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LINES HAVING CONTACT FOUR WITH A PROJECTIVE HYPERSURFACE

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Let $X \subset \mathbf{P}^{n+1}(\mathbf{C})$ be a projective hypersurface and $p \in X$. The *third contact cone* of X at p, C_p^3, is the set of all lines in $\mathbf{P}^{n+1}$ having contact ≥ 4 with X at p. If $\dim X \geq 3$ then the map $p \mapsto$ (projective moduli of C_p^3) usually is a local immersion (answering a conjecture of Griffiths and Harris), and one can prove a rigidity theorem: X is determined by the projective moduli of its C_p^3's and certain fourth order invariants. This immersion property may fail e.g. if X is a homogeneous space. We study this case also.

Introduction. In [**G-H**, pp. 450], Griffiths and Harris remark that the local geometry of a projective hypersurface can be described in terms of certain third order invariants (the C_p^3's) which occur in moving frames computations. It is known (see [**J.2**, Cor. 15]) that C_p^3 usually is a smooth complete intersection of type (2, 3) in the projectivized tangent space $\mathbf{P}T_pX$ of X at p. For example, Griffiths and Harris observe that, if X is a general hypersurface of dimension 4 then C_p^3 is a canonical curve of genus 4 in $\mathbf{P}T_pX$ (at a general p) and they conjecture that the associated map $X \to$ (moduli of curves of genus 4) should be locally injective. Is this true? To what extent does this map determine the geometry of X? What happens in other dimensions?

The present study grew out of an attempt to verify the Griffiths-Harris conjecture. In §1 we give the basic definitions, and work some examples. Section 2 is concerned with computing the ideals of the C_p^3's in terms of a moving frame, and a proof of the Griffiths-Harris conjecture in the general case (Prop. (2.10)). Novel features of our moving frames computation are the explicit general formulas (2.6) and (2.8) which greatly simplify the usual inductive procedure for computing invariants, and the fact that the invariants appear naturally as polynomials instead of simply as lists of coefficients (as in [**G-H**] and [**C**]).

In §3 we prove a rigidity theorem (Th. (3.3)) involving the map into moduli space and a quartic related to its first derivative, and conclude with a theorem (Th. (3.6)) relating failure of the Griffiths-Harris conjecture (in certain cases) to group actions.

This paper is a version of part of my thesis at UCLA. I would like to thank my advisor, Mark Green, for his expert assistance and warm friendship during its preparation.

1. Definitions and examples. Let $\mathcal{U}$ be open in $\mathbf{P}^{n+1} = \mathbf{P}^{n+1}(\mathbf{C})$, and $X \subset \mathcal{U}$ a smooth analytic hypersurface. Let $\mathbf{G}(1, n+1)$ be the Grassmannian of lines $\mathbf{P}^{n+1}$, and $J = \{(p, l) \in \mathbf{P}^{n+1} \times \mathbf{G}(1, n+1) \mid p \in l\}$ the incidence correspondence with projection $\pi\colon J \to \mathbf{P}^{n+1}$.

DEFINITION. For each $r = 0, 1, \ldots,$ the rth contact cone of X is

$$C^r = \{(p, l) \in \pi^{-1}\mathcal{U} \mid l \text{ has contact } \geq r+1 \text{ with } X \text{ at } p\}, \text{ with}$$

$$C_p^r = C^r \cap \pi^{-1}(p).$$

The contact cones measure the local geometry of the embedding $X \hookrightarrow \mathcal{U}$. Identify J with the projectivized tangent space $\mathbf{P}T\mathbf{P}^{n+1}$ via the relation "v is tangent to l". Then $C^0 = \mathbf{P}T\mathbf{P}^{n+1}|_X$, $C^1 = \mathbf{P}TX$, $C_p^2 = \{$asymptotic lines through $p\}$, etc. The following facts are proved in **[J.2]**.

(i) C^r is an analytic subscheme of J.

(ii) If X is not ruled then, at a generic point $p \in X$, C_p^{n+1} is empty and, for all $r = 1, \ldots, n$, $C_p^r \subset \mathbf{P}T\mathbf{P}^{n+1}$ is a smooth complete intersection of type $(1, 2, \ldots, r)$.

(iii) Moreover if, for some $r \leq n$, C_p^r is a smooth complete intersection of type $(1, \ldots, r)$ at generic $p \in X$, then C_p^s is a smooth complete intersection of type $(1, \ldots, s)$ at generic $p \in C$ for all $s = 0, \ldots, r$ (even if X is ruled).

Throughout this paper we shall assume that C_p^3 is a smooth complete intersection of type $(1, 2, 3)$ for generic $p \in X$.

View C_p^r, $r \geq 1$, as an abstract algebraic subvariety of $\mathbf{P}^{n-1} \cong \mathbf{P}T_pX$ defined modulo "projective equivalence" = linear change of coordinates in $\mathbf{P}^{n-1}$. If r, $n \geq 3$ then the projective equivalence class $[C_p^r]$ has nontrivial projective moduli. Let $[C^3]\colon X \to$ (moduli of C_p^3's) be the map into projective moduli space.

(1.2) EXAMPLE. If $n = 3$ then $C_p^2 \cong \mathbf{P}^1$ is a plane conic, and C_p^3 is six points on C_p^2. Thus C_p^3 has ∞^3 projective moduli. If $n = 4$, $C_p^3 \subset \mathbf{P}^3 \cong \mathbf{P}T_pX$ is a canonical curve of genus 4 with ∞^9 moduli. If $n = 5$, $C_p^3 \subset \mathbf{P}^4$ is an algebraic $K-3$ surface with ∞^{19} moduli.

Despite the large number of moduli the map $[C^3]$ may not be injective. In fact it is constant if, as in the following examples, the group $\mathrm{GL}(n+2)$, which acts linearly on $\mathbf{P}^{n+1}$, contains a subgroup which acts transitively on X. See Th. (3.6).

