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A STUDY OF REGULARITY PROBLEM OF HARMONIC MAPS

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In this paper we study the regularity problem of harmonic maps between closed compact manifolds (M^n, g) and (N^m, h) in dimensions $n \geq 3$.

1. Introduction. Harmonic maps are critical points of the energy functional. For technical convenience we assume, by virtue of the Nash imbedding theorem, that the target manifold N is isometrically imbedded in the smallest Euclidean space $\mathbf{R}^k$. At the end of this section we will discuss the independence of our definitions on the imbedding of N in $\mathbf{R}^k$.

DEFINITION (1.1). A map $u = (u^1, u^2, \dots, u^k)$: $M \to \mathbf{R}^k$ is said to belong to the Sobolev space $L_1^2(M, \mathbf{R}^k)$ if for $i = 1, 2, \dots, k$

$$\int_M |\nabla u^i|^2 \, dV < \infty$$

where $|\nabla u^i|$ is the covariant derivative, in local coordinates,

$$|\nabla u^i|^2(x) = g^{\alpha\beta}(x) \cdot \partial_\alpha u^i \cdot \partial_\beta u^i,$$

dV is the volume element of M. For $u \in L_1^2(M, \mathbf{R}^k)$ one defines its energy as

$$E(u) = \sum_{i=1}^{k} \int_M |\nabla u^i|^2 \, dV = \int_M |\nabla u|^2 \, dV.$$

DEFINITION (1.2). A map u is said to belong to $L_1^2(M, N)$ if $u \in L_1^2(M, \mathbf{R}^k)$ and if $u(x) \in N$, a.e. $x \in M$.

REMARK. $L_1^2(M, \mathbf{R}^k)$ with the usual norm

$$|u|_{1,2} = \left(E(u) + \sum_{i=1}^{k} \int_M |u^i|^2 \, dV \right)$$

is a separable Hilbert space. $L_1^2(M, N)$ has strong and weak topologies induced from that of $L_1^2(M, \mathbf{R}^k)$. Moreover, the set

$$\left\{ u \in L_1^2(M, N) : |u|_{1,2} \leq C \right\}$$

is weakly compact in $L_1^2(M, \mathbf{R}^k)$.

DEFINITION (1.3). A weak solution $u \in L_1^2(M, N)$ to the formal Euler-Lagrange equations of the energy functional is called a harmonic map from M into N. The equations in local coordinates form a nonlinear elliptic system

$$\Delta u^i(x) = g^{\alpha\beta}(x) \cdot A^i_{u(x)}(\partial_\alpha u, \partial_\beta u), \qquad i = 1, 2, \ldots, k,$$

where $A_u(X, Y) \in (T_u N)^\perp$ for vectors X, $Y \in T_u N$ is the second fundamental form of N given by

$$A_u(X, Y) = (D_X Y)^\perp,$$

X and Y are considered to have been extended to vector fields on N in a neighborhood of $u \in N$.

It is easy to see that u is harmonic if and only if

$$\left.\frac{d}{dt}\right|_{t=0} E(u_t) = 0,$$

where u_t is a 1-parameter family of maps defined by

$$u_t(x) = \Pi(u(x) + \eta_t(x)),$$

$$\eta_t \in C_0^\infty(M, \mathbf{R}^k), \qquad t \in [0, 1].$$

Π is the orthogonal projection of $\mathbf{R}^k$ into N. Next we introduce the concept of stationarity. One takes $u_t = u \cdot \phi_t$ for ϕ_t a 1-parameter family of compactly supported C^1 diffeomorphisms of M with $\phi_0 = \text{Id}$. Clearly $E(u_t)$ is differentiable in t. If u is critical for all variations of this type and if u is harmonic then u is called a stationary map.

By our definitions a stationary map is harmonic. The converse is not known. What is known to be true is that a C^2 harmonic map is stationary. One of the properties enjoyed by stationary maps is the monotonicity formula. We will show that the monotonicity formula still holds under some assumptions about the singular set of harmonic maps.

In this paper we will study stationary maps whose singular set is of codimension greater than 2. Our main theorem is

THEOREM. *Suppose u is a stationary map, whose singular set is contained in the graph of a $C^{1,\alpha}$ function with dimension $d < n - 2$. There exists an $\varepsilon > 0$ such that u is regular if $E(u) \leq \varepsilon$.*

The main theorem will be proved by a blow up argument which seems to work well for general elliptic functionals of quadratic type (cf. [**Kin, Har, Lin**]).

