PACIFIC JOURNAL OF MATHEMATICS
Vol. 129, No. 2, 1987

4-FIELDS ON $(4k + 2)$-DIMENSIONAL MANIFOLDS

TZE BENG NG

Let M be a closed, connected, smooth and 2-connected mod 2 (i.e., $H_i(M, \mathbf{Z}_2) = 0$, $0 < i \leq 2$) manifold of dimension $n = 4k + 2$ with $k > 1$. We obtain some necessary and sufficient conditions for the span of an n-plane bundle η over M to be greater than or equal to 4. For instance for k odd span $M \geq 4$ if and only if $\chi(M) = 0$. Some applications to immersion are given. In particular if $n = 2 + 2^l$, $l \geq 3$ and $w_4(M) = 0$ then M immerses in $\mathbf{R}^{2n-4}$.

1. Introduction. Let M be a smooth manifold, assumed throughout the paper to be closed and connected and of dimension $n = 4k + 2$ with $k > 1$.

If $k > 2$ and M is $(t - 2)$-connected mod 2 where $t = 5$ or 6, then Thomas in **[20]** gave necessary and sufficient conditions for span $M \geq t$. We shall give necessary and sufficient conditions for a 2-connected mod $2M$ to have span ≥ 4.

The Main Result. Recall the Euler-Poincaré characteristic of M is given by

$$\chi(M) = \sum_{i=0}^{n} (-1)^i \operatorname{Rank} H_i(M; \mathbf{Z}),$$

where $n = \dim M = 4k + 2$. We state our main theorem as follows:

THEOREM 1.1. *Suppose M is 2-connected* mod 2 *and* $\dim M = n \equiv 2 \bmod 4$ *and* $n \geq 10$.

(a) *If* $n \equiv 6 \bmod 8$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

(b) *If* $n \equiv 10 \bmod 16$ *and* $w_4(M) = 0$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

(c) *If* $n \equiv 2 \bmod 16$ *and* $w_4(M) = 0$ *then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\delta w_{n-4}(M) = 0$ *and* $\chi(M) = 0$.

In Theorem 1.1 δ is the co-boundary operator associated with the sequence $0 \to Z \to Z \to Z_2 \to 0$.

Notation. Let BSpin_j be the classifying space of orientable j-plane bundles ξ satisfying $w_2(\xi) = 0$. Let $\widetilde{\mathrm{BSO}_j}\langle 8\rangle$ (cf. **[13]**) be the classifying space for orientable j-plane bundles ξ satisfying $w_2(\xi) = w_4(\xi) = 0$. Then

$\widetilde{\mathrm{BSO}}_j\langle 8\rangle$ fibres over BSpin_j with k-invariant $w_4 \in H^4(\mathrm{BSpin}_j;\ \mathbf{Z}_2)$. Throughout the remainder of the paper cohomology would be ordinary cohomology with coefficients in the mod 2 integers unless otherwise specified. We denote Eilenberg-MacLane spaces of type $(\mathbf{Z}_2, j)$ and $(\mathbf{Z}, j)$ by K_j and K_j^* respectively and their fundamental classes by ι_j and ι_j^* respectively.

2. The n-MPT **for the fibration** π: $\mathrm{BSpin}_{n-4} \to \mathrm{BSpin}_n$. We list the k-invariants for the modified Postnikov tower for the fibration π: $\mathrm{BSpin}_{n-4} \to \mathrm{BSpin}_n$ through dimension n (abbreviated n-MPT see [**4**]). For the computation the reader can refer to Thomas [**17**]. Because of the fact that the indeterminacy $\mathrm{Indet}^n(k_3^2, M)$ is trivial, although our choice of k_2^2 and k_3^2 for $n \equiv 2 \bmod 8$ are not independent k-invariants, it does not affect our computation. Note that $\binom{n-4}{4} \equiv 1 \bmod 2 \Leftrightarrow (\mathrm{Sq}^4 + w_4 \cdot\,)w_{n-4} = w_n$.

TABLE 1. k invariant for π

	k-invariant	Dim	Defining relation
Stage 1	k_1^1	$n-3$	$k_1^1 = \delta w_{n-4}$
	k_2^1	$n-2$	$k_2^1 = w_{n-2}$
Stage 2	k_1^2	$n-2$	$Sq^2k_1^1 + Sq^1k_2^1 = 0$
	k_2^2	n	$(Sq^4 + w_4)k_1^1 + \binom{n-4}{4}Sq^3k_2^1 = 0$
	k_3^2	n	$(\delta Sq^2)k_2^1 = 0$
Stage 3	k^3	n	$Sq^2Sq^1k_1^2 + Sq^1k_2^2 = 0.$

We shall denote the n-MPT by

$$\begin{tikzcd} & & \mathrm{BSpin}_{n-4} \arrow[dll, "q_2"'] \arrow[dl, "q_1"] \arrow[d, "\pi"] \\ E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p_1"] & \mathrm{BSpin}_n \end{tikzcd}$$

Since we shall be considering manifolds which are 2-connected mod 2, to realize k_1^3 we shall identify $(Sq^1k_1^2, k_2^2)$ in stage 2 instead of (k_1^2, k_2^2). Let $E_1 \xrightarrow{p_1} \mathrm{BSpin}_n$ be the 1st stage n-MPT for the fibration. From the defining relation for k_3^2, the fact that $Sq^2w_{n-2} = w_n = \chi_n$ mod 2 where χ_n is the Euler class for BSpin_n, and the Peterson-Stein formula we deduce (via functional operation considerations). (See also [**6**, page 337].)