(1.3) EXAMPLE. (Abelian). Let $X \subset \mathbf{P}^{m+n-1}$ be defined by the homogeneous polynomial $F(Y,Z) = \prod_{i=1}^{m} Y_i^n - \prod_{j=1}^{n} Z_j^m$. X is the closure of the orbit of $[1,\dots,1]$ under the group of diagonal matrices of the form

$$A = \begin{pmatrix} a_1 & & & & & \\ & \ddots & & & 0 & \\ & & a_m & & & \\ & & & b_1 & & \\ & 0 & & & \ddots & \\ & & & & & b_n \end{pmatrix} \quad \text{such that } 1 = \prod a_i = \prod b_j.$$

If m, n are relatively prime then X is not ruled, so C_p^3 is a smooth complete intersection of type $(1,2,3)$ for generic $p \in X$.

(1.4) EXAMPLE. (Non-Abelian). Regard $\mathbf{P}^4 \cong \mathbf{P}H^0(\mathbf{P}^1, \mathcal{O}(4))$ as the linear system of quartics on $\mathbf{P}^1$. Let $G \subset \mathrm{SL}(5)$ be the image of the representation S^4: $\mathrm{SL}(2) \to \mathrm{SL}(5)$ arising from the action of $\mathrm{SL}(2)$ on quartics:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot F(X,Y) = F(aX + bX, cX + dY).$$

Then $G \cong \mathrm{SL}(2)/\{\sqrt[4]{1}\}$ is non-Abelian.

If F has at least three distinct roots then the orbit $G \cdot F$ is a hypersurface in $\mathbf{P}^4$. Let $p_1, \dots, p_4$ be the roots of F,

$$\lambda = (p_4 - p_1)(p_2 - p_3)/(p_4 - p_3)(p_2 - p_1)$$

their cross ratio, $J = 4(1 - \lambda + \lambda^2)^3/27\lambda^2(1-\lambda)^2$ (which does not depend on the order of the p_i's), $D(F)$ the discriminant, and $E(F)$ the cubic in the coefficients of F such that $E(F) = 0$ whenever $\partial^2 F/\partial X^1$, $\partial^2 F/\partial X \partial Y$, and $\partial^2 F/\partial Y^2$ are linearly dependent quadrics, with $E(F)$ normalized so that $E(F)^2 = D(F)$ when $J(F) = 0$. Then the closure of the orbit $G \cdot F$ is the algebraic hypersurface

$$\overline{G \cdot F} = \left\{ F' \in \mathbf{P}^4 \middle| E(F')^2 = (1 - J) D(F') \right\}$$

where $J = J(F)$.

If $J(F) \neq 0, 1$, or ∞ then the six cross ratios gotten by permuting the roots of F are distinct, and $\overline{G \cdot F}$ is an algebraic hypersurface of degree six which is not ruled. The exceptional cases are:

$J = 0$: There are only two cross ratios. $\overline{G \cdot F}$ is a smooth quadric threefold, ruled by lines.

$J = 1$: There are only three cross ratios. $\overline{G \cdot F}$ is a cubic—the secant variety of the rational normal curve $[s, t] \to [(sX - tY)^4]$ in $\mathbf{P}^4$. Thus $\overline{G \cdot F}$ is ruled by lines.

$J = \infty$: F has a multiple root. $\overline{G \cdot F}$ has degree six. Its dual in $\mathbf{P}^{4*}$ is the rational normal curve $[X, Y] \to [X^4, X^3Y, X^2Y^2, XY^3, Y^4]$. Thus $\overline{G \cdot F}$ is ruled by planes.

2. Moving frames and the Griffiths-Harris conjecture. Let $(x) = (x_1, \ldots, x_{n+1})$ be an *affine* coordinate system on a neighborhood of p, $(dx) = (dx_1, \ldots, dx_{n+1})$, $g \in \mathcal{O}_p$. Expand g in a power series:

$$g(x(p) + t) = g^0(x(p)) + g^1(x(p); t) + g^2(x(p); t) + \cdots, \tag{2.1}$$

where $t = (t_1, \ldots, t_{n+1})$ are indeterminants and $g^r(x(p); t)$ is the rth order part. If g generates the ideal $\mathcal{I}_p X$ of X at p, then $v \in T_p\mathbf{R}^{n+1}$ is tangent to a line l with $(p, l) \in C_p^r$ iff $0 = g^0(x(p); dx(v)) = \cdots = g^r(x(p); dx(v))$. Viewing C^r as a subscheme of $\mathbf{P}T\mathbf{P}^{n+1}$ it follows that the ideal sheaf $\mathcal{I}^r$ of C^r is generated by

$$\mathcal{I}^r = \big(g^s(x; dz) \,|\, s = 0, \ldots, r,\ g \in \mathcal{I}X\big)$$

in coordinates.

Let $(x, y) = (x_1, \ldots, x_n, y)$, $t = (t_1, \ldots, t_n)$.

If $p \in X$ is a smooth point and $dx_1, \ldots, dx_n$ are linearly independent on T_pX then we may assume g is of the form

$$g(x, y) = f(x) - y \tag{2.2}$$

for some local analytic function $f(x_1, \ldots, x_n)$ uniquely determined by X and the choice of coordinates. The ideal

$$J_p^r = \big(f^1(x(p); t), \ldots, f^r(x(p); t)\big)$$

determines a variety in $\mathbf{P}^{n-1} \cong \mathbf{P}T_pX$ which is projectively equivalent to C_p^r.

Let $e = (e_0, \ldots, e_{n+1})$ be a basis for $\mathbf{C}^{n+2}$. The set $\mathcal{F}\mathbf{P}^{n+1}$ of all such e is a principal bundle over $\mathbf{P}^{n+1}$ with projection $\pi e = e_0$ (projectivized). $\mathcal{F}\mathbf{P}^{n+1} \cong \mathrm{GL}(n+2)$ by right multiplication: $e \cdot A = (\Sigma_r e_r \cdot A_0^r, \ldots, \Sigma_r e_r \cdot A_{n+1}^r)$. The structure group $H_0 \subset \mathrm{GL}(n+2)$ is the group of matrices of the form

$$A = \begin{pmatrix} * & * & \cdots & * \\ 0 & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}. \tag{2.3}$$

The Maurer-Cartan forms ω_j^i are defined by

$$de_i = \sum_r e_r \cdot \omega_i^r. \tag{2.4}$$

Each $e \in \mathscr{F}\mathbf{P}^{n+1}$ determines a unique homogeneous coordinate system $[Z_0, \ldots, Z_{n+1}]$, such that $Z_i(e_j) = \delta_j^i$, and a unique affine coordinate system (x, y) such that $x_i = Z_i/Z_0$, $i = 1, \ldots, n$, and $y = Z_{n+1}/Z_0$. Clearly

$$(x, y)(e_0) = 0, \quad \text{and} \quad \pi^* dx_r = \omega_0^r, \qquad r = 1, \ldots, n. \tag{2.5}$$

If $e_0 \in X$ is a smooth point and $dx_1, \ldots, dx_n$ are linearly independent on $T_{e_0}X$ then e determines a unique function f as in (2.2); write

$$f = f(e; \cdot), \quad \text{and} \quad f^r(x(e_0); t) = f^t(e; t), \qquad r = 1, 2, \ldots,$$

for the power series as in (2.1).