Schoen and Uhlenbeck (1982) developed a statisfying regularity theory for energy-minimizing harmonic maps. They showed that the singular set of an energy minimizing map has codimension bigger than 2. For general harmonic maps, it was shown in **[Liao]** that the isolated singularities are removable if the total energy is small. In the case $n = 2$, the result was due to Sacks and Uhlenbeck (1981).

To conclude this introduction we remark on the independency of our harmonic map definition on the imbedding of N in $\mathbf{R}^k$.

Suppose there is another isometric imbedding $\tilde{N}$ in $\mathbf{R}^k$. The situation is this: we have (weakly)

$$\Delta u^i(x) = g^{\alpha\beta}(x) \cdot A^i_{u(x)}(\partial_\alpha u, \partial_\beta u) \qquad i = 1, 2, \dots, k.$$

We want to show that the same system is satisfied by $\tilde{u} = h \circ u$, where h is a smooth isometry from N onto $\tilde{N}$.

Extend h arbitrarily to a smooth map from an open neighborhood of N into $\mathbf{R}^k$. We compute that (in the weak sense)

$$\begin{aligned} (1.3) \qquad \Delta\tilde{u}^i(x) &= g^{-1/2}\partial_\alpha\left(g^{-1/2}g^{\alpha\beta}\partial_\beta\left(h^i(u(x))\right)\right) \\ &= g^{-1/2}\partial_\alpha\left(g^{-1/2}g^{\alpha\beta}\partial_\beta u^j \partial_j h^i\right) \\ &= \partial_j h^i \cdot \Delta u^j(x) + g^{\alpha\beta} \cdot \partial_\beta u^j \cdot \partial_l \partial_j h^i \cdot \partial_\alpha u^l \\ &= g^{\alpha\beta} \cdot \left(\partial_j h^i \cdot A^j(\partial_\alpha u, \partial_\beta u) + \partial_\beta u^j \cdot \partial_l \partial_j h^i \cdot \partial_\alpha u^l\right). \end{aligned}$$

Consider X, $Y \in T_{u(x)}N$. One can easily check that

$$(1.4) \qquad \tilde{A}(Dh(X), Dh(y)) = Dh(A(X,Y)) + D^2h(X,Y)$$

where D is the usual differentiation in $\mathbf{R}^k$, $\tilde{A}$ is the second fundamental form of $\tilde{N} \subset \mathbf{R}^k$. Indeed, we have

$$(1.5) \qquad D^2h(X,Y) = XYh - (D_XY)h.$$

By a rigid motion in $\mathbf{R}^k$, we may assume $dh(X) = X$, $dh(Y) = Y$, denoting by $\tilde{\nabla}$ the covariant derivative in $\tilde{N}$, by ∇ the covariant derivative in N. We have

$$(1.6) \qquad \tilde{A}(Dh(X), Dh(Y)) = XYh - (\tilde{\nabla}_XY)h.$$

Subtracting (1.5) from (1.6), we get

$$\tilde{A}(Dh(X), Dh(Y)) = D^2h(X,Y) + (D_XY - \tilde{\nabla}_XY)h.$$

Notice that $\tilde{N}$ and N are isometric. We can replace $\tilde{\nabla}_X Y$ by $\nabla_X Y$. Thus, we get (1.4).

By assumption $u \in L_1^2(M, N)$. This ensures that $\partial_\alpha u$, $\alpha = 1, 2, \ldots, n$, exist almost everywhere. Hence, we come to the conclusion that

$$\Delta \tilde{u}^i(x) = g^{\alpha\beta}(x) \cdot \tilde{A}^i_{u(x)}(\partial_\alpha \tilde{u}, \partial_\beta \tilde{u})$$

weakly.

In the next section we will make some general remarks about the regularity problem of harmonic maps. Section 3 consists of a collection of preliminary results. In the last two sections we study stationary maps whose singular set is of codimenion greater than 2.

2. Regularity problem of harmonic maps. In this section we want to make precise the concept of regularity of harmonic maps.

A harmonic map u is by our definition a weak solution to a nonlinear elliptic system in local coordinates. Even in the unconstrained case there is no reason to hope that it always will be continuous (cf. [**Fr**]).

