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FUCHSIAN MODULI ON A RIEMANN SURFACE —ITS POISSON STRUCTURE AND POINCARÉ-LEFSCHETZ DUALITY

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The moduli space of Fuchsian projective connections on a closed Riemann surface admits a Poisson structure. The moduli space of projective monodromy representations on the punctured Riemann surface also admits a Poisson structure which arises from the Poincaré-Lefschetz duality for cohomology. We shall show that the former Poisson structure coincides with the pull-back of the latter by the projective monodromy map. This result explains intrinsically why a Hamiltonian structure arises in the monodromy preserving deformation.

Introduction. It has been known that a Hamiltonian structure arises in the theory of monodromy preserving deformation of meromorphic differential equations. See [KO], [O]. However it has not yet been known why such a Hamiltonian structure does arise. Our result in the present paper will explain the reason clearly and intrinsically. Rather it will be even self-evident from our point of view why such a Hamiltonian structure arises.

In this introduction we shall explain only idea of the present paper. As for rigorous statements written by using precise notation, see the later sections.

Let M be a closed Riemann surface of genus $g \geq 0$. Let m be a positive integer such that $n = m + 3g - 3$ is positive. In the previous paper [I] we constructed a moduli space $\mathscr{E}$ of a certain class of Fuchsian differential equations L on M such that L has m-generic singular points and n-apparent singular points and such that L has *fixed* characteristic exponents at each generic singular point. As for the definition of generic singular point and apparent singular point, see §1. Let B be the configuration space of m-points in M. We have the natural projections $\varpi : \mathscr{E} \to B$ which assigns to each differential equation in $\mathscr{E}$ its generic singular points.

In [I] we also constructed a moduli space of projective monodromy representations of the punctured Riemann surface $M \backslash \{m\text{-points}\}$. More precisely, we constructed a space R together with a projection

$ls\colon R \to B$ with the following property: For each $S \in B$, the fiber R_S of $l_s\colon R \to B$ over S is the moduli space of representations ρ of the fundamental group $\pi_1(M\backslash S)$ into the projective linear group such that the local representation around each point in S induced by ρ is *fixed*. Notice that $ls\colon R \to B$ is a local system whose characteristic homomorphism is given by a natural action of the "braid group" $\pi_1(B)$ on the moduli space R_S of projective representations.

We define the projective monodromy map $PM\colon \mathscr{E} \to R$ which assigns to each differential equation in $\mathscr{E}$ its projective monodromy representation. By definition, the apparent singular points have no effect on the projective monodromy representation. So the projective monodromy map is well-defined. It is known that PM is a local biholomorphism. We have the commutative diagram:

$$\begin{tikzcd} \mathscr{E} \arrow[rr, "PM"] \arrow[dr, "\varpi"'] & & R \arrow[dl, "ls"] \\ & B & \end{tikzcd}$$

A Poisson structure $\{\cdot, \cdot\}$ on the analytic space P is a Lie algebra structure on the structure sheaf $\mathscr{O}_P$ such that, for each germ f at $p \in P$, $\{f, \cdot\}$ acts on the stalk $\mathscr{O}_{P,p}$ at p as a derivation. See, e.g., **[LM]**.

In **[I]** we showed that the moduli space $\mathscr{E}$ of differential equations admits a Poisson structure. On the other hand the moduli space R of projective representations also admits a natural Poisson structure which arises from the Poincaré-Lefschetz duality for cohomology. We shall recall the Poisson structure on $\mathscr{E}$ in §1. We shall describe the Poisson structure on R in §3. Our main theorem in the present paper is the following:

MAIN THEOREM. *The Poisson structure on $\mathscr{E}$ coincides with the pull-back of that of R by the projective monodromy map $PM\colon \mathscr{E} \to R$.*

This main theorem will be restated more rigorously as Theorem 5 in §3 after precise terminology and notation will be introduced in §1–§3. This theorem was conjectured in **[I]**.

