received January 11, 1985. This research was supported by NSERC grant A5232.

UNIVERSITY OF PRINCE EDWARD ISLAND
CHARLOTTETOWN, P. E. I,
C1A 4P3 CANADA