Next we remark that if a harmonic map has small oscillation then it is smooth. This is a well known fact to specialists in this area but had not appeared explicitly in its full scale. It was shown in [**H, W**] that a weakly harmonic map with small oscillation is Hölder continuous. Recently a proof was given in [**Sch**] to show that if a weakly harmonic map u is Hölder continuous then ∇u is locally bounded, i.e., $\nabla u \in L^\infty_{\text{loc}}$. By the harmonic map system, one then gets $\Delta u \in L^\infty_{\text{loc}}$. By the L^p theory of linear elliptic systems one deduces that $u \in L^p_{2,\text{loc}}$ for $p < \infty$. From the harmonic map system we see that $\Delta u \in L^p_{1,\text{loc}}$. Hence, $u \in L^p_{3,\text{loc}}$. Repeating this procedure, we get that $u \in L^p_{k,\text{loc}}$ for $p < \infty$ and $k = 1, 2, 3, \ldots$. By Sobolev imbedding theorem $u \in C^\infty$.

We say that a point $x \in M$ is a *regular point* if there is a neighborhood U of x such that x is Hölder continuous on U. In view of the above remark, we could assume small oscillations in place of Höder continuity as well. Let Ω be the set of all regular points in M. Ω is an open set. Its complement is called the singular set of u, denoted by $\Sigma(u)$. Clearly, $\Sigma(u)$ is a closed set.

3. Preliminary results. As mentioned in the introduction, for stationary maps, we have a monotonicity inequality.

LEMMA (3.1) (*Monotonicity inequality*). *Suppose u is a stationary map from $B(O)$ into $N \subset \mathbf{R}^k$. $B(O)$ is the unit ball in $\mathbf{R}^n$ equipped with a*

Riemannian metric. For $n > 2$ we have for $0 < \sigma < \rho < \operatorname{dist}(x_0, \partial B)$

$$(3.1)\quad e^{C\Lambda\rho_2} \cdot \rho_2^{2-n} \cdot \int_{B_{\rho_2}(x_0)} |\nabla u|^2 \, dV - e^{C\Lambda\rho_1} \cdot \rho_1^{2-n} \cdot \int_{B_{\rho_1}(x_0)} |\nabla u|^2 \, dV$$
$$\geq 2 \int_{B_{\rho_2}(x_0) \setminus B_{\rho_1}(x_0)} e^{C\Lambda r} \cdot |x - x_0|^{2-n} \cdot |\partial_r u|^2 \, dV$$

where Λ and C are constants.

For a proof one can read **[Pr]**. This type of inequality gives some useful information about the map u. In particular, we see that

$$(3.2)\qquad \int_{B(O)} |x - x_0|^{2-n} \cdot |\partial_r u|^2 \, dV \leq C \cdot E(u)$$

where $E(u)$ is the total energy of u.

The basic a priori estimates used in this work were obtained by R. Schoen and K. Uhlenbeck. We state it here as a Lemma.

LEMMA (3.2) **[Sch]**. *Suppose $u \in C^2(B_r^n, N)$ is harmonic with respect to a metric g on B_r^n. Suppose that*

$$\Lambda^{-1} \cdot (\delta_{\alpha\beta}) \leq g_{\alpha\beta} \leq \Lambda \cdot (\delta_{\alpha\beta}),$$
$$|\partial_\nu g_{\alpha\beta}| \leq \Lambda \cdot r^{-1}.$$

There exists $\varepsilon = \varepsilon(\Lambda, n, N) > 0$ such that if

$$r^{2-n} \cdot \int_{B_r} |\nabla u|^2 \, dV \leq \varepsilon$$

then

$$(3.3)\qquad r^2 \cdot \sup_{B_{r_2}} \{|\nabla u|^2\} \leq C \cdot r^{2-n} \int_{B_r} |\nabla u|^2 \, dV.$$

We outline its proof because of its importance in this paper. The proof of this lemma given in **[Sch]** makes use of Lemma 1, noticing that 'C^2 harmonic' implies 'stationary', to construct a scaled version v of u. The map v satisfies

$$|\nabla v|^2(0) = 1, \qquad \sup\{|\nabla v|^2\} \leq 4$$

in a ball B_{r_0} with $r_0 < 1$ if the energy of u is small. Then it follows from the Bochner formula that in B_{r_0}

$$\Delta(|\nabla v|^2) \geq -C \cdot |\nabla v|^2.$$

The conclusion (2.3) is an immediate consequence of the mean value inequality of C. B. Morrey [**M**, 5.3.1].