The local system structure on $ls\colon R \to B$ induces a foliation $\mathscr{F}_R$ on R which is transverse to each of its fibers. The fundamental 2-form Ω_R associated to the Poisson structure on R is horizontal with respect to $\mathscr{F}_R$. Let $\mathscr{F}_{\mathrm{mp}}$ be the foliation on $\mathscr{E}$ which is the pull-back of $\mathscr{F}_R$ by the local biholomorphism $PM\colon \mathscr{E} \to R$. The monodromy preserving deformation on $\mathscr{E}$ is given by $\mathscr{F}_{\mathrm{mp}}$. So we call $\mathscr{F}_{\mathrm{mp}}$ the monodromy preserving foliation. $\mathscr{F}_{\mathrm{mp}}$ is transverse to each fiber of $\varpi : \mathscr{E} \to B$. By Main Theorem, the fundamental 2-form $\Omega_{\mathscr{E}}$ associated to the Poisson structure on $\mathscr{E}$ coincides with the pull-back of Ω_R by PM. Hence we immediately obtain the following:

COROLLARY. *The monodromy preserving foliation $\mathscr{F}_{\mathrm{mp}}$ is an $\Omega_{\mathscr{E}}$-Lagrangian foliation. Namely we have $L_X\Omega_{\mathscr{E}} = 0$ for any $\mathscr{F}_{\mathrm{mp}}$-horizontal vector field X on $\mathscr{E}$, where L_X denotes the Lie derivative with respect to X.*

Since any local horizontal vector field X is a horizontal lift of a local vector field on B and vice versa, the equation $L_X\Omega_{\mathscr{E}} = 0$ gives us a completely integrable Hamiltonian system with B as the space of independent variables.

1. Moduli of differential equations. Let us recall the notation and results in the previous paper [I] which will be necessary in what follows. For details see [I].

Let M be a closed Riemann surface of genus $g \geq 0$, κ the canonical line bundle over M, ξ a holomorphic line bundle over M with the first Chern class $c_1(\xi) = 1 - g \in H^2(M;\mathbb{Z}) = \mathbb{Z}$. We denote by $\mathscr{M}(\xi)$ the sheaf of meromorphic sections of ξ. Then there exist differential operators $L: \mathscr{M}(\xi) \to \mathscr{M}(\xi \otimes \kappa^{\otimes 2})$ such that the following condition holds: In terms of a local coordinate x of M and a local trivialization of ξ at any point of M, L is represented by

$$L = -\left(\frac{d}{dx}\right)^2 + Q, \tag{1}$$

where Q is a locally defined meromorphic function. Geometrically speaking, these differential operators are identified with meromorphic projective connections on M.

The Riemann surface M admits a projective structure subordinate to its complex structure, i.e., it admits a complex coordinate system all of whose transition functions are projective linear transformations (e.g., **[G1]**). Fix a projective structure on M; then $Q(dx)^{\otimes 2}$ becomes a meromorphic quadratic differential globally defined on M. We denote this quadratic differential also by Q. This abuse of notation will cause no confusion. Hereafter we identify a meromorphic differential operator L with the corresponding meromorphic quadratic differential Q.

In this paper we assume that all differential operators are of the form (1) and of *Fuchsian type*. Consider a differential operator L

and let S be the set of singular points of L. By assumption all singular points are regular singular. Since the solution sheaf of the differential operator L is a local system over $M\backslash S$, it determines the *linear monodromy represenation* $\rho_L\colon \pi_1(M\backslash S) \to \mathrm{SL}_2(\mathbb{C})$ of the fundamental group of $M\backslash S$ up to conjugacy.

A regular singular point $p \in S$ is said to be *generic* if the difference of charactersitic exponents of L at p is different from integers. A regular singular point $q \in S$ is said to be *apparent* if the local circuit matrix at q induced by ρ_L is in the center $\{\pm I\}$ of $\mathrm{SL}_2(\mathbb{C})$. An apparent singular point is non-generic. The non-generic singular points are divided into two categories; one is the *logarithmic* singular points and the other is the *apparent* singular points. In this paper we assume that all singular points are either generic or apparent. We shall not consider logarithmic singular points.