PROPOSITION 2.2.

$$k_3^2 = \frac{1}{2} p_1^* \chi_n$$

(*cf. Atiyah-Dupont* **[3]** *Theorem* 1.1 *page* 3.)

COROLLARY 2.3. *Suppose η is an n-plane bundle over M. Suppose $\delta w_{n-4}(\eta) = 0$ and $w_{n-2}(\eta) = 0$. Then modulo zero indeterminacy $k_3^2(\eta) = 0$ if, and only if $\chi(\eta) = 0$, where $\chi(\eta)$ denotes the Euler class of η.*

3. The case $w_{n-4}(M) = 0$. Throughout this section we assume that $w_{n-4}(M) = 0$.

Consider the following relations:

$$(3.1)\qquad \begin{cases} \tilde{\phi}_3\colon Sq^2Sq^2 + Sq^3\delta = 0 \quad \text{and} \\ \tilde{\phi}_4\colon \left(1 \otimes Sq^4 + \iota_4^* \otimes \rho_2\right)\delta + Sq^1\left(1 \otimes Sq^4 + \iota_4^* \otimes 1\right) \\ \qquad + \left(Sq^2Sq^1\right)Sq^2 = 0 \end{cases}$$

where ι_4^* is the fundamental class of $K(\mathbf{Z}, 4)$, ρ_2 is reduction mod 2, δ is the Bockstein operator associated with the exact sequence $0 \to Z \to Z \to Z_2 \to 0$. In (3.1), the tensor product is to be interpreted as for the Massey-Peterson algebra $\mathfrak{A}(K(\mathbf{Z}, 4))$ for the mod 2 steenrod algebra $\mathfrak{A}$. The multiplication for ρ_2 and δ is obvious. By abuse of notation and to save space we sometimes write α for $1 \otimes \alpha$ for $\alpha \in \overline{\mathfrak{A}} \cup \{\delta\}$. Consider the vector cohomology operation defined by (3.1). Its existence follows from the method of universal example as in Thomas **[18]**. Moreover it is easily seen that if we denote the operator by $(\tilde{\phi}_3, \tilde{\phi}_4)$ we have the following relation

$$(3.2)\qquad \Lambda_4\colon Sq^2\tilde{\phi}_3 + Sq^1\tilde{\phi}_4 = 0.$$

Hence we have a tertiary operation associated with the relation (3.2). Let us denote such an operation also by the symbol Λ_4. In the terminology of **[18]**, $(\tilde{\phi}_3, \tilde{\phi}_4)$ and Λ_4 are twisted cohomology operations.

Let ζ_j: $\mathrm{BSpin}_j \to K_4^*$ represent a generator of $H^4(\mathrm{BSpin}_j; \mathbf{Z}) \approx \mathbf{Z}$. Then we have

THEOREM 3.3. *Let $j \geq 5$ and let U_j be the Thom class of the universal spin j-plane bundle over* BSpin_n. *Then*

$$(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(U_j, \zeta_j) \quad \textit{and}$$

$$0 \in \Lambda_4(U_j, \zeta_j).$$

Proof. Since $H^3(\mathrm{BSpin}_j) \approx \{0\}$ and $H^4(\mathrm{BSpin}_j)$ is generated by the 4th mod 2 universal Stiefel-Whitney class w_4, trivially we can choose $(\tilde{\phi}_3, \tilde{\phi}_4)$ such that $(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(U_j, \zeta_j)$. If necessary we can replace $(\tilde{\phi}_3, \tilde{\phi}_4)$ by $(\tilde{\phi}_3, \tilde{\phi}_4 + Sq^4)$. Similarly we can choose the stable tertiary operation Λ_4 such that $0 \in \Lambda_4(U_j, \zeta_j)$.

Instead of writing ζ_j, by abuse of notation we shall confuse ζ_j with the class $Q \in H^4(\mathrm{BSpin}_j; \mathbf{Z})$ which it represents. Notice that $2Q = P_1$ the first Pontrjagin class of the universal spin j-plane bundle over BSpin_j.

Let w_{n-4} be the $(n-4)$th mod 2 universal Stiefel-Whitney class considered as in $H^{n-4}(\mathrm{BSpin}_{n-4})$. Then $(\mathrm{Sq}^4 + Q \cdot)w_{n-4} = 0$, $Sq^2 w_{n-4} = 0$ and $\delta w_{n-4} = 0$. Thus an immediate corollary to Theorem 3.3 is

PROPOSITION 3.4.
(a) $(0,0) \in (\tilde{\phi}_3, \tilde{\phi}_4)(w_{n-2}, Q) \subset H^{n-1}(\mathrm{BSpin}_{n-4}) + H^n(\mathrm{BSpin}_{n-4})$.
(b) $0 \in \Lambda_4(w_{n-4}, Q) \subset H^n(\mathrm{BSpin}_{n-4})$.