In our regularity proof, this lemma is used in the following way. Suppose that a point x is away from the singular set $\Sigma(u)$. Apply the Lemma to $B_r(x)$ where $r = \operatorname{dist}(x, \Sigma(u))$. Then

$$r^2 \cdot |\nabla u|^2(x) \leq C \cdot r^{2-n} \int_{B_r} |\nabla u|^2 \, dV.$$

In some cases we can bound

$$C \cdot r^{2-n} \int_{B_r} |\nabla u|^2 \, dV$$

by the total energy $E(u)$. Thus we get an a priori estimate of $|\nabla u|^2$:

$$|\nabla u|^2 \leq \frac{CE(u)}{r^2}. \tag{3.4}$$

LEMMA (3.3). (First variation formula.) *For a smooth family ϕ_t of diffeomorphisms which are the identity near ∂B we let $u_t = u \circ \phi_t$. We then have*

$$\frac{d}{dt} E(u_t)\bigg|_{t=0} = -\int_B \left[|du|^2 \cdot \operatorname{div} X - 2\left\langle du(\nabla_{e_1} X), du(e_i) \right\rangle \right] dV$$

where $X =$ the variation vector field $= (d/dt)\varphi_t|_{t=0}$, e_i, $i = 1, \dots, n$ form an orthonormal basis on B.

This is a standard result. One can prove it by a change of coordinates. We mention the following regularity lemma by C. B. Morrey.

LEMMA (3.4). (C. B. Morrey). *Suppose $0 < \alpha < 1$ and $c < \infty$. If $u \in L_1^2(M, N)$ and*

$$\gamma^{2-n} \int_{B_\gamma[x]} |\nabla u|^2 \, dx \leq c\gamma^{2\alpha} \tag{3.5}$$

for any $x \in B$ and $\gamma \in (0, \frac{1}{4})$, then u is Hölder continuous on B.

4. Higher dimensional singular set. In [**Liao**] it was proved that isolated singular points are removable if the total energy is small. A natural question arises, i.e., what can we say if the apparent singular set has higher dimension? In this and the next sections we will take on this problem and prove our main theorem.

By rescaling and by taking normal coordinates, we can work on a ball B in $\mathbf{R}^n$ with a Riemannian metric g, which is almost Euclidean. We prove the following

PROPOSITION (4.1). *Suppose for any* $K > 0$, *there exist numbers* ε_0, $\sigma \in (0,1)$ *depending only on the metric g, K and N so that if*

(1) $\Sigma \subset$ *the graph of a* $C^{1,\alpha}$ *vector valued function*

$$f\colon \mathbf{B}^n \cap \mathbf{R}^d \to \mathbf{R}^n, \quad n - d > 2, \quad |f|_{1,\alpha} \le K, \tag{4.1}$$

and

(2) $u \in C^\infty(B \setminus \Sigma, N)$ *is a stationary map with* $E(u) \le \varepsilon_0$, *then*

$$\sigma^{2-n} E_\sigma(u) \le \tfrac{1}{2} E(u),$$

where $E_\sigma(u) = \int_{B_\sigma} |\nabla u|^2 \, dv$, B_σ *is the geodesic ball of radius* σ, *where* u_σ *is defined by* $u_\sigma(x) = u(\sigma x)$. *Thus Proposition* (4.1) *asserts that*

$$E(u_\sigma) \le \tfrac{1}{2} E(u). \tag{4.3}$$

This is an energy improving type of inequality.

We prove it by contradiction.

Proof. Assume that the conclusion is false. Then for $i = 1, 2, 3, \ldots$, there is a stationary map

$$u_i \in C^\infty(\mathbf{B}^n \setminus \Sigma_i, N),$$

whose singular set $\Sigma_i \subset$ the graph of a $C^{1,\alpha}$-vector valued function f_i on $\mathbf{R}^d$, $|f_i|_{1,\alpha} \le K$, such that $E(u_i) \le 1/i$ but

$$\sigma^{2-n} E_\sigma(u_i) \ge \tfrac{1}{2} E(u_i). \tag{4.4}$$

Define a scaled map v_i by

$$v_i = (u_i - \bar{u}_i)[E(u_i)]^{-1/2},$$

where $\bar{u}_i$ is the average of u_i. Note that

$$E(v_i) = E(u_i) \cdot E(u_i)^{-1} = 1.$$

By the weak compactness, there is a subsequence (again denoted by v_i) such that (weakly)

