PACIFIC JOURNAL OF MATHEMATICS
Vol. 130, No. 2, 1987

ON THE CONGRUENCE LATTICE OF A FRAME

B. BANASCHEWSKI, J. L. FRITH AND C. R. A. GILMOUR

Recall that the Skula modification SkX of a topological space X is the space with the same underlying set as X whose topology is generated by the topology ΩX of X and the closed subsets of X. R. E. Hoffmann characterizes the spaces X for which SkX is compact Hausdorff as the noetherian sober spaces. The object of this note is to give a simple proof of the analogue of this characterization for frames and to show how our result for frames applies to the original one for spaces.

For basic information on frames and further references, see the second chapter of Johnstone [7]. Here we just recall the following:

A *frame* (also: locale) is a complete lattice in which $x \wedge \bigvee x_i = \bigvee x \wedge x_i$ for binary meet ($\wedge$) and arbitrary join ($\bigvee$), and the terms *subframe* and frame *homomorphism* refer to finite meets and arbitrary joins. The category of frame and frame homomorphisms is called Frm. For any frame L, 0 will be its zero (= bottom) and e its unit (= top). An element c of a frame L will be called *compact* (Johnstone [7]: finite) whenever $c \leq \bigvee x_i$ implies that already $c \leq x_{i_1} \vee \cdots \vee x_{i_n}$ for suitable $i_1, \ldots, i_n$; if the unit $e \in L$ is compact, one also calles L compact. A *coherent* frame is one in which (i) every element is a join of compact elements, and (ii) e is compact and finite meets of compact elements are compact.

For any frame L, its *congruence lattice* CL consists of the congruences on L, that is, the equivalence relations on L which are subframes of $L \times L$, partially ordered by inclusion. The meet in CL is then intersection, so that CL is evidently a complete lattice. The more subtle and interesting fact that CL is again a frame (this was observed by Funayama and Nakayama for the congruence lattice of a distributive lattice see Birkhoff VI, 4 [2]). Dowker and Papert [4] used the isomorphism of CL with the lattice of quotient frames of L to investigate the latter. That CL is a frame can also be seen from the fact that it is isomorphic to the frame NL of nuclei on L, the latter being the $\wedge$-preserving closure operators on L (Johnstone, [7]), by the map $CL \to NL$ taking each congruence θ to its associated nucleus defined by $k(a) = \bigvee\{x \mid (x, a) \in \theta\}$.

Particular congruences on L associated with each $a \in L$ are $\nabla_a = \{(x, y) \mid x \vee a = y \vee a\}$ and $\Delta_a = \{(x, y) \mid x \wedge a = y \wedge a\}$, also characterized as the congruences generated by $(0, a)$ and (a, e), respectively.

Similarly, for any $a \leq b$, the congruence generated by (a, b) is $\Delta_a \cap \nabla_b$, which shows that each $\theta \in CL$ is the join of such $\Delta_a \cap \nabla_b$. Further, the map $a \rightsquigarrow \nabla_a$ $(a \in L)$ is a frame embedding ν_L: $L \to CL$, natural in L, and ∇_a and Δ_a are complementary to each other, that is $\nabla_a \cap \Delta_a = \Delta = \{(x, x) \mid x \in L\}$, the zero (= bottom) of CL and $\nabla_a \vee \Delta_a = \nabla = L \times L$, the unit (= top) of CL. The latter implies that ν_L: $L \to CL$ is an epimorphism of frames.

The equivalence of the first two properties in the following proposition is the analogue for frames of Hoffmann's result for spaces. This characterization is the solution to a problem of Macnab **[9]**. We thank the referee for drawing our attention to this paper. In the following a frame is called *noetherian* whenever each of its elements is compact. Using the Axiom of Choice, this is easily seen to be equivalent to the Ascending Chain Condition which says that every sequence $a_1 \leq a_2 \leq \cdots \leq a_n \leq a_{n+1} \leq \cdots$ in L is eventually constant.

PROPOSITION 1. *The following are equivalent*:
(1) *CL is compact*,
(2) *L is Noetherian*,
(3) $CL = \operatorname{Cong} L$ (*the congruence lattice of L as a lattice*),
(4) *the complemented elements of CL are precisely the compact ones*,
(5) *CL is coherent*.

Proof. If CL is compact then every complemented element of CL, and hence in particular each ∇_a, $a \in L$, is compact. Since $a \rightsquigarrow \nabla_a$ is a frame embedding, this makes $a \in L$ compact. This establishes the implications (1) $\Rightarrow$ (4) $\Rightarrow$ (2).

If L is noetherian then so is $L \times L$. This implies that arbitrary joins in $L \times L$ are actually finite joins, and hence any *sublattice* of $L \times L$ (including top and bottom) is already a subframe. In particular, any lattice congruence on L is actually a frame congruence, and CL is just the congruence lattice of L as a lattice. It follows that CL is closed under up-directed unions, and since ∇ is generated by $(0, e)$ this makes it compact. It follows that (2) $\Rightarrow$ (3) $\Rightarrow$ (1). That (5) is an equivalent: any CL is generated by its complemented elements. If these are compact then it follows that CL is coherent.