Since π^* maps $\mathrm{Indet}^{n-1,n}(\mathrm{BSpin}, (\tilde{\phi}_3, \tilde{\phi}_4))$ onto $\mathrm{Indet}^{n-1,n}(\mathrm{BSpin}_{n-4}, (\tilde{\phi}_3, \tilde{\phi}_4))$, $w_{n-4} \in H^{n-4}(\mathrm{BSpin}_n)$ is a generating class (see [**18**, §5]) for $(Sq^1 k_1^2, k_2^2)$. Thus by the generating class theorem [**18**, Theorem 5.9] we have

$$(Sq^1 k_1^2, k_2^2) \in (\tilde{\phi}_3, \tilde{\phi}_4)(p_1^* w_{n-4}, p_1^* Q). \tag{3.5}$$

Consider the commutative diagram

$$\begin{array}{ccccc} E_2 & \xrightarrow{P_1} & E_1 & \xrightarrow{(k_1^2, k_2^2, k_3^2)} & K_{n-2} \times K_n \times K_n^* \\ \downarrow f & & \downarrow \| & & \downarrow j \\ \tilde{E}_2 & \xrightarrow{\xi} & E_1 & \xrightarrow{(k_1^2, k_2^2)} & K_{n-2} \times K_n \end{array}$$

where j is the projection and ξ is the principal fibration with k-invariant (k_1^2, k_2^2) and f is the natural map induced by the commutative right-hand square. Then there is a class $\tilde{k} \in H^n(\tilde{E}_2)$ associated with the relation $Sq^2 Sq^1 k_1^2 + Sq^1 k_2^2 = 0$ such that $f^* \tilde{k} = k^3$. Since $\mathrm{Ker}\, \pi^* \subset \mathrm{Ker}\, P_1^*$ in dimension $\le n$, q_1^* maps $\mathrm{Indet}^n(E_1, \Lambda_4, Q)$ onto $\mathrm{Indet}^n(\mathrm{BSpin}_{n-4}, \Lambda_4, Q)$. Thus we have by Proposition 3.4 and (3.5) the following

PROPOSITION 3.6. $w_{n-4} \in H^{n-4}(\mathrm{BSpin}_n)$ *is a generating class for* $\tilde{k}$. *Here* $\tilde{k}$ *is considered as a coset modulo* $\mathrm{Ker}\, \tilde{q}_1^* \cap \mathrm{Im}\, \xi^*$ *where* $\tilde{q}_1 = f \circ q_2$: $\mathrm{BSpin}_{n-4} \to \tilde{E}_2$.

By the connectivity condition on M, the ith Wu class is trivial unless $i \equiv 0\,(4)$. We can easily show with the help of S-duality that $\mathrm{Indet}^n(M, k^3) = \mathrm{Indet}^n(M, \Lambda_4, \eta^*Q)$ for any map $\eta\colon M \to \mathrm{BSpin}_n$ classifying a spin n-plane bundle over M.

PROPOSITION 3.7. *Suppose* $\eta\colon M \to \mathrm{BSpin}_n$ *is a map such that* $\eta^*(\delta w_{n-4}) = 0$, $0 \in \tilde{\phi}_4(\eta^* w_{n-4}, \eta^*(Q))$ *and* $\eta^*(\chi) = 0$, *then*

$$k^3(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

Proof. Note that $\mathrm{Indet}^n(M, \tilde{k}) = \mathrm{Indet}^n(M, k^3)$. Since M is 2-connected mod 2, $(k_1^2, k_2^2)(\eta) = (0, k_2^2)(\eta)$. Thus $(0, k_2^2)(\eta) = (0, \tilde{\phi}_4)(\eta^* w_{n-4}, \eta^* Q)$. Since $0 \in \tilde{\phi}_4(\eta^* w_{n-4}, \eta^* Q)$, $(0,0) \in (0, k_2^2)(\eta)$. Thus $\tilde{k}(\eta)$ is defined. Since $\eta^*(\chi) = 0$, then by Corollary 2.3 $k_3^2(\eta) = 0$ modulo zero indeterminacy. Therefore $k^3(\eta)$ is defined. By Proposition 3.6 and the generating class theorem, there exists an element h in $H^n(E_1)$ such that $h \in \operatorname{Ker} q_1^*$ and

$$(\tilde{k} + h)(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

Since $\operatorname{Ker} q_1^* \subset \operatorname{Ker} p_2^*$ through dimension $\leq n$ and $k_3^2(\eta) = 0$

$$k^3(\eta) = (f^*\tilde{k})(\eta) = (\tilde{k} + h)(\eta) = \Lambda_4\big(\eta^* w_{n-4}, \eta^* Q\big).$$

For an n-plane bundle η over M with classifying map also denoted by η, let $w_j(\eta) = \eta^* w_j$ and $Q(\eta) = \eta^* Q$. We have from Proposition 3.7 the following

THEOREM 3.8. *Suppose* η *is an n-plane bundle over M. Then* $\operatorname{span} \eta \geq 4$ *if, and only if* $\delta w_{n-4}(\eta) = 0$, $0 \in \tilde{\phi}_4(w_{n-4}(\eta), Q(\eta))$, $\chi(\eta) = 0$ *and* $0 \in \Lambda_4(w_{n-4}(\eta), Q(\eta))$

THEOREM 3.9. *Suppose M is 2-connected* mod 2 *and* $w_{n-4}(M) = 0$. *Then* $\operatorname{span}(M) \geq 4$ *if, and only if* $\chi(M) = 0$.