$$v_i \to v_\infty \in L_1^2(M, \mathbf{R}^k).$$

Dividing (4.4) by $E(u_i)$, we get

$$\sigma^{2-n} E_\sigma(v_i) \ge \tfrac{1}{2}. \tag{4.5}$$

The harmonic map u_i satisfies elliptic system

$$\Delta u_i = g^{\alpha,\beta} A(\partial_\alpha u_i, \partial_\beta u_i), \tag{4.6}$$

where A is a quadratic form. Dividing (4.6) by $E(u_i)^{1/2}$, we get

$$\Delta v_i = E(u_i)^{1/2} \cdot g^{\alpha\beta} \cdot A(\partial_0 v_i, \partial_\beta v_i).$$

Letting $i \to \infty$, since $E(u_i) \to 0$, we get

$$\Delta v_\infty = 0.$$

By the Weyl Lemma v_∞ is smooth. Our plan is to show

$$E_\sigma(v_i) \to E_\sigma(v_\infty) \quad \text{as } i \to \infty.$$

Let $\eta > 0$ be an arbitrary constant. The $C^{1,\alpha}$ boundedness of f_i enables us to extract a subsequence (again denoted by f_i) so that $f_i \to f$ uniformly in C^1 norm. The limit f is $C^{1,\alpha}$. Let the graph of f be Σ.

$$\Sigma = \{(x', f(x')) \in \mathbf{R}^n : x' \in \mathbf{R}^d \cap B\}.$$

Consider the tube neighborhood

$$\Sigma_\lambda = \{x \in \mathbf{R}^n : \operatorname{dist}(x, \Sigma) \le \lambda\}.$$

Fix $\lambda_0 > 0$ such that for $\lambda \le \lambda_0$

$$\int_{B_\sigma \cap \Sigma_\lambda} |\nabla v_\infty|^2 \le \frac{1}{3} \cdot \eta. \tag{4.7}$$

Cover $\Sigma_\lambda \cap B_\sigma$ by balls centered at points x_j, $j = 1, 2, \ldots, N(\lambda)$, with radii $\mu = C_1 \cdot \lambda$. Because the metric g on B is close to the Euclidean metric, we can arrange so that the number of these balls is bounded. More explicitly

$$N(\lambda) \le C_i \cdot \lambda^{-d}.$$

By the monotonicity inequality

$$\mu^{2-n} \int_{B_\mu(x_j)} |\nabla u_i|^2 \, dV \le C_3 \cdot E(u_i).$$

In terms of v_i we have

$$\int_{B_\mu(x_j)} |\nabla v_i|^2 \, dV \le C_3 \cdot \mu^{n-2}.$$

From these estimates, we get

$$\int_{B_\sigma \cap \Sigma_\lambda} |\nabla v_i|^2 \, dV \le N(\lambda) \cdot \int_{B_\mu(x_j)} |\nabla v_i|^2 \, dV \le C_4 \cdot \lambda^{-d} \cdot \lambda^{n-2}.$$

By assumption $n - d > 2$, we can take $\lambda_1 > 0$ such that

$$\int_{B_\sigma \cap \Sigma_\lambda} |\nabla v_i|^2 \, dV \leq \frac{1}{3}\eta, \tag{4.8}$$

for $i = 1, 2, 3, \ldots$ and all $0 < \lambda \leq \lambda_1$. Fix $\lambda > 0$ such that $\lambda \leq \min(\lambda_0, \lambda_1)$. There is an integer I_1 such that

$$\Sigma_i \cap B_\sigma \subset \Sigma_\lambda \cap B_\sigma$$

if $i \geq I_1$. Using the estimate (3.4), we get

$$|\nabla v_i| \leq C_5$$

on $B_\sigma \setminus \Sigma_\lambda$. In particular, there is a subsequence (again denoted by v_i) such that

$$\int_{B_\sigma \setminus \Sigma_\lambda} |\nabla v_i|^2 \, dV \to \int_{B_\sigma \setminus \Sigma_\lambda} |\nabla v_\infty|^2 \, dV, \quad \text{as } i \to \infty.$$

Thus we can take $I_2 > I_1$ so that for $i \geq I_2$

$$\left| \int \left(|\nabla v_i|^2 - |\nabla v_\infty|^2 \right) dV \right| \leq \frac{1}{3}\eta. \tag{4.9}$$