Let S_{ge} be the set of generic singular points of L, S_{ap} the set of apparent singular points of L: $S = S_{\mathrm{ge}} \cup S_{\mathrm{ap}}$. Passing to the quotient $\mathrm{SL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C})/\{\pm I\}$, the linear monodromy representation ρ_L induces the *projective monodromy representation* $\rho_P\colon \pi_1(M\backslash S_{\mathrm{ge}}) \to \mathrm{PSL}_2(\mathbb{C})$. Remark that the apparent singular points S_{ap} have no effect on the projective monodromy representation. This is the reason why we call these singular points apparent singular points.

At an apparent singular point the difference of characteristic exponents takes an integer not less than 2. An apparent singular point is said to be *of ground state* if the difference of characteristic exponents takes the minimal possible value 2. In this paper we assume that all apparent singular points are of ground state.

Given a differential operator L, let m be the number of generic singular points of L, n the number of apparent singular points of L.

Assumption (A). We assume that

$$n = m + 3g - 3 > 0,$$

namely n is the moduli number of Riemann surfaces of genus g with m punctures.

The assumption $n > 0$ implies that the universal covering space of the punctured Riemann surface $M\backslash S_{\mathrm{ge}}$ is the unit disk $\{z \in \mathbb{C};\, |z| < 1\}$.

From now on, we shall consider differential operators marked by *ordering* of their singular points.

Given $\theta = (\theta_1, \dots, \theta_m) \in (\mathbb{C}\backslash\mathbb{Z})^m$, let $E(m; \theta)$ be the set of differential operators L with $m+n$ *ordered* regular singular points such that the following two conditions hold:

(i) For $j = 1, \dots, m$, the difference of characteristic exponents of L at the jth singular point is θ_j.

(ii) The last n-singular points of L are apparent and of ground state.

The first condition implies that the first m-singular points are generic.

Let $B(l)$ be the set of mutually distinct ordered l-points in M. Then we have the natural map $\pi\colon E(m; \theta) \to B(m+n)$ which assigns to each element of $E(m; \theta)$ its ordered singular points in $B(m+n)$. Let $\mathfrak{p}\colon B(m+n) \to B(m)$ be the projection into the first m-components, $\varpi := \mathfrak{p} \circ \pi$ the composition of $\mathfrak{p}$ and π. Then we have the following commutative diagram:

$$\begin{tikzcd} & E(m;\theta) \arrow[dl, "\pi"'] \arrow[d, "\varpi"] \\ B(m+n) \arrow[r, "\mathfrak{p}"'] & B(m) \end{tikzcd} \tag{2}$$

$B(m)$ and $B(m+n)$ are naturally complex manifolds of dimension m and $m+n$, respectively.

THEOREM 1 [I]. *E admits a natural analytic space structure of pure dimension $m+2n$ such that $\pi\colon E(m; \theta) \to B(m+n)$ is a holomorphic map. All maps in the diagram* (2) *are surjective. Each ϖ-fiber is an analytic subspace of $E(m; \theta)$ of pure dimension $2n$.*

We denote by $E(\mathbf{p}; \theta)$ the ϖ-fiber of $E(m; \theta)$ over $\mathbf{p} \in B(m)$.

Given $\mathbf{r} = (p_1, \dots, p_m, q_1, \dots, q_n) \in B(m+n)$, let $\xi_\mathbf{r}$ be the holomorphic line bundle over M defined by

$$\xi_\mathbf{r} := \kappa^{\otimes 2} \otimes [p_1 + \cdots + p_m - (q_1 + \cdots + q_n)].$$

Remark that Assumption (A) and the Riemann-Roch formula imply the Fredholm alternative:

$$\dim H^0(M; \mathscr{O}(\xi_\mathbf{r})) = \dim H^1(M; \mathscr{O}(\xi_\mathbf{r})).$$