Proof. Immediate from Theorem 3.8.

4. The case $w_4(M) = 0$. In this section we shall assume that $w_4(M) = 0$.

Consider the following relations:

$$(4.1)\quad \begin{cases} \phi_1\colon Sq^3(\delta Sq^{n-4}) + Sq^2(Sq^2 Sq^{n-4}) = 0, \\ \phi_2\colon Sq^4(\delta Sq^{n-4}) + Sq^1(Sq^4 Sq^{n-4}) + Sq^2 Sq^1(Sq^2 Sq^{n-4}) = 0. \end{cases}$$

Choose stable secondary cohomology operation associated with ϕ_1 and ϕ_2 of Hughes-Thomas type **[5]**, also denoted by the same symbols such that on the fundamental class d_{n-4} of D_{n-4}, the principal bundle over K_{n-4} with classifying map $(Sq^1\iota_{n-4}, Sq^2\iota_{n-4})$

$$0 \in \phi_1(d_{n-4}) \quad \text{and} \quad Sq^4 d_{n-4} \cup d_{n-4} \in \phi_2(d_{n-4}).$$

Moreover we can choose (ϕ_1, ϕ_2) such that $(0,0) \in (\phi_1, \phi_2)(\iota_{n-5})$. By the Leray-Serre exact sequence for the universal example tower for (ϕ_1, ϕ_2), we see that

$$\phi_1 = \phi_3^* \circ Sq^{n-4} \text{ modulo } \{Sq^{n-1}, Sq^{n-2}Sq^1\} \quad \text{and}$$
$$\phi_2 = \phi_4^* \circ Sq^{n-4} \text{ modulo } \{Sq^n, Sq^{n-1}Sq^1, Sq^{n-2}Sq^2\}$$

where ϕ_3^* and ϕ_4^* are defined by the following relations

$$Sq^3\delta + Sq^2Sq^2 = 0 \quad \text{and} \quad Sq^4\delta + Sq^1Sq^4 + (Sq^2Sq^1)Sq^2 = 0.$$

Furthermore (ϕ_1, ϕ_2) can be chosen in such a way that

$$\Omega\colon Sq^2\phi_1 + Sq^1\phi_2 = 0. \tag{4.2}$$

Consider now the fibration $\tilde{\pi}\colon \widetilde{\mathrm{BSO}}_{n-4}\langle 8\rangle \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$ where $\widetilde{\mathrm{BSO}}_j\langle 8\rangle$ is the classifying space for n-plane bundles ξ satisfying $w_2(\xi) = w_4(\xi) = 0$. The k-invariants for the n-MPT is as defined before in Table 1. Then $(\phi_1, \phi_2)(T\tilde{\pi})^*U_n = s^4(\phi_3^*, \phi_4^*)(U_{n-4} \cup U_{n-4})$ where s is the suspension homomorphism and U_j is the Thom class of the universal bundle over $\widetilde{\mathrm{BSO}}_j\langle 8\rangle$. Therefore $(\phi_1, \phi_2)(T\tilde{\pi})^*U_n = 0$ modulo zero indeterminacy by a Cartan formula for (ϕ_3^*, ϕ_4^*).

Now observe that $\tilde{\pi}^*\colon H^*(\widetilde{\mathrm{BSO}}_n\langle 8\rangle) \to H^*(\widetilde{\mathrm{BSO}}_{n-4}\langle 8\rangle)$ is an epimorphism in dimension $\leq n$ for $n \geq 30$ and $n \neq 34$. For $n < 30$ and $n = 34$ think of the n-MPT over $\widetilde{\mathrm{BSO}}_n\langle 8\rangle$ as the induced tower from the n-MPT over BSO_n. With this in mind it can be easily verified that $(\delta w_{n-4}, w_{n-2})$ is admissible for $(Sq^1k_1^2, k_2^2)$ via (ϕ_1, ϕ_2) [**12**, §3.2].

Let $E_2 \to E_1 \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$ be the Postnikov tower for $\tilde{\pi}$. Then by the admissible class theorem [**12**, Theorem 3.3] we have

THEOREM 4.3.

$$U(E_1)(Sq^1k_1^2, k_2^2) \in (\phi_1^*, \phi_2^*)U(E_1),$$

where $U(E_i)$ is the Thom class of the bundle over E_i induced from the universal n-plane bundle over $\widetilde{\mathrm{BSO}}_n\langle 8\rangle$ by the map $E_i \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$.