If $\mathscr{W} \subset \mathrm{GL}(n+2)$ is sufficiently small so that, for all $A \in \mathscr{W}$, $f(e; \cdot)$ and $f(e \cdot A; \cdot)$ are both defined on a common neighborhood of e_0 and $(e \cdot A)_0$, then, after substituting $y = f(x)$ into the affine coordinate transition functions, one has an identity

$$\begin{aligned} &\frac{A_0^{n+1} + \sum_{j=1}^n A_j^{n+1} t_j + A_{n+1}^{n+1} f(e \cdot A; t)}{A_0^0 + \sum_{j=1}^n A_j^0 t_j + A_{n+1}^0 f(e \cdot A; t)} \\ &\qquad = f\left(e; \ldots, \frac{A_0^i + \sum_{j=1}^n A_j^i t_j + A_{n+1}^i f(e \cdot A; t)}{A_0^0 + \sum_{j=1}^n A_j^0 t_j + A_{n+1}^0 f(e \cdot A; t)}, \ldots\right) \end{aligned}$$

where i runs from 1 to n. Differentiate at $A = I$ and take rth order parts:

$$\begin{aligned} df^r = {} & (1-r) f^r \omega_0^0 + (2-r) f^{r-1} \sum_j t_j \omega_j^0 + \sum_{s=2}^{r-1} (1-s) f^s f^{r-s} \omega_{n+1}^0 \\ & + \sum_i \frac{\partial f^{r+1}}{\partial t_i} \omega_0^i + \sum_{ij} \frac{\partial f^r}{\partial t_i} t_j \omega_j^i + \sum_{s=1}^{r} \sum_i \frac{\partial f^s}{\partial t_i} f^{r-s+1} \omega_{n+1}^i \\ & - \delta_0^r \omega_0^{n+1} - \delta_1^r \sum_j t_j \omega_j^{n+1} - f^r \omega_{n+1}^{n+1} \end{aligned} \tag{2.6}$$

where i, j run from 1 to n, $f^{-1} = 0$, and df^r is the derivative of the polynomial-valued map $e \mapsto f^r(e; t)$.

By (2.5), $f^0(e) = 0$ for all $e \in \pi^{-1}(p)$. By making a change of frame of the form $e \mapsto e \cdot A$, $A \in H_0$ (2.3), one may arrange that $e_0, \ldots, e_n$ span T_pX, i.e. that $f^1(e; t) \equiv 0$. Assuming C_p^2 is a smooth complete intersection a further normalization makes $f^2(e; t) = Q$, where Q is the standard quadric

$$Q = \frac{1}{2} \sum_{i=1} t_i^2.$$

The set of all such frames forms a principal bundle $\mathscr{F}_Q X$ over X whose group H is all nonsingular $(n+2) \times (n+2)$ matrices

$$\text{(2.7)} \qquad A = \begin{pmatrix} a & * & * \\ 0 & B & * \\ 0 & 0 & c \end{pmatrix} \quad \text{such that} \begin{cases} a, c \in \mathbf{C} - \{0\}, \text{ and} \\ B/(ac)^{1/2} \in O(n) \end{cases}$$

where $O(n) \subset \mathrm{GL}(n)$ is the complex orthogonal group.

Set

$$R = f^3(e; t), \quad T = f^5(e; t) \quad \text{and} \quad G = f^4 - \frac{1}{4} \sum_{i=1} \left(\frac{\partial f^3}{\partial t_i} \right)^2.$$

On $\mathscr{F}_Q X$, $f^0 = 0$, $f^1 = 0$, $f^2 = Q$, so $df^0 = 0$, $df^1 = 0$, $df^2 = 0$. Using these relations (2.6) becomes

$$\text{(2.8)} \qquad \omega_0^{n+1} = 0, \qquad \omega_i^{n+1} = \omega_0^i,$$

$$\delta_{ij}\left(\omega_0^0 + \omega_{n+1}^{n+1}\right) - \left(\omega_j^i + \omega_i^j\right) = \sum_k R_{ijk} \omega_0^k,$$

$$dR = \sum_k G_k \omega_0^k + \frac{1}{2} \sum_{ij} R_i Q_j \left(\omega_j^i - \omega_i^j\right) + Q_i\left(\omega_{n+1}^i - \omega_i^0\right)$$
$$+ \frac{1}{2} R\left(\omega_{n+1}^{n+1} - \omega_0^0\right),$$

$$dG = \sum_k \left\{ \left(T - \frac{1}{2} \sum_i G_i R_i \right)_k - \frac{1}{4} \sum_{ij} R_i R_{ij} R_{jk} \right\} \omega_0^k$$
$$+ \frac{1}{2} \sum_{ij} G_i Q_j \left(\omega_j^i - \omega_i^j\right) + \frac{1}{2} \sum_i (Q R_i - R Q_i)\left(\omega_{n+1}^i + \omega_i^0\right)$$
$$- Q^2 \omega_{n+1}^0 + G\left(\omega_{n+1}^{n+1} - \omega_0^0\right),$$

where i, j, k run from 1 to n, and R_i denotes $\partial R / \partial t_i$ etc.