Let $X(m)$ be the subset of $B(m+n)$ consisting of all $\mathbf{r} \in B(m+n)$ such that $\dim H^0(M; \mathscr{O}(\xi_\mathbf{r})) = 0$. Then $X(m)$ is a nonempty Zariski open subset of $B(m+n)$ such that the restriction of the projection $\mathfrak{p}\colon B(m+n) \to B(m)$ to $X(m)$ is surjective. Let $\mathscr{E}(m;\theta) := \pi^{-1}(X(m)) \subset E(m;\theta)$. Then, instead of (2), we have the following commutative diagram:

$$
\begin{tikzcd}
 & \mathscr{E}(m;\theta) \arrow[dl, "\pi"'] \arrow[d, "\varpi"] \\
X(m) \arrow[r, "\mathfrak{p}"'] & B(m)
\end{tikzcd}
\tag{3}
$$

THEOREM 2 [I]. *The open analytic subspace* $\mathscr{E}(m;\theta)$ *of* $E(m;\theta)$ *is smooth. All maps in the diagram* (3) *are surjective. Each* ϖ*-fiber is a complex submanifold of* $\mathscr{E}(m;\theta)$.

We denote by $\mathscr{E}(\mathbf{p};\theta)$ the ϖ-fiber of $\mathscr{E}(m;\theta)$ over $\mathbf{p} \in B(m)$. A differential operator is said to be *reducible* if it is "decomposable into a product of two first order differential operators." Otherwise it is said to be *irreducible.* For a precise definition of (ir)reducibility, see [I]. It is (ir)reducible if and only if its monodromy representation is (ir)reducible. Given a subset D of $E(m;\theta)$, we denote by D_{irr} the subset of D consisting of all irreducible elements of D. Let $D = E(m;\theta), E(\mathbf{p};\theta), \mathscr{E}(m;\theta)$ or $\mathscr{E}(\mathbf{p};\theta)$. Then D_{irr} is a nonempty Zariski open subset of D. Moreover $D_{\mathrm{irr}} = D$ for θ in a nonempty Zariski open subset of $(\mathbb{C}\backslash\mathbb{Z})^m$

REMARK 3. All statements of Theorem 1 and Theorem 2 are still valid even if $E(m;\theta)$ and $\mathscr{E}(m;\theta)$ are replaced by $E(m;\theta)_{\mathrm{irr}}$ and $\mathscr{E}(m;\theta)_{\mathrm{irr}}$, respectively.

We shall discuss a local coordinate system of the complex manifold $\mathscr{E}(m;\theta)$. Fix a projective structure on M subordinate to its complex structure. Let $\mathbf{r}^* = (p_1^*, \dots, p_m^*, q_1^*, \dots, q_n^*)$ be any point in $X(m)$. Hereafter we assume that the suffixes i and j run through $1, \dots, m$ and $1, \dots, n$, respectively. Let (U_i, x_i) and (V_j, y_j) be sufficiently small projective coordinate charts of M centered at p_i^* and q_j^*, respectively. "Sufficiently small" means that these charts are disjoint. Then $W := \prod_{i=1}^m U_i \times \prod_{j=1}^n V_j$ is a product neighbourhood of $\mathbf{r}^*$ in $X(m)$. We shall give a local coordinate of $\mathscr{E}(m;\theta)$ in $\mathscr{E}(m;\theta)|W := \pi^{-1}(W)$.

Let L be any element of $\mathscr{E}(m;\theta)|W$ and set $\pi(L) = (p_1, \dots, p_m, q_1, \dots, q_n) \in W$. We set $t_i = x_i(p_i)$ and $\lambda_j = y_j(q_j)$. Then the meromorphic quadratic differential Q corresponding to L admits the following Laurent expansion at each singular point:

$$Q = \left\{ \frac{\theta_i^2 - 1}{4(x_1 - t_i)^2} + \frac{H_i}{x_i - t_i} + O(1) \right\} (dx_i)^{\otimes 2} \quad \text{at } p_i,$$

$$Q = \left\{ \frac{3}{4(y_j - \lambda_j)^2} - \frac{\mu_j}{y_j - \lambda_j} + \mu_j^2 + O(y_j - \lambda_j) \right\} (dy_j)^{\otimes 2} \quad \text{at } q_j.$$

It was shown in [I] that (t_i, λ_j, μ_j) is a local coordinate of $\mathscr{E}(m; \theta)$ in $\mathscr{E}(m; \theta)|W$.