From the relation (4.2) we can choose an operation associated with the relation (4.2) denoted by Ω such that on the fundamental class b_{n-4} of Y_{n-4}, the principal bundle over K_{n-4}^* with classifying map

$(Sq^2\iota_{n-4}^*, Sq^4\iota_{n-4}^*)$

$$\tilde{\phi}_4^*(b_{n-4}) \cup (b_{n-4}) \in \Omega(b_{n-4}) \tag{4.4}$$

where $\tilde{\phi}_4^*$ is the secondary operation on integral classes associated with the relation

$$\tilde{\phi}_4^*\colon Sq^2Sq^3 + Sq^1Sq^4 = 0$$

and K_j^* is an Eilenberg-MacLane space of type $(\mathbf{Z}, j)$ and ι_j^* its fundamental class. By the methods of [**12**] (see for example. [**12**, §4.20] we can easily derive (4.4). The details are left to the reader. Thus (4.4) and the admissible class theorem give us

THEOREM 4.5.

$$U(E_2) \cdot \big(k_1^3 + p_2^* p_1^*(w_{n-4}\theta_4)\big) \in \Omega(U(E_2)),$$

where $\theta_4 \in H^4(\widetilde{\mathrm{BSO}}_n\langle 8\rangle)$ *is defined by* $\phi_4 U(\widetilde{\mathrm{BSO}}_n\langle 8\rangle) = U(\widetilde{\mathrm{BSO}}_n\langle 8\rangle)\cdot\theta_4$. *Indeed by Proposition* 3.4 *of* [**12**] *treating* $\widetilde{\mathrm{BSO}}_n\langle 8\rangle$ *as a principal fibration over* BSO_n *we see that* $\phi_4(U(\widetilde{\mathrm{BSO}}_n\langle 8\rangle)) = U(\widetilde{\mathrm{BSO}}_n\langle 8\rangle)\cdot\theta_4$ *where* θ_4 *is such that* $i^*\theta_4 = sq^1\iota_3$ *where* $i\colon K_3 \to \widetilde{\mathrm{BSO}}_n\langle 8\rangle$ *is the inclusion of the fibre. Thus* θ_4 *is a generator of* $H^4(\widetilde{\mathrm{BSO}}_n\langle 8\rangle) \approx \mathbf{Z}_2$.

REMARK. Notice that by a spectral sequence argument $q_1^*\colon H^*(E_1) \to H^*(\widetilde{\mathrm{BSO}}_{n-4}\langle 8\rangle)$ is an epimorphism through dimension n. Also

$$U(E_1)\cdot\big(\mathrm{Indet}^{n-1,n}\big(Sq^1k_1^2, k_2^2, E_1\big)\big) = \mathrm{Indet}^{2n-1,2n}(\phi_1, \phi_2, TE_1).$$

Hence we can apply the admissible class theorem.

Let ξ be an n-plane bundle over M such that $w_4(\xi) = 0$.

THEOREM 4.6. (a) *Suppose* $\mathrm{Indet}^n(k^3, M) \neq 0$. *Then* $\mathrm{span}(\xi) \geq 4$ *if, and only if* $\delta w_{n-4}(\xi) = 0$, *and* $\chi(\xi) = 0$.

(b) *Suppose* $\mathrm{Indet}^n(k^3, M) = 0$ *and* $w_{n-4}(\xi)\theta_4(\xi) = 0$ *where* $\theta_4(\xi) = g^*\theta_4$, g *a classifying map into* $\widetilde{\mathrm{BSO}}_n\langle 4\rangle$ *for* ξ. *Suppose* $\theta_4(\xi) = \theta_4(\nu)$, *where* ν *is the normal bundle of* M. *Then* $\mathrm{span}(\xi) \geq 4$ *if, and only if* $\delta w_{n-4}(\xi) = 0$, $\chi(\xi) = 0$, $\phi_2(U(\xi)) = 0$ *and* $\Omega(U(\xi)) = 0$ *modulo zero indeterminacy*.

Proof. This follows from Theorem 4.5. The details are left to the reader.

5. Evaluation on the manifold. Let $g\colon M \times M \to T(M)$ be the map that collapses the complement of a tubular neighborhood of the diagonal to a point. Then let

$$\bar{U} = g^*(U(\tau)) \bmod 2 \in H^n(M \times M).$$

We want to give a decomposition of $\overline{U}$. Note that for any $x \in H^{n/2}(M)$, $x^2 = 0$. Thus $\mathbf{Z}_2$ rank of $H^{n/2}(M)$ is even. Suppose rank $H^{n/2}(M) = 2q$. Then we have the following.

PROPOSITION 5.1. *Suppose $H^{n/2}(M) \neq \{0\}$. There exists a basis $\{x_1, \ldots, x_q, y_1, \ldots, y_q\}$ for $H^{n/2}(M)$ and an integer $r \geq 0$ such that*

$$Sq^1 x_i = 0, \qquad i = 1, \ldots, q, \qquad Sq^1 y_{r+i} = 0, \qquad i = 1, \ldots, q-r,$$
$$Sq^1 y_i \neq 0, \qquad i = 1, \ldots, r,$$

and $x_i y_j = \delta_{ij}\mu$ where δ_{ij} is the Kronecker function and $\mu \in H^n(M)$ is a generator. In particular $\{x_1, \ldots, x_r\} \subseteq Sq^1 H^{n/2-1}(M)$.