Let $e,\ e' \in \mathscr{F}_Q X,\ \pi(e) = e') = p'$. Then by definition, $C_p^3,\ C_{p'}^3$, are projectively equivalent iff $Q \cap R_e$ and $Q \cap R_{e'}$ are, i.e., iff there exists a complex orthogonal map B such that $B \cdot R_{e'} \equiv 0 \bmod Q,\ R_e$. Differentiating this condition, it follows that the tangent space at $[C_p^3]$ to the projective moduli space $\mathscr{M}$ of C_p^3 may be identified with

$$T_{[C_p^3]}\mathscr{M} \cong H^0(\mathbf{P}^n, \mathscr{O}(3))/\big(Q, R, Q_i, R_j - Q_j R_i \,|\, i, j = 1, \ldots, n\big).$$

Then using (2.8), the derivative of the map $p \mapsto [C_p^3] \in \mathscr{M}$ may be identified with

$$(2.9)\quad d[C^3] = \sum_k G_k dx_k \bmod \big\{Q, R, Q_i R_j - Q_j R_i \big| i, j = 1, \ldots, n\big\}.$$

The following answers a conjecture of Griffiths and Harris [**G-H**, pp. 450].

(2.10) Proposition. *For a generic algebraic hypersurface $X \subset \mathbf{P}^{n+1}$ of dimension $n \geq 3$ and degree ≥ 3, the map $[C^3]$: $X \to \mathscr{M}$ is a local immersion near a generic $p \in X$.*

Proof. By (2.9) we need to show that $G_1, \ldots, G_n$ are linearly independent modulo Q, R, $\{Q_i R_j - Q_j R_i\}$ for some (hence any) $e \in \pi^{-1}(p) \cap \mathscr{F}_Q X$. This is an open condition on the pair (p, X), so it suffices to produce a single example at $p \in X$ where X is a cubic.

An example which does the trick is the cubic

$$y\Big(1 + \sum_{ij} (\delta_{ij} + 1) x_i x_j\Big) = \frac{1}{2} \sum_i x_i^2 + \frac{1}{3} \sum_i x_i^3,$$

with $p = (0, 0, \ldots, 0)$.

Remark. The above analysis can be generalized easily, leading to an analogue of formula (2.6) for projective varieties of any codimension. See [**J.1**].

3. Rigidity Theorem. Orbits. Elie Cartan proved a rigidity theorem which pertains to the following situation [**C**, §50]. Let X and Y be (local) hypersurfaces in $\mathbf{P}^{n+1}$, and g: $X \to Y$ a holomorphic map. Assume that for each $p \in X$ there exist affine coordinate systems $(x_1, \ldots, x_{n+1})$ on a neighborhood in $\mathbf{P}^{n+1}$ of p and $(y_1, \ldots, y_{n+1})$ on a neighborhood of $g(p)$ so that, for each $i = 1, \ldots, n + 1$, the restriction to X of the coordinate functions x_i and the pullbacks $y_i \circ g$ agree on X through second order at p. In this case X and Y are said to be "projectively applicable." Cartan's theorem says that, if $n \geq 3$ and X is not developable (X is not developable if its second contact cones are nonsingular), then X and Y are projectively applicable iff they are projectively equivalent (in other words, if X and Y are projectively applicable then g extends to a linear map on $\mathbf{P}^{n+1}$). (For $n = 2$ see **[C**, §16] or **[F-C**, §65B]).

We seek to replace the hypothesis on g with one involving equivalence of contact cones. Let g: $X \to Y$ be a holomorphic map. For each $p \in X$ ($q \in Y$) let $C_p^r X$ ($C_q^r Y$) be the rth contact cone. Suppose one knows only that for each $p \in X$ the contact cones $C_p^2 X$ and $C_p^3 X$ are nonsingular and isomorphic to $C_{g(p)}^2 Y$ and $C_{g(p)}^3 Y$ (we do not require the isomorphism to come from g). As the following example shows, X and Y may not be projectively equivalent.

(3.1) Example. Let $X \subset \mathbf{P}^{n+1}$ and let $Y \subset \mathbf{P}^{n+1*}$ be its dual: $Y = \{$hyperplanes $H \subset \mathbf{P}^{n+1}$ such that H is tangent to $X\}$. Let g: $X \to Y$ be the Gauss map: $g(p) = H$ if H is tangent to X at p. If $(e_0, \ldots, e_{n+1})$ is a frame in $\mathscr{F}_Q X$ over p and $e_0^*, \ldots, e_{n+1}^* \in \mathbf{C}^{n+1}$ is the dual basis ($e_i^*(e_j) = \delta_{ij}$) then one checks that $(e_{n+1}^*, e_1^*, e_2^*, \ldots, e_n^*, e_0^*)$ is a frame in $\mathscr{F}_Q Y$ over $g(p)$. The corresponding affine coordinates are related by $y_i \circ g|_X = -x_i|_X$ through *first* order at p, and the defining polynomials for the contact cones agree: $Q(e^*) = Q(e)$ and $R(e^*) = R(e)$. Thus the (projectivized) differential dg: $\mathbf{P}T_p X \to \mathbf{P}T_{g(p)} Y$ restricts to an isomorphism from $C_p^r X$ to $C_{g(p)}^r Y$ ($r = 2, 3$). But X and Y are not in general projectively equivalent.

(In some cases X and its dual are projectively equivalent—for instance, if X is the variety in Example (1.3), or if X is a smooth quadric.)

(3.2) Definition. Triples (Q, R, G) and (Q', R', G') of polynomials of degrees $(2, 3, 4)$ are *frame equivalent* if there exist a matrix $A \in \mathrm{GL}(n)$ and a scalar $\lambda \in \mathbf{C} - \{0\}$ such that

$$Q \circ A = Q,\ R' \circ A \equiv \lambda R \bmod Q \text{ for some } \lambda \neq 0, \quad \text{and}$$
$$G' \cdot A \equiv \lambda^2 G \bmod \{Q^2, QR_i - RQ_i | i = 1, \ldots, n\}.$$

(Here $(Q \cdot A)(t) = Q(\Sigma_r A_r^0 t_r, \ldots, \Sigma_r A_r^n t_r)$ similarly for R, G.)