Consider the closed 2-form Ω in $\mathscr{E}(m; \theta)|W$ defined by

$$\Omega := \sum_{j=1}^{n} \delta\mu_j \wedge \delta\lambda_j - \sum_{i=1}^{m} \delta H_i \wedge \delta t_i, \tag{4}$$

where δ denotes the exterior differential on $\mathscr{E}(m; \theta)$. The following theorem is fundamental in the previous paper.

THEOREM 4 [I]. *Ω is a global closed 2-form on $\mathscr{E}(m; \theta)$ and defines a Poisson structure on $\mathscr{E}(m; \theta)$.*

The purpose of the present paper is to give an intrinsic description of this Poisson structure in terms of the Poincaré-Lefschetz duality for cohomology.

2. Moduli of monodromy representations. Given a topological space X, let ΠX be the fundamental groupoid of X. Given a Lie group G, let $\mathrm{Hom}(\Pi X, G)$ be the set of all groupoid homomorphisms of ΠX into G, $\mathrm{Map}(X, G)$ the set of all maps of X into G. $\mathrm{Map}(X, G)$ acts on $\mathrm{Hom}(\Pi X, G)$; if $\varpi \in \mathrm{Map}(X, G)$ and $\rho \in \mathrm{Hom}(X, G)$, then $\varpi \cdot \rho$, defined by $(\varpi \cdot \rho)(\gamma) := \varpi(q)\rho(\gamma)\varpi(p)^{-1}$ for $\gamma \in \Pi X$ with initial point $p \in X$ and terminal point $q \in X$, is an element of $\mathrm{Hom}(X, G)$. The correspondence $\rho \mapsto \varpi \cdot \rho$ defines a left action of ϖ on $\mathrm{Hom}(X, G)$. We denote by $R_G(X)$ the quotient set $\mathrm{Map}(X, G)\backslash\mathrm{Hom}(\Pi X, G)$ with respect to this action. A continuous map $f: X \to Y$ induces a groupoid homomorphism $\Pi f: \Pi X \to \Pi Y$ and this in turn induces a map $R_G(f): R_G(Y) \to R_G(X)$. $R_G(\cdot)$ is a contravariant functor of the category of homotopy equivalence classes of topological spaces into the category of sets.

Given a topological space with base point (X, p), let $\mathrm{Hom}(\pi_1(X, p), G)$ be the set of all group homomorphisms of the fundamental group $\pi_1(X, p)$ into G. The G acts on $\mathrm{Hom}(\pi_1(X, p), G)$; if $g \in G$ and $\rho \in \mathrm{Hom}(\pi_1(X, p), G)$, then $g \cdot \rho$, defined by $(g \cdot \rho)(\gamma) := g\rho(\gamma)g^{-1}$ for $\gamma \in \pi_1(X, p)$, is an element of

$\mathrm{Hom}(\pi_1(X, p), G)$. The correspondence $\rho \mapsto g \cdot \rho$ defines a left action of g on $\mathrm{Hom}(\pi_1(X, p), G)$. We denote by $R_G(X, p)$ the quotient set $G\backslash\mathrm{Hom}(\pi_1(X, p), G)$ with respect to this action. Since $\pi_1(X, p)$ is a subset of ΠX, there is a natural restriction map $\mathfrak{r}_p\colon \mathrm{Hom}(\Pi X, G) \to \mathrm{Hom}(\pi_1(X, p), G)$. This map $\mathfrak{r}_p$ is compatible with the actions described above and hence defines the natural restriction map