Proof. First we remark that for $n = 4k + 2$ $\operatorname{Ker} Sq^1$: $H^{2k+1}(M) \to H^{2k+2}(M)$ is non-trivial unless $H^{2k+1}(M) = \{0\}$. For if $Sq^1 x \neq 0$ then for any $y \in H^{2k}(M)$ with $Sq^1 x \cdot y \neq 0$, y satisfies $Sq^1 y \neq 0$ and $Sq^1 y \in H^{2k+1}(M) \cap \operatorname{Ker} Sq^1$. Choose generators

$$\{\alpha_1, \ldots, \alpha_r, \alpha_{r+1}, \ldots, \alpha_{r+p}, \beta_1, \ldots, \beta_r, \beta_{r+1} \cdots \beta_{r+p}\}, \quad r + p = q$$

such that $\{\alpha_1, \ldots, \alpha_r\} \subseteq \operatorname{Im} Sq^1 \cap H^{2k+1}(M)$ and $\{\alpha_{r+1}, \ldots, \alpha_{r+p}\} \subseteq \operatorname{Cok} Sq^1 \cap \operatorname{Ker} Sq^1 \cap H^{2k+1}(M)$ and $\{\beta_1, \ldots, \beta_{r+p}\}$ are their corresponding duals (i.e. $\beta_i \cdot x = 0$ for all $x \in H^{2k+1}(M)$ and $x \neq \alpha_i$, $\beta_i \cdot \alpha_i \neq 0$). Notice this choice is possible by the above remark, for $Sq^1 x \neq 0$ and $x \in H^{2k+1}(M)$ implies that x is dual to $Sq^1 y$ for some $y \in H^{2k}(M)$. Now $Sq^1 \beta_{r+i} = 0$, $1 \leq i \leq p$ for otherwise β_{r+i} is dual to some α_i, $1 \leq i \leq r$. Of course now letting $x_i = \alpha_i$, $y_i = \beta_i$ gives the required basis. Let

$$A = \sum_{i=0}^{2k} \sum_{l=1}^{n(i)} \alpha_i^l \otimes \beta_{n-i}^l + \sum_{i=1}^{q} x_i \otimes y_i$$

where $\dim H^i(M) = n(i)$ and $\{x_1, \ldots, x_q, y_1, \ldots, y_q\}$ are given by Proposition 4.1. Here $\alpha_i^k \cup \beta_{n-i}^j = \delta_{kj}\mu$. Then we have

THEOREM 5.2.
(i) $\overline{U} = A + tA$
(ii) $Sq^1 A = 0$
(iii) $A \cup tA = \hat{\chi}_2(M)\mu \otimes \mu$

where

$$\hat{\chi}_2(M) = \frac{1}{2}\left(\sum_{i=0}^{4k+2} \dim H^i(M)\right) \operatorname{mod} 2 = \frac{1}{2}\chi(M) \operatorname{mod} 2.$$

Proof. Assertion (i) follows from the fact that

$$\{\alpha_i^l, \beta_{n-i}^l\}_{i=1,\dots,2k;\, l=1,\dots,n(i)} \cup \{x_i, y_i\}_{i=1,\dots,q}$$

is a basis for $H^*(M)$ and Milnor [**11**].

$$Sq^1\overline{U} = 0 \quad \text{and}$$

$$Sq^1A = \sum_{i=0}^{2k}\left(\sum_{l=1}^{n(i)} Sq^1\alpha_i^l \otimes \beta_{n-i}^l + \alpha_i^l \otimes Sq^1\beta_{n-1}^l\right) + \sum_{i=1}^{r} x_i \otimes Sq^1y_i$$

is a sum of terms of bidegree $(j, n+1-j)$, $j \leq 2k+1$. Now $n+1-j = 4k+3-j \geq 4k+3-(2k+1) \geq 2k+2$. Therefore $Sq^1A + Sq^1tA = 0$ implies that $Sq^1A = Sq^1tA = 0$. Assertion (iii) is obvious.

PROPOSITION 5.3.
(i) $\delta Sq^{n-4}(A) = \delta w_{n-4}(M) \otimes \mu$
(ii) $Sq^4Sq^{n-4}(A) = 0$ *if* $w_4(M) = 0$.

Proof. (i)

$$Sq^{n-4}(A) = Sq^{4k-2}(A) = \sum_{l=1}^{n(2k)} Sq^{2k-2}\alpha_{2k}^l \otimes Sq^{2k}\beta_{2k+2}^l.$$

Now $Sq^{2k}\beta_{2k+2}^l = v_{2k}\cdot\beta_{2k+2}^l \neq 0$ if β_{2k+2}^l is dual to v_{2k} the $2k$th Wu class of M. We can choose for some α_{2k}^j to be v_{2k}. Thus $Sq^{2k}\beta_{2k+2}^l = 0$ for $l \neq j$. Thus

$$Sq^{n-4}(A) = Sq^{2k-2}v_{2k} \otimes \mu = w_{4k-2}(M) \otimes \mu = w_{n-4}(M) \otimes \mu,$$

and so $\delta Sq^{n-4}(A) = \delta w_{n-4}(M) \otimes \mu$.
(ii) is obvious.