REMARK. Evidently, a frame L is noetherian iff the natural homomorphism σ_L: $\mathscr{I}L \to L$ from its ideal lattice by taking joins is an isomorphism, that is, iff every ideal of L is principal. Hence the above proposition may be paraphrased thus: CL is compact iff σ_L: $\mathcal{J}L \to L$ is an isomorphism. This is the frame counterpart of the early result by Brümmer [**3**] that SkX is compact Hausdorff iff the natural embedding of X into the prime spectrum of ΩX (given by *all* lattice homomorphisms $\Omega X \to 2$) is a homeomorphism. For a further development of related ideas see also Künzi-Brümmer [**8**].

In order to relate Proposition 1 to topological spaces, we have to consider the spectrum functor Σ from the category Frm to the category TOP of topological spaces and continuous maps. For any frame L, ΣL is the space whose elements, called the *points* of L, are the frame homomorphisms ξ: $L \to 2$, and whose topology $\Omega\Sigma L$ consists of the sets $\Sigma_a = \{\xi \mid \xi(a) = 1\}$. Recall that L is called *spatial* whenever its points separate its elements, which is equivalent to the requirement that the frame homomorphism $L \to \Omega\Sigma L$ given by $a \rightsquigarrow \Sigma_a$ be an isomorphism. Proving the spatiality of certain types of frames usually requires some choice principle such as the Ultrafilter Theorem for Boolean algebras which we shall assume whenever needed. A particular class of frames to which this applies are the coherent frames (Banaschewski [**1**]).

The following description of the spectrum of the congruence lattice of a frame appears in [**10**] and is used by Simmons in [**11**] to study the properties of $N\Omega(X)$.

Proposition 2. *There exists a homeomorphism* γ_L: $\Sigma CL \to Sk\Sigma L$, *natural in* L.

Remark 1. The homeomorphism γ_L is determined by the continuous one-one map $\Sigma\nu_L$: $\Sigma CL \to \Sigma L$ (remember that ΣL and $Sk\Sigma L$ have the same underlying set). Moreover γ_L determines a frame isomorphism $\Omega Sk\Sigma L \to \Omega\Sigma CL$, and since the latter is the spatial reflection of CL this says: the Skula topology of the spectrum ΣL is the spatial reflection of the congruence lattice CL.

Remark 2. Frith [**5**] shows that ν_L: $L \to CL$ is the universal frame homomorphism from L with the property that each element in the image is complemented. Using this, one has an alternative proof that every $L \to 2$ factors through ν_L.

We are now in the position to give the simple proof of the sufficiency of the noetherian condition in Hoffmann's result for sober spaces quoted earlier [**6**]. For this, recall that a space X is called *noetherian* whenever its frame ΩX of open sets is *noetherian*, and *sober* whenever the usual continuous map ε_X: $X \to \Sigma\Omega X$, taking $x \in X$ to the point $\tilde{x}$ of ΩX given by $\tilde{x}(U) = \text{card}(U \cap \{x\})$, is one-one and onto. Note that the latter implies X is T_0.

PROPOSITION 3. *For any topological space X, SkX is compact Hausdorff iff X is sober and noetherian.*

Proof. ($\Rightarrow$) This part of the argument is entirely topological, and the reader is referred to the straightforward proof given in [**6**].

($\Leftarrow$) Since X is sober and hence T_0, SkX is Hausdorff by its definition and we need only check compactness. For noetherian X, $C\Omega X$ is coherent (Proposition 1) and consequently spatial (Johnstone [**7**]) and hence $\Sigma C\Omega X$ is compact. On the other hand, if X is also sober one has $SkX \cong Sk\Sigma\Omega X$, and then the homeomorphism $Sk\Sigma\Omega X \cong \Sigma C\Omega X$ of Proposition 2 for $L = \Omega X$ shows SkX is compact.

REFERENCES

[1] B. Banaschewski, *Coherent Frames*, "Continuous Lattices" Springer LNM 871, 12–19. Springer-Verlag Berlin Heidelberg New York, 1981.

[2] G. Birkhoff, *Lattice Theory*, Amer. Math. Soc. Colloquium Publications vol. 25, Third edition, American Mathematical Society 1967.

[3] G. C. L. Brümmer, *On some bitopologically induced monads in* TOP, Bremen Mathem. Arbeitspapiere, **18** (1979), 13–30a.

[4] C. H. Dowker and D. Papert, *Quotient frames and subspaces*, Proc. London Math. Soc., (3) **16** (1966), 275–296.

[5] J. L Frith, *Structured frames*, Doctoral Diss. University of Cape Town 1986.

[6] R. E. Hoffmann, *On the sobrification remainder* ${}^sX \setminus X$, Pacific J. Math., **83** (1979), 145–156.

[7] P. T. Johnstone, *Stone Spaces*, Cambridge University Press 1982.

[8] H.-P. A. Künzi and G. C. L. Brümmer, *Sobrification and bicompletion of totally bounded quasi-uniform spaces*, to appear.

[9] D. S. Macnab, *Modal operators on Heyting algebras*, Algebra Universalis, **12** (1981), 5–29.

[10] H. Simmons, *A framework for topology*, Logic Colloquium 77, Studies in Logic 96 North-Holland 1978, 239–251.

[11] ______, *Spaces with Boolean assemblies*, Colloq. Math., **43** (1980), 23–39.

Received September 25, 1986 and in revised form January 13, 1987. The first author acknowledges financial assistance from the Natural Sciences and Engineering Council of Canada. The second and third authors acknowledge financial support from the South African Council for Scientific and Industrial Research via the Topology Research Group at the University of Cape Town.

McMaster University
Hamilton, Ontario
Canada L8S 4K1

and

University of Cape Town
Rondebosch
7700 South Africa