$$\mathfrak{r}_p\colon R_G(X) \to R_G(X, p). \tag{5}$$

One observes that this map is bijective. If G is a linear group or projective linear group, then $R_G(X, p)$ is the set of all linear or projective linear representation classes of the fundamental group $\pi_1(X, p)$. In this case let $R_G(X, p)_{\mathrm{irr}}$ be the subset of $R_G(X, p)$ consisting of all irreducible representation classes, $R_G(X)_{\mathrm{irr}}$ the subset of $R_G(X)$ corresponding to $R_G(X, p)_{\mathrm{irr}}$ by the bijection (5).

Given $\mathbf{p} = (p_1, \dots, p_m) \in B(m)$, let $M_\mathbf{p}$ be the punctured Riemann surface $M\backslash\{p_1, \dots, p_m\}$. Let $\mathrm{q}\colon \mathfrak{M} \to B(m)$ be the universal family of punctured Riemann surfaces, i.e., $\mathfrak{M} := \{(p, \mathbf{p}) \in M \times B(m); p \in M_\mathbf{p}\}$, q being the projection into the second component. Hereafter we identify $M_\mathbf{p}$ with $M_\mathbf{p} \times \{\mathbf{p}\} \subset \mathfrak{M}$. Let $\mathrm{q}'\colon \mathfrak{M}^* \to M \times B(m)$ be the real blow-up of $M \times B(m)$ along the locus $(M \times B(m))\backslash\mathfrak{M}$ and set $M_\mathbf{p}^* := \mathrm{q}'^{-1}(M \times \{\mathbf{p}\})$. We denote by $\mathrm{q}^*\colon \mathfrak{M}^* \to B(m)$ the natural projection. $M_\mathbf{p}^*$ is homeomorphic to the compact surface with boundary obtained from M by deleting small open disks centered at p_i's, q' maps the interior $\mathrm{Int}\, M_\mathbf{p}^*$ of $M_\mathbf{p}^*$ onto $M_\mathbf{p}$ homeomorphically. The boundary $\partial M_\mathbf{p} = \mathrm{q}^{*-1}(\{p_1, \dots, p_m\} \times \{\mathbf{p}\})$ of $M_\mathbf{p}^*$ is a disjoint union of m-copies of the unit circle S^1. Let $h\colon M_\mathbf{p} \to M_\mathbf{p}^*$ be the composition of the inverse map of $\mathrm{q}'\colon \mathrm{Int}\, M_\mathbf{p}^* \to M_\mathbf{p}$ with the inclusion $\mathrm{Int}\, M_\mathbf{p}^* \hookrightarrow M_\mathbf{p}^*$. Then h gives a homotopy equivalence. Hence we have the natural bijection

$$R_G(h)\colon R_G(M_\mathbf{p}^*) \to R_G(M_\mathbf{p}). \tag{6}$$

For any $\mathbf{q} = (q_1, \dots, q_m) \in B(m)$, let $U = U_1 \times \cdots \times U_m$ be a product open neighbourhood of $\mathbf{q}$ in B. Suppose that each U_i is simply connected and sufficiently small. Then there exists a differentiable map $\Phi\colon M \times U \to M$ such that (i) $\Phi(\cdot, \mathbf{q})\colon M \to M$ is the identity and (ii) for each point $\mathbf{p} = (p_1, \dots, p_m) \in U$, $\Phi_\mathbf{p} := \Phi(\cdot, \mathbf{p})$ maps M onto itself diffeomorphically and takes q_i to p_i. This map gives a local trivialization $\Phi\colon M_\mathbf{q}^* \times U \to \mathrm{q}^{*-1}(U)$ of $\mathfrak{M}^*$. For each $\mathbf{p} \in U$, we set $\Phi_\mathbf{p} := \Phi(\cdot, \mathbf{p})$. $\Phi_\mathbf{p}$ maps $(M_\mathbf{q}^*, \partial M_\mathbf{q}^*)$ to $(M_\mathbf{p}^*, \partial M_\mathbf{p}^*)$