PROPOSITION 5.4. *Suppose* $w_4(M) = 0$ *and* $\delta w_{n-4}(M) = 0$. *Then*
(i) (ϕ_1, ϕ_2) *is defined on* A, *and*
(ii) *Modulo zero indeterminacy,*

$$\big(0, \phi_4^*(w_{n-4}(M) \otimes \mu)\big) = (\phi_1, \phi_2)(A).$$

Hence
(iii) $(0,0) = (\phi_1, \phi_2)(U(\tau))$.

Proof. Part (i) follows from 5.3. Part (iii) follows from Part (ii) since g^* is injective. Note that $Sq^{n-2}Sq^2A = 0$ so that

$$\phi_2(A) = \phi_4^*Sq^{n-4}A = \phi_4^*(w_{n-4}(M) \otimes \mu).$$

Let $P \to K_n$ be a universal example tower for (ϕ_1, ϕ_2). Consider A as a map A: $M \times M \to K_n$. Since $\delta w_{n-4}(M) = 0$, A has a lifting $\bar{A}$ to P. Let m: $P \times P \to P$ be the multiplication map. Then the map $h = m \circ (\bar{A}, \bar{A} \circ t)$ is a lifting of $A + t^*A$ regarded as a map $m \circ (A, A \circ t)$. Let ϕ be a representative for the operation ϕ_2. Then $m^*\phi = 1 \otimes \phi + \phi \otimes 1$. Thus

$$h^*\phi = \bar{A}^*\phi + t^*\bar{A}^*\phi.$$

But t^*: $H^{2n}(M \times M) \to H^{2n}(M \times M)$ is an identity homomorphism. Therefore $h^*\phi = 0$.

Let U: $T(M) \to K_n$ represent the Thom class of the tangent bundle of M reduced mod 2. Let $\bar{U}$: $T(\tau) \to P$ be any lifting of U to P. Then $f = \bar{U} \circ g$ is a lifting of $A + t^*A$. Since g^* is injective, $\phi_2(U(\tau))$ vanishes if and only if $g^*\phi_2(U(\tau)) = f^*(\phi) = 0$. Since $\text{Indet}^{2n}(M \times M, \phi_2) = 0$, $h^*\phi = 0 \Rightarrow f^*(\phi) = 0$ since both h and f are liftings of $A + t^*A$. By the connectivity condition on M, this shows that $(\phi_1, \phi_2)(U(\tau)) = (0, 0)$. This completes the proof of Proposition 5.4.

Consider $\text{Indet}^{2n}(\Omega, T(M))$. By the connectivity condition on M $\text{Indet}^{2n}(\Omega, T(M))$ is a sum of secondary operations defined below

$$\begin{aligned}&\text{Indet}^{2n}(\Omega, T(M))\\ &\quad = \left\{ \tilde{\phi}_4^*(x) + \zeta_3(y) \,|\, x \in H^{2n-4}(T(M); \mathbf{Z}),\ y \in H^{2n-3}(T(M)) \right\}\end{aligned}$$

where ζ_3 is associated with

$$\zeta_3\colon Sq^2Sq^2 + Sq^1(Sq^2Sq^1) = 0.$$

By Atiyah-James duality the S-dual of $T(M)$ is the Thom space of the stable bundle $\alpha = -\tau - \tau$. Thus ζ_3 is trivial on $H^{2n-3}(T(M))$. $\tilde{\phi}_4^*$ is also trivial on $H^{2n-4}(T(M);\ \mathbf{Z}))$ since $\tilde{\phi}_4^*(x) = \phi_4^*(x)$ and $\theta_4(\alpha) = 0$. Thus if $\text{Indet}^n(k^3, M) = 0$ then $\text{Indet}^{2n}(\Omega, T(M)) = \text{Indet}^n(k^3, M) = \text{Indet}^{2n}(\Omega, M \times M) = 0$.

THEOREM 5.5. *Suppose* $\delta w_{n-4}(M) = 0$ *and* $w_4(M) = 0$. *Suppose further that* $\text{Indet}^n(k^3, M) = 0$. *Then*

$$\Omega(U(\tau)) = 0$$

modulo zero indetermicacy.

Proof. From Theorem 4.6 and the fact that $\text{Indet}^n(k^3, M) = 0$, $\phi_4^*(w_{n-4}(M)) = 0$. Therefore Ω is defined on A hence on tA. Thus $\Omega(A + tA) = \Omega(A) + t^*(\Omega A) = 0$ modulo zero indeterminacy.

5.6. *Proof of Theorem* 1.1.

1.1(a) follows from Theorem 3.9 since $w_{n-4}(M) = 0$ for $n \equiv 6 \bmod 8$. Similarly 1.1(b) follows from Theorem 3.9 since $n \equiv 10 \bmod 16$ and $w_4(M) = 0$ implies $w_{n-4}(M) = 0$. 1.1(c) follows from Theorem 4.6 and Theorem 5.5.

6. Immersions of manifolds. As an application of Theorem 3.8 and Theorem 4.6 we derive some immersion results. Note that for immersion we don't need the unstable k-invariants.