In particular, if $(Q, R, G) = (Q, R, G)(e)$ for some frame $e \in \mathscr{F}_Q(X)$, then the frame equivalence class of (Q, R, G) is just the set of all (Q', R', G') of the form $(Q', R', G') = (Q, R, G)(e')$ for some e' in the same fiber $\pi^{-1}\pi e \subset \mathscr{F}\mathbf{P}^{n+1}|_X$. (To see this, integrate equations (2.8) along the fiber.) It follows that the notion of frame equivalence does not depend on the partcular frame in a given fiber.

Let $X, Y \subset \mathbf{P}^{n+1}$ be (locally defined) irreducible hypersurfaces, ϕ: $X \to Y$ a holomorphic map, C^rX, C^rY the respective rth contact cones. If ϕ extends to a nonsingular linear map ϕ: $\mathbf{P}^{n+1} \to \mathbf{P}^{n+1}$ then $C_p^rX \cong C_{\phi(p)}^rY$ are projectively equivalent for all $p \in X$, $r \geq 0$. Conversely:

(3.3) THEOREM. *With X, Y, ϕ as above: assume also that $n \geq 3$, C_p^3X is a smooth complete intersection of type $(1,2,3)$ for all $p \in X$, $[C^3X]$: $X \to$ (Moduli of C^3) is a local immersion, and $(Q, R, G)_{(p)}$, $(Q, R, G)(\phi(p))$ are frame equialent for all $p \in X$. Then ϕ extends to a nonsingular linear map on $\mathbf{P}^{n+1}$.*

(REMARK. In Example (3.1), $G(e^*) = -G(e)$, so this does not contradict the theorem.)

The proof of Theorem (3.3) depends on the following two lemmas, whose proofs are given in the appendix.

(3.4) LEMMA. *If $n \geq 2$ and Q is smooth and $C_p^3 = Q \cap R$ is a smooth complete intersection then there are no linear relations among the cubics R, QQ_i, $i = 1, \ldots, n$, and $Q_iR_j - Q_jR_i$, $1 \leq i \leq j \leq n$.*

(3.5) LEMMA. *If Q is smooth and $Q \cap R$ is a smooth complete intersection in $\mathbf{P}^{n-1}$ and $n \geq 3$ then any quintic T in $\mathbf{P}^{n-1}$ satisfying*

$$T_i \equiv 0 \bmod \left\{Q^2, QR_k - RQ_k \middle| k = 1, \ldots, n\right\}, \quad \text{for all } i = 1, \ldots, n,$$

is zero.

Proof of Theorem (3.3). Shrinking X, Y if necessary one may choose sections e: $X \to \mathscr{F}_QX$, e': $Y \to \mathscr{F}_QY$ such that $(Q, R, G)(e(p)) = (Q, R, G')(e'(f(p)))$ for all $p \in X$. Define a map Φ: $\mathscr{F}_QX \to \mathscr{F}_QY$ by $\Phi(e(p) \circ h) = e'(f(p)) \circ h$ for all $p \in X$, $h \in H$ (2.7). Then since Φ is H-equivariant,

$$\Phi^*\omega_j^i \equiv \omega_j^i \bmod \omega_0^1, \ldots, \omega_0^n \qquad \text{for all } i,\ j = 0, \ldots, n+1$$

and $(Q, R, G) = (Q, R, G) \circ \Phi$ on $\mathscr{F}_QX$.

Set $\psi_j^i = \omega_j^i - \Phi^*\omega_j^i$, i, $j = 0, \ldots, n+1$. Then

$$0 = dR - \Phi^*dR = \sum_k G_k\psi_0^k + \frac{1}{2}\sum_{ij} R_iQ_j\left(\psi_j^i - \psi_i^j\right)$$

$$+ Q\sum_i Q_i\left(\psi_{n+1}^i - \psi_i^0\right) + \frac{1}{2}R\left(\psi_{n+1}^{n+1} - \psi_0^0\right).$$

Since $[C^3X]$ is a local immersion into moduli space, $G_1, \dots, G_k$ are linearly independent $\mathrm{mod}\{Q, R, Q_iR_j - Q_jR_i\}$, hence $\psi_0^k = 0$ for all $k = 1, \dots, n$. Then by Lemma (3.4), $\psi_j^i - \psi_i^j$, $\psi_{n+1}^i - \psi_i^0$, $\psi_{n+1}^{n+1} - \psi_0^0$ all vanish i, $j = 1, \dots, n$.

Plugging this into the dG-equation (2.8),

$$0 = dG - \Phi^* dG = \sum_k (T - T \circ \Phi)_k \omega_0^k$$

$$+ \frac{1}{2} \sum_i (QR_i - RQ_i)(\psi_{n+1}^i + \psi_i^0) - Q^2 \psi_{n+1}^0.$$

Apply Lemma (3.5). Then $T = T \circ \Phi$, so $\psi_{n+1}^i + \psi_i^0$, ψ_{n+1}^0 vanish for all $i = 1, \dots, n$. Next, by (2.8), ψ_0^{n+1}, ψ_i^{n+1}, $\delta_{ij}(\psi_0^0 + \psi_{n+1}^{n+1}) - (\psi_j^i + \psi_i^j)$ all vanish, i, $j = 1, \dots, n$. Finally, by multiplying e' by an appropriate scalar-valued function, one may arrange things so that $\sum_{r=0}^{n+1} \psi_r^r = 0$ (this doesn't affect the other forms).

Now

$$\Phi^* \omega_j^i = \omega_j^i \quad \text{for all } i,\ j = 0, \dots, n+1.$$

Let $\iota\colon \mathscr{F}_Q X \to \mathscr{F}\mathbf{P}^{n+1} \cong \mathrm{GL}(n+2)$ be the inclusion. By the Frobenius theorem for maps into a Lie group [**S**, pp. 10–40, 41], ι and $\Phi \circ \iota$ differ by a left translation: $\Phi \circ \iota = A \circ \iota$ for some fixed $A \in \mathrm{GL}(n+2)$. Clearly the induced map $A\colon \mathbf{P}^{n+1} \to \mathbf{P}^{n+1}$ extends ϕ. □

Theorem (3.3) does not apply in examples such as (1.3) and (1.4), since there $[C^3]$ is a constant map. It would be nice to know whether all such examples are homogeneous spaces. A partial result along these lines is the following.