diffeomorphically. Let $\imath\colon \partial M_{\mathbf{p}}^* \hookrightarrow M_{\mathbf{p}}^*$ be the inclusion map. Then we have the following commutative diagram for each $\mathbf{p} \in U$:

$$\begin{tikzcd} R_G(M_{\mathbf{p}}^*) \arrow[r, "R_G(\Phi_{\mathbf{p}})"] \arrow[d, "R_G(\imath)"'] & R_G(M_{\mathbf{q}}^*) \arrow[d, "R_G(\imath)"] \\ G_G(\partial M_{\mathbf{p}}^*) \arrow[r, "R_G(\Phi_{\mathbf{p}})"'] & R_G(\partial M_{\mathbf{q}}^*) \end{tikzcd}$$

The horizontal arrows in this diagram are bijective and independent of the choice of Φ described above. From this observation one obtains the following commutative diagram for each $\gamma \in \Pi B(m)$ with initial point $\mathbf{q}$ and terminal point $\mathbf{p}$:

$$\begin{tikzcd} R_G(M_{\mathbf{p}}^*) \arrow[r, "S(\gamma)"] \arrow[d, "R_G(\imath)"'] & R_G(M_{\mathbf{q}}^*) \arrow[d, "R_G(\imath)"] \\ R_G(\partial M_{\mathbf{p}}^*) \arrow[r, "S(\gamma)"'] & R_G(\partial M_{\mathbf{q}}^*) \end{tikzcd} \tag{7}$$

This commutative diagram gives a covariant functor of the fundamental groupoid $\Pi B(m)$ of $B(m)$ into the category of maps of sets and determines a local system over $B(m)$.

Hereafter we assume that G is the projective linear group $\mathrm{PSL}_2(\mathbb{C})$.

Let p be a point in $M_{\mathbf{p}}$ and consider the fundamental group $\pi_1(M_{\mathbf{p}}, p)$ of $M_{\mathbf{p}}$. We regard $\pi_1(M_{\mathbf{p}}, p)$ as a discrete group. We equip $\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)$ with the compact-open topology and $R_G(M_{\mathbf{p}}, p) = G\backslash\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)$ with the quotient topology. Its subspace $R_G(M_{\mathbf{p}}, p)_{\mathrm{irr}}$ carries a natural structure of complex manifold of dimension $2n$, which we shall describe below.

The fundamental group $\pi_1(M_{\mathbf{p}}, p)$ has a generator

$$\{\alpha_1, \dots, \alpha_g, \beta_1, \dots, \beta_g, \gamma_1, \dots, \gamma_m\}$$

subjecting one relation $[\alpha_1, \beta_1]\cdots[\alpha_g, \beta_g]\gamma_1\cdots\gamma_m = 1$, where $[\alpha, \beta]$ is the commutator of α and β. One observes that $\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)$ is homeomorphic to the subvariety of G^{2g+m} defined by the equation $[A_1, B_1]\cdots[A_g, B_g]C_1\cdots C_m = I$ for $(A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_m) \in G^{2g+m}$. This subvariety is evidently smooth. Hence we can regard $\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)$ as a complex manifold. By Shur's lemma, G acts on $\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)_{\mathrm{irr}}$ freely. One can show as in Theorem 27 **[G2]** that the quotient space

$$R_G(M_{\mathbf{p}}, p)_{\mathrm{irr}} = G\backslash\mathrm{Hom}(\pi_1(M_{\mathbf{p}}, p), G)_{\mathrm{irr}}$$

has a natural structure of complex manifold. It is independent of the choice of the generator of $\pi_1(M_{\mathbf{p}}, p)$.