From inequalities (4.7), (4.8), and (4.9), we get for $i \geq I_2$

$$\left| \int \left(|\nabla v_i|^2 - |\nabla v_\infty|^2 \right) dV \right| \leq \frac{1}{3}\eta + \frac{1}{3}\eta + \frac{1}{3}\eta = \eta.$$

Hence we have

$$E_\sigma(v_i) - E_\sigma(v_\infty) \quad \text{as } i \to \infty.$$

In particular, from (4.5) we get

$$\sigma^{2-n} E_\sigma(v_\infty) \geq \tfrac{1}{2}. \tag{4.10}$$

On the other hand we have from linear theory (cf. **[M]**)

$$\sigma^{2-n} E_\sigma(v_\infty) \leq \sigma^{2-n} \sigma^n \cdot \sup_{B_{1/2}} |v_\infty|^2 \leq \sigma^2 \cdot C_6,$$

where C_6 is a universal constant (depending on n). If we fix σ small such that

$$\sigma^2 < \tfrac{1}{2} C_6^{-1}$$

we would have a contradiction to (4.10). Thus the conclusion of the proposition is true.

5. Proof of the main regularity theorem. We proceed to prove our main result that a stationary map u, whose singular set Σ has codimension at least 2, is regular if its total energy is small.

First let us consider a scaling property. Define

$$u_{\lambda,x_0}: B - \mathbf{R}^k$$

by

$$u_{\lambda,x_0}(x) = u(x_0 - \lambda x)$$

as before. We want to show that condition (1) in (4.1) is preserved under scaling. To see this, let

$$u \in C^\infty(B \setminus \Sigma, N),$$
$$|f|_{1,\alpha} \leq K,$$
$$\Sigma = \{(v,w) \in \mathbf{R}^n: v = f(w) \in \mathbf{R}^{n-d}, w \in \mathbf{R}^d \cap B\}.$$

Let

$$\Sigma_\lambda = \{(v,w) \in \mathbf{R}^n: v = \lambda^{-1} f(\lambda w),\ w \in \mathbf{R}^d \cap B\}.$$

Suppose $x = (w,v) \in$ the singular set of u_λ. Then

$$(\lambda w, \lambda v) \in \Sigma.$$

Thus $\lambda v = f(\lambda w)$. That is $v = \lambda^{-1} f(\lambda w)$. Hence x must belong to Σ_λ. Denoting $\lambda^{-1} \cdot f(\lambda w)$ by $f_\lambda(w)$, we get

$$\nabla f_\lambda(w) = \nabla f(\lambda w),$$

$$|\nabla f_\lambda(w_1) - \nabla f_\lambda(w_2)| = |\nabla f(\lambda w_1) - \nabla f(\lambda w_2)|$$
$$\leq K|\lambda w_1 - \lambda w_2|^\alpha \leq K \cdot |w_1 - w_2|^\alpha$$

for $0 < \lambda < 1$. Thus f_λ is again $C^{1,\alpha}$ and

$$|f_\lambda|_{1,\alpha} \leq K.$$

Next we assume that u satisfies the assumptions of the main theorem. Let $\sigma \in (0, \frac{1}{2})$ be given in Proposition (4.1). We write $u_\sigma = u_{\sigma,x_0}$ for $x_0 \in \Sigma$. Observe that

$$E(u_\sigma) = \sigma^{2-n} \cdot E_\sigma(u) \leq \tfrac{1}{2} E(u).$$

By the scaling property we remarked above, u_σ satisfies all conditions in Proposition (4.1). Hence

$$\sigma^{2-n} E_\sigma(u_\sigma) \leq \tfrac{1}{2} E(u_\sigma).$$

We can write this as

$$\sigma^{2-n} \cdot \sigma^{2-n} \cdot E_{\sigma^2}(u) \leq \tfrac{1}{2} \cdot \tfrac{1}{2} \cdot E(u),$$

i.e.,

$$(\sigma^2)^{2-n} \cdot E_{\sigma^2}(u) \le 2^{-2} \cdot E(u).$$

Observe that one can write this inequality as

$$E(u_{\sigma^2}) \le 2^{-2} \cdot E(u).$$

Again u_{σ^2} satisfies the conditions in Proposition (4.1). Thus

$$\sigma^{2-n} E_\sigma(u_{\sigma^2}) \le \tfrac{1}{2} E(u_{\sigma^2}),$$

i.e.,

$$\sigma^{2-n} \cdot (\sigma^2)^{2-n} \cdot E_{\sigma^3}(u) \le \frac{1}{2} \cdot 2^{-2} \cdot E(u).$$