(3.6) THEOREM. *Let $X \subset \mathscr{U} \subset \mathbf{P}^{n+1}$, $n \geq 3$ be a smooth irreducible analytic hypersurface such that C_p^3 is a smooth complete intersection of type* $(1, 2, 3)$ *for all $p \in X$. Then*:

(a) *the following are equivalent*:

  (i) *X extends to an immersed orbit $\tilde{X}$ of a connected Lie group $\mathscr{G} \subset \mathrm{SL}(n+2)$ acting on $\mathbf{P}^{n+1}$ in the usual way.*

  (ii) *$(Q, R, G)(e)$ and $(Q, R, G)(e')$ are frame equivalent for all e, $e' \in \mathscr{F}_Q(X)$.*

(b) *If* (a)(i) *holds then* $\dim \mathscr{G} = n$.

(c) *The following are equivalent*:

  (i) (a)(i) *holds and $\mathscr{G}$ is Abelian*

  (ii) $G \equiv 0 \,\mathrm{mod}\{Q^2, QR_i - RQ_i | i = 1, \dots, n\}$ *for all $e \in \mathscr{F}_Q(X)$.*

Proof. Given (a)(i), replace X by $\tilde{X}$. The map $g \mapsto g \cdot e = (g \cdot e_0, \dots, g \cdot e_{n+1})$ embeds $\mathcal{G}$ in $\mathcal{F}_G(X)$. Identify $\mathcal{G}$ with its image.

Let ρ: $\mathbf{C}^{n+2} - \{0\} \to \mathbf{P}^{n+1}$ be the usual projection. $f(e; t)$ is determined by the condition: $\rho(e_0 + e_1 t_1 + \cdots + e_n t_n + e_{n+1} f(e; t)) \in X$ for all t small. Multiply on the left by $g \in \mathcal{G}$. Since g acts on X, $\rho((ge_0) + (ge_1)t_1 + \cdots + (ge_{n+1})f(e; t)) \in gX = X$, hence

$$f(g \circ e; t) = f(e; t) \quad \text{for all } g \in \mathcal{G}.$$

In particular Q, R, G are constant on $\mathcal{G}$. This implies (a)(ii).

Now let $p = \pi(e)$, $\mathcal{G}_p = \{ y \in \mathcal{G} \mid g \circ p = p \}$. e_0 is constant up to scalar multiples on $\pi^{-1}(p)$. Thus, if $v \in T_e(\mathcal{G}_p)$ then $\omega_0^0, \dots, \omega_0^{n+1}$ vanish on v, as do $\sum_{r=0}^{n+1} \omega_r^r$ (since $\mathcal{G}_p \subset \mathrm{SL}(n+2)$) and dQ, dR, and dG (since, by the above, Q, R, G are constant on $\mathcal{G}_p$).

Applying the forms in (2.8) and using Lemma (3.4), it follows that $v = 0$. This proves (b).

Suppose that (a)(ii) holds. Let $e \in \mathcal{F}_Q(X)$. Identify $\mathcal{F}\mathbf{P}^{n+1}$ with $\mathrm{GL}(n+2)$ by left multiplication $A \leftrightarrow A \cdot e$. Then the Maurer-Cartan forms ω_j^i are just the left invariant 1-forms on $\mathrm{GL}(n+2)$.

Let

$$\mathcal{G}^0 = \mathrm{SL}(n+2) \cap \left\{ e' \in \mathcal{F}_Q(X) \middle| (Q, R, G)(e') = (Q, R, G)(e) \right\}.$$

(a)(ii) implies that π: $\mathcal{G}^0 \to X$ is surjective. Replace $\mathcal{G}^0$ by the connected component containing e. Then by the same argument as above, $\dim \mathcal{G}^0 = n$. Since $\mathcal{G}^0 \to X$ is surjective it follows (2.4) that the forms $\omega_0^1, \dots, \omega_0^n$ are a basis for $T^*\mathcal{G}^0$. So there are relations:

$$\begin{aligned} (3.7) \qquad \omega_j^i - \omega_i^j &= \sum_k a_{kj}^k \omega_0^k, \\ \omega_{n+1}^i - \omega_i^0 &= \sum_k b_i^k \omega_0^k, \\ \omega_{n+1}^{n+1} - \omega_0^0 &= \sum_k c^k \omega_0^k, \\ \omega_{n+1}^i + \omega_i^0 &= \sum_k d_i^k \omega_0^i, \\ \omega_{n+1}^0 &= \sum_k e^k \omega_0^k \end{aligned}$$

in $T^*\mathcal{G}^0$ for some functions $a_{ij}^k, \dots, e^k$ where i, j, k run from 1 to n. Since $dR = 0$ on $T\mathcal{G}^0$, (2.8) implies

$$(3.8) \qquad 0 = \sum_k \left\{ G_i + \frac{1}{2} \sum_{ij} a_{ij}^k R_i Q_j + \sum_i b_i^i Q_i + \frac{1}{2} R c^k \right\} \omega_0^i$$

on $T\mathcal{G}_0$, hence the expression inside the parentheses vanishes for all k. Since G is constant on $\mathcal{G}^0$ Lemma (3.4) implies that a_{ij}^k, b_i^k, c^k are constant.

Similarly, the dG equation says that

$$0 = \left(T - \frac{1}{2}\sum_i G_i R_i\right)_k - \frac{1}{4}\sum_{ij} R_i R_{ij} R_{jk} + \frac{1}{2}\sum_{ij} a_{ij}^k G_i R_j$$

$$+ \frac{1}{2}\sum_i d_i^k (QR_i - RQ_i) - e^k Q^2 + c^k G, \qquad k = 1, \ldots, n.$$

If there were any other solution T', $d_i^{k\prime}$, $e^{k\prime}$, i, $k = 1, \ldots, n$, to these equations then $T - T'$ would be a quintic satisfying

$$0 = (T - T')_k + \frac{1}{2}\sum_i \left(d_i^k - d_i^{k\prime}\right)(QR_i - RQ_i) - (e^k - e^{k\prime})Q^2$$

for all k, hence T' by Lemma (3.5).