Suppose M is a spin-manifold. Then by Massey **[9]** it can be easily shown that if $\dim m = n \equiv 2 \bmod 4$ then $\overline{w}_{n-2}(M) = 0$ and $\delta \overline{w}_{n-4}(M) = 0$. In particular if $\dim M = n \equiv 6 \bmod 8$, $\overline{w}_{n-4}(M) = 0$. Also if $\dim M = n \equiv 10 \bmod 16$ and $w_4(M) = 0$, then $\overline{w}_{n-4}(M) = 0$.

Thus using the proof of Theorem 3.8, letting η be the stable normal bundle of M, we have:

THEOREM 6.1. *Suppose M is 2-connected* mod 2 *and $n > 6$. If* $\dim M = n \equiv 6 \bmod 8$ *or if $n \equiv 10 \bmod 16$ and $w_4(M) = 0$, then M immerses in* R^{2n-4}.

As an application of Theorem 4.6 bearing in mind that the condition $\chi(\xi) = 0$ does not apply to stable bundle we have:

THEOREM 6.2. *Suppose M is 2-connected* mod 2, $\operatorname{Dim} M = n \equiv 2 \bmod 16$ *and $w_4(M) = 0$. Then M immerses in* R^{2n-4}.

Proof. If $\operatorname{Indet}^n(k^3, M) \neq 0$, we have nothing to prove since $k^3(\nu)$ is defined and $0 \in k^3(\nu)$, where ν is the Spivak normal bundle. If $\operatorname{Indet}^n(k^3, M) = 0$, then $\tilde{\phi}_4^*$ is trivial on $H^{n-4}(M, \mathbf{Z})$. Since $\overline{w}_{n-4}(M)$ is an integral class, $\tilde{\phi}_4^*(\overline{w}_{n-4}(M)) = 0$ modulo zero indeterminacy. Therefore $\overline{w}_{n-4}(M) \cdot \theta_4(\nu) = 0$. Thus by Theorem 4.6(b) M immerses in $\mathbf{R}^{2n-4}$ since $\phi_2(U(\nu)) = \Omega(U(\nu)) = 0$ being operation mapping into the top class of $T(\nu)$.

REFERENCES

[1] Adém and S. Gitler, *Secondary characteristic classes and the immersion problem*, Bol. Soc. Mat. Mexicana, **8** (1963), 53–78.

[2] M. F. Atiyah, *Thom complexes*, Proc. London. Math. Soc., (3) **11** (1961), 291–310.

[3] M. F. Atiyah and J. L. Dupont, *Vector fields with finite singularities*, Acta Mathematica, **128** (1972), 1–40.

[4] S. Gitler and M. E. Mahowald, *The geometric dimension of real stable vector bundles*, Bol. Soc. Mat. Mexicana **11** (1966), 85–106.
[5] A. Hughes and E. Thomas, *A note on certain secondary cohomology operations*, Bol. Soc. Mat. Mexicana, **13** (1968), 1–17.
[6] M. E. Mahowald, *On obstruction theory in orientable fiber bundles*, Trans. Amer. Math. Soc., **110** (1964), 315–349.
[7] _____, *The index of a tangent* 2*-field*, Pacific J. Math., **58** (1975), 539–548.
[8] C. R. F. Maunder, *Cohomology operations of the N-th kind*, Proc. London Math. Soc., (3) (1960), 125–154.
[9] W. S. Massey and F. P. Peterson, *On the dual Stiefel-Whitney classes of a manifold*, Bol. Soc. Mat. Mexicana, (2) **8** (1963), 1–13.
[10] J. Milgram, *Cartan formulae*, Illinois J. Math., **75** (1971), 633–647.
[11] J. Milnor, *Lectures on Characteristic Classes*, Princeton University 1957.
[12] Tze Beng Ng, *The existence of* 7*-fields and* 8*-fields on manifolds*, Quart. J. Math. Oxford, (2) **30** (1979), 197–221.
[13] _____, *A note on the* mod 2 *cohomology of* $\widehat{\mathrm{BSO}}_n\langle 16\rangle$, Canad. J. Math., **XXXVII** (1985), 893–907.
[14] _____, *Vector bundles over* $(8k+3)$*-dimensional manifolds*, Pacific J. Math., **121** (1986), 427–443.
[15] _____, *Frame fields on manifolds*, to appear in Canad. J. Math.
[16] D. G. Quillen, *The* mod 2 *cohomology rings of extra-special* 2*-groups and the spinor groups*, Math. Ann., **194** (1971), 197–212.
[17] E. Thomas, *Real and complex vector fields on manifolds*, J. Math. and Mechanic, **16** (1967), 1183–1205.
[18] _____, *Postnikov invariants and higher order cohomology operations*, Ann. of Math., (2) **85** (1967), 184–217.
[19] _____, *The index of a tangent* 2*-field*, Comment. Math. Helv., **42** (1967), 86–110.
[20] _____, *The span of a manifold*, Quart. J. Math., Oxford (2) **19** (1968), 225–244.
[21] _____, *Vector fields on manifolds*, Bull. Amer. Math. Soc., **75** (1969), 643–683.

Received March 24, 1986.

NATIONAL UNIVERSITY OF SINGAPORE
KENT RIDGE
SINGAPORE 0511