Hence

$$(\sigma^3)^{2-n} \cdot E_{\sigma^3}(u) \le 2^{-3} \cdot E(u),$$

$$E(u_{\sigma^3}) \le 2^{-3} \cdot E(u).$$

Repeating this procedure, we get for $i = 1, 2, 3, \ldots$

$$E(u_{\sigma^i}) \le 2^{-i} \cdot E(u). \tag{5.1}$$

We claim that there is a $\beta > 0$ such that for any $r \in (0, \sigma)$

$$r^{2-n} E_r(u) \le C \cdot r^\beta \cdot E(u). \tag{5.2}$$

To see that (5.2) holds, take $\beta = \ln 2 / \ln(\sigma^{-1})$ where σ is obtained by Proposition (4.1) and is less than $1/2$. Given any $r \in (0, \sigma)$, there is an integer i such that

$$\sigma^{1+i} \le r \le \sigma^i.$$

We have

$$\begin{aligned}\ln(\sigma^{-1} r^\beta) &= \ln(\sigma^{-1}) + \beta \ln r \ge \ln(\sigma^{-1}) + \frac{\ln 2}{\ln(\sigma^{-1})} \cdot \ln(\sigma^{i+1}) \\ &\ge \ln(\sigma^{-1}) + \ln(2^{-1}) + \ln(2^{-i}) \\ &\ge -\ln(2\sigma) + \ln(2^{-i}) \ge \ln(2^{-i})\end{aligned}$$

since $0 < 2\sigma < 1$. Thus

$$\sigma^{-1} r^\beta \ge 2^{-i}. \tag{5.3}$$

We get from (5.1) that

$$r^{2-n} E_r(u) \le C_1 \cdot (\sigma^i)^{2-n} \cdot E_{\sigma^i}(u) \le C_2 \cdot 2^{-i} E(u).$$

By (5.3), we have

$$r^{2-n} \cdot E_r(u) \le C_2\sigma^{-1} \cdot r^{\beta} \cdot E(u).$$

Since σ is a constant we can take $C = C_2\sigma^{-1}$ and (5.2) is proved.

By a theorem of C. B. Morrey (cf. [**M**]) u is regular on a small ball centered at x_0. Apply this argument to every point of Σ. We then have the desired result that u is regular provided its total energy is less than ε given by the Proposition (4.1). Thus the main theorem has been proved.

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References

[E, L] J. Eells and L. Lemaire, *A report on harmonic maps*, Bull. London Math. Soc., **10** (1978), 1–68.

[F] J. Frehse, *A discontinuous solution to a mildly nonlinear elliptic system*, Math. Z., **134** (1973), 229–230.

[G, R, S, B] W. Barger, S. N. M. Ruijsenaars, E. Seiler and D. Buren, *On finite action solutions of the nonlinear σ-model*, Annals of Physics, **119** (1979), 305–325.

[H, W] S. Hildebrandt and K. O. Widman, *On the Hölder continuity of weak solutions of quasilinear elliptic systems of second order*, Ann. Sci. Norm. Suppisa, **IV** (1977), 145–178.

[Liao] G. Liao, *A regularity theorem for harmonic maps with small energy*, J. Differential Geom., **22** (1985), 233–241.

[M] C. B. Morrey, Jr., *Multiple integrals in the calculus of variations*, (1966), Springer-Verlag, New York,

[P] P. Price, *A monotonicity formula for Yang-Mills fields*, Manuscripta Math., **43** (1983), 131–166.

[Sa.U] J. Sacks and K. Uhlenbeck, *The existence of minimal immersions of 2-spheres*, Ann. of Math., **113** (1981), 1–24.

[Sch] R. Schoen, *Analytic aspects of the harmonic map problem*, Seminar on Nonlinear Partial Differential Equations, ed. S. S. Chern, Springer-Verlag, (1985), 321–358.

[Sch, U] R. Schoen and K. Uhlenbeck, J. Differential Geom., **17** (1982), 307–335.

[Sch.U] _____, *Regularity of minimizing harmonic maps into the spheres*, Invent. Math., **78** (1984), 89–100.

[Kin.Har.Lin] D. Kinderlehrer, R. Hardt and F.-H. Lin, *Existence and partial regularity of static liquid crystal configuration*, Comm. Math. Physics, **105** (1986), 547–570.

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