This implies that $d_i^k = d_i^{k\prime}$, $e^k = e^{k\prime}$, hence all of the coefficients in (3.7) are constant. By (2.8), $\delta_{ij}(\omega_0^0 + \omega_{n+1}^{n+1}) - (\omega_j^i + \omega_i^j) = \sum_k R_{ijk}\omega_0^k$, $\omega_i^{n+1} = \omega_0^i$, ω_0^{n+1} all have constant coefficients on $\mathcal{G}^0$, and since $\mathcal{G}^0 \subset$ SL$(n+1)$, so does $\sum_{r=0}^{n+1} \omega_r^r = 0$.

Therefore every left invariant 1-form ω_j^i, i, $j = 0, \ldots, n+1$, satisfies a relation $\omega_j^i = \sum_k f_{ij}^k \omega_0^k$ with constant coefficients on $\mathcal{G}^0$. Let $\mathcal{I}$ be the space of 1-forms on GL$(n+2)$ spanned by $\omega_j^i - \sum_k f_{ij}^k \omega_0^k$, i, $j = 0, \ldots, n+1$. Then $\mathcal{I}$ is integrable on $\mathcal{G}^0$, hence, since the coefficients are constant and $\dim \mathcal{G}^0 = n$, $\mathcal{I}$ is integrable on GL$(n+2)$. Let $\mathcal{G} \subset$ GL$(n+2)$ be a maximal connected integral manifold extending $\mathcal{G}^0$. Since the coefficients are constant and $\mathcal{I}$ is integrable, $\mathcal{G}$ is a subgroup [S, pp. 10–40, 41]. $\pi\mathcal{G}$ extends X, hence (a)(ii) implies (a)(i).

It remains to show: (c)(i) iff (c)(ii). If (c)(ii) holds then

$$G_k \equiv 0 \bmod \{QQ_k, Q_k R_i - R_k Q_i + QR_{ik} - RQ_{ik}, i, k = 1, \ldots, n\}$$

so (a)(ii) follows by (2.9). Thus if either (c)(i) or (c)(ii) holds then everything above holds.

Let $H' \subset H$ (eq. (2.7)) be the subgroup consisting of all matrices:

$$h(\lambda, \tau) = \begin{pmatrix} 1 & \lambda_1 & \cdots & \lambda_n & \tau \\ & 1 & & & \lambda_1 \\ & & \ddots & & \vdots \\ & & & 1 & \lambda_n \\ & & & & 1 \end{pmatrix}$$

in block form (blank spaces are zero). Integrating formulas (2.8) it follows that $Q(e \cdot h) = Q(e)$, $R(e \cdot h) = R(e)$, but

$$(3.9) \quad G(e \cdot h(\lambda, \tau)) = G(e) + \sum \lambda_i (QR_i) + \left(\frac{1}{2}\sum \lambda_i^2 - \tau\right) Q^2.$$

We may choose a constant $h(\lambda, \tau)$ so that replacing e by $e \cdot h$,

$$(3.10) \quad \left\{\begin{array}{ll} \text{if (c)(i)} & \text{hold then the } c^k\text{'s are all zero in (3.8)} \\ & \text{(differentiate (3.9) with respect to } t_k \\ & \text{and apply (3.8))} \\ \text{if (c)(ii)} & \text{holds then } G = 0 \text{ (on } G\text{),} \\ & \text{and in particular the } c^k\text{'s are all zero.} \end{array}\right\} \text{for all } e \in \mathcal{G}^0.$$

Replace $\mathcal{G}^0 \subset \mathcal{F}_Q X$ by $\mathcal{G}^0 \cdot h$. In GL$(n+1)$, this just amounts to conjugating $\mathcal{G}^0$ by h, so it doesn't affect commutativity. $\omega_0^1, \dots, \omega_0^n$ are a basis for the left invariant 1-forms on $\mathcal{G}^0$. So $\mathcal{G}^0$ is Abelian iff $0 = d\omega_0^i$, $i = 1, \dots, n$, in $\Lambda^2 T^* \mathcal{G}^0$. Differentiating (2.4) on $\mathcal{F}\mathbf{P}^{n+1}$,

$$d\omega_0^i = -\sum_{j=0}^{n+1} \omega_j^i \wedge \omega_0^j.$$

Using (2.8), (3.7), (3.10) this reduces to

$$d\omega_0^i = \frac{1}{2}\sum_{j,k=1}^{n} a_{ij}^k \omega_0^k \wedge \omega_0^j, \qquad i = 1, \dots, n.$$

If (c)(ii), then all $a_{ij}^k = 0$ by (3.8), (3.10), (3.4) so (c)(ii) $\Rightarrow$ (c)(i).

If (c)(i) holds then $a_{ij}^k = a_{ik}^j$ for all $i, j, k = 1, \dots, n$. Since $a_{ij}^k = -a_{ji}^k$ for all i, j, k, and $n \geq 3$, it follows that $a_{ij}^k = 0$ for all i, j, k. Then (3.8) becomes

$$G_i' \equiv 0 \bmod Q, \qquad k = 1, \dots, n.$$

Symmetry of mixed partials now forces $G \equiv 0 \bmod Q^2$, which is (c)(ii). □

Appendix. *Proofs of Lemmas* (3.4) *and* (3.5).

Proof of (3.4). Suppose that

$$(\text{A.1}) \qquad 0 = \sum a_{ij} Q_i R_j + \sum b_i Q Q_i + cR, \qquad a_{ij} = -a_{ji},$$

is a relation. Set

$$v = \sum (a_{ij} + c\delta_{ij}/3) Q_i \frac{\partial}{\partial t_j}.$$

It suffices to show $v \equiv 0$, since then (A.1) and the fact that Q is smooth also imply $b_i = 0$ for all i.

Let $dQ = \Sigma Q_i dt_i$, $dR = \Sigma R_i dt_i$ be the formal differentials. Since $Q_i = t_i$, (A.1) says

$$0 \equiv dQ(v) \equiv dR(v) \bmod Q \tag{A.2}$$

(by the Euler relation).

Let $\Sigma \to \mathbf{P}^{n-1}$ be the vector bundle

$$\Sigma = \bigoplus_{i=1}^{n} \mathcal{O}(1).$$

Since Q, $Q \cap R$ are smooth there exist vector bundles $\Sigma' \to \mathbf{P}^{n-1}$, $\Sigma'' \to Q$ defined by exact sequences

$$0 \to \Sigma' \hookrightarrow \Sigma \xrightarrow{dQ} \mathcal{O}(2) \to 0, \quad \text{and} \tag{A.3}$$

$$0 \to \Sigma'' \hookrightarrow \Sigma'|_Q \xrightarrow{dR} \mathcal{O}(3)|_Q \to 0, \tag{A.4}$$

where $dQ(\sigma_1, \ldots, \sigma_n) = \Sigma Q_i \sigma_i$ and similarly for dR. Then $v|_Q \in H^0(Q, \Sigma'')$, by (A.2). The restriction map $H^0(\mathbf{P}^{n-1}, \Sigma) \to H^0(Q, \Sigma|_Q)$ is injective, so it's enough to show that $H^0(Q, \Sigma'') = 0$.

Let $\Lambda^k \Sigma(d) = (\lambda^k \Sigma) \otimes \mathcal{O}(d)$. Similarly for Σ', Σ''. By (A.3), there is an exact sheaf sequence:

$$\begin{aligned} 0 \to \Lambda^n \Sigma(d - 2(n-k)) &\xrightarrow{dQ} \Lambda^{n-1}\Sigma(d - 2(n-k-1)) \\ &\xrightarrow{dQ} \cdots \xrightarrow{dQ} \Lambda^{k+1}\Sigma(d-2) \xrightarrow{dQ} \Lambda^k \Sigma'(d) \to 0 \end{aligned} \tag{A.5}$$

over $\mathbf{P}^{n-1}$ for each $d \in \mathbf{Z}$, $k = 0, \ldots, n$, where "dQ" means "contract with dQ". Since the sheaf sequence is exact, the spectral sequences of hypercohomology [**G-H2** pp. 445] abut to zero. Since

$$H^p\big(\mathbf{P}^{n-1}, \Lambda^k \Sigma(d)\big) \cong \Lambda^k \mathbf{C}^n \otimes H^p\big(\mathbf{P}^{n-1}, \mathcal{O}(k+d)\big) = 0$$
$$\text{for all } p = 1, \ldots, n-2,\ d,\ k \in \mathbf{Z},$$

it follows that (A.5) is exact at the global section level if $k \geq 1$.

Similarly, in the spectral sequence associated to the exact sheaf sequence

$$0 \to \Lambda^k \Sigma'(d) \hookrightarrow \Lambda^k \Sigma(d) \xrightarrow{dQ} \Lambda^{k-1}\Sigma(d+2) \to \cdots \to \mathcal{O}(d+2k) \to 0$$

the $'d_{p+1}$ map induces isomorphisms $'E_1^{0,p} \cong 'E_2^{p+1,0}$ for $p = 1, \dots, n-2$. Since (A.5) is exact at the global section level if $k \geq 1$, it follows that

(A.6) $H^p(\mathbf{P}^{n-1}, \Lambda^k\Sigma'(d)) = 0$ if $p \neq k$ and $p = 1, \dots, n-2$,

for all k, d. Apply the restriction sequence $0 \to \Lambda^k\Sigma'(d-2) \overset{\otimes Q}{\to} \Lambda^k\Sigma'(d) \to \Lambda^k\Sigma'(d)|_Q \to 0$ and get:

(A.7) $H^p(Q, \Lambda^k\Sigma'(d)) = 0$ if $k \neq p$, $p+1$, and $p = 1, \dots, n-3$,

for all k, d.

By (A.4) one has an exact sheaf sequence on Q:

$$0 \to \Lambda^{n-1}\Sigma'\Big|_Q(-3(n-2)) \overset{dR}{\to} \Lambda^{n-2}\Sigma'\Big|_Q(-3(n-3))$$

$$\to \cdots \overset{dR}{\to} \Lambda^2\Sigma'\Big|_Q(-3) \to \Sigma'' \to 0.$$

Computing spectral sequences and applying (A.7) one has that $'d_r$: $'E_r^{n-2-r,r-1} \to 'E_r^{n-2,0}$ is the zero map $r = 2, \dots, n-1$. Hence $dR: H^0(Q, \Lambda^2\Sigma'(-3)|_Q) \to H^0(Q, \Sigma'')$ is surjective. By (A.6), and the restriction sequence, the restriction map $H^0(\mathbf{P}^{n-1}, \Lambda^2\Sigma'(-3)) \to H^0(Q, \Lambda^2\Sigma'(-3)|_Q)$ is surjective. By global exactness in (A.5), dQ: $H^0(\mathbf{P}^{n-1}, \Lambda^3\Sigma(-5)) \to H^0(\mathbf{P}^{n-1}, \Lambda^2\Sigma'(-3))$ is surjective. Finally, $H^0(\mathbf{P}^{n-1}, \Lambda^3\Sigma(-5)) \cong \Lambda^3\mathbf{C}^n \otimes H^0(\mathbf{P}^{n-1}, \mathcal{O}(-2)) = 0$. It follows that $H^0(Q, \Sigma'') = 0$.

Proof of (3.5). If

$$T_i = \sum_{k=1}^n a_{ik}(QR_k - RQ_k) + b_iQ^2 \quad \text{then}$$

$$T_{ij} = \sum_{k=1}^n a_{ik}(Q_jR_k - R_jQ_k) + Q\left(\sum_{k=1}^n a_{ik}R_{kj} + 2b_iQ_j\right) - R\sum_{k=1}^n a_{ik}Q_{jk}.$$

Since $0 = T_{ij} - T_{ji}$, for all i, j, Lemma (3.4) says that

$$a_{ij} = a_{jk} = 0 \quad \text{if } i, j, k \text{ are distinct, and}$$

$$a_{ii} = -a_{jj} \quad \text{if } i \neq j.$$

So $a_{ij} = 0$ for all i, j, since $n \geq 3$. Thus

$$0 = T_{ij} - T_{ji} = Q(b_iQ_j - b_jQ_i) \quad \text{for all } i, j$$

which is impossible unless all $b_i = 0$ since $Q_1, \dots, Q_n$ are linearly independent.

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