Since $\mathfrak{r}_p$ in (5) maps $R_G(M_{\mathbf{p}})_{\mathrm{irr}}$ to $R_G(M_{\mathbf{p}}, p)_{\mathrm{irr}}$ bijectively, it induces a structure of complex manifold on $R_G(M_{\mathbf{p}})_{\mathrm{irr}}$. Moreover the bijection (6) induces a structure of complex manifold on $R_G(M_{\mathbf{p}}^*)_{\mathrm{irr}}$ from that on $R_G(M_{\mathbf{p}})_{\mathrm{irr}}$. It is easy to see that this complex structure is independent of the choice of the base point p and hence canonical.

We remark here that $R_G(S^1)$ is the space of all conjugacy classes of the group $G = \mathrm{PSL}_2(\mathbb{C})$ equipped with the natural quotient topology induced from that of G. This space is described as follows: Let $\mathfrak{C}$ be the space obtained by attaching a single point named 1^* to the Gauss plane $\mathbb{C}$ such that open neighbourhoods of 1^* in $\mathfrak{C}$ are of the form: $\{1^*\} \cup U$, where U is any open neighbourhood of 1 in $\mathbb{C}$. Then $R_G(S^1)$ is homeomorphic to $\mathfrak{C}$. A homeomorphism $f\colon R_G(S^1) \to \mathfrak{C}$ is given as follows: For any conjugacy class c in G, if c is the class consisting of the unit element then we set $f(c) = 1^*$; otherwise $f(c) = \frac{1}{2}\mathrm{trace}\, A^2$, where A is any representative of c. We identify $R_G(S^1)$ with $\mathfrak{C}$.

Since $\partial M_{\mathbf{p}}^*$ is the disjoint union of m-copies of S^1, $R_G(\partial M_{\mathbf{p}}^*)$ is naturally bijective to $R_G(S^1)^m = \mathfrak{C}^m$. Now recall that there is the restriction map $R_G(\iota)\colon R_G(M_{\mathbf{p}}^*)_{\mathrm{irr}} \to R_G(\partial M_{\mathbf{p}}^*) = \mathfrak{C}^m$. Given $\theta = (\theta_1, \dots, \theta_m) \in (\mathbb{C}\backslash\mathbb{Z})^m$, let $R_G(M_{\mathbf{p}}^*, \theta)_{\mathrm{irr}}$ be the inverse image of $(\cos 2\pi\theta_1, \dots, \cos 2\pi\theta_m) \in \mathfrak{C}^m$ by the above restriction map. Then one can show that $R_G(M_{\mathbf{p}}^*; \theta)_{\mathrm{irr}}$ is a complex submanifold of $R_G(M_{\mathbf{p}}^*)_{\mathrm{irr}}$ of dimension $2n$. The commutative diagram (6) induces a biholomorphism $S(\gamma)\colon R_G(M_{\mathbf{p}}^*; \theta) \to R_G(M_{\mathbf{q}}^*; \theta)$ for each $\gamma \in \Pi B$ with initial point $\mathbf{q}$ and terminal point $\mathbf{p}$, which gives $\bigsqcup_{\mathbf{p}\in B} R_G(M_{\mathbf{p}}^*; \theta)_{\mathrm{irr}}$ a structure of local systems over $B(m)$ with values in the category of complex manifolds.

Hereafter we set

$$R(\mathbf{p})_{\mathrm{irr}} := R_G(M_{\mathbf{p}}^*)_{\mathrm{irr}} \quad \text{and} \quad R(\mathbf{p}; \theta)_{\mathrm{irr}} := R_G(M_{\mathbf{p}}^*; \theta)_{\mathrm{irr}}$$

for simplicity. Moreover we set $R(m; \theta)_{\mathrm{irr}} := \bigsqcup_{\mathbf{p}\in B(m)} R(\mathbf{p}, \theta)_{\mathrm{irr}}$. Since $R(m; \theta)_{\mathrm{irr}} \to B(m)$ is a local system as mentioned above and each of its fibers $R(\mathbf{p}; \theta)_{\mathrm{irr}}$ is a complex manifold of dimension $2n$, $R(m; \theta)_{\mathrm{irr}}$ is naturally a complex manifold of dimension $m + 2n$.

Now recall the moduli space $\mathscr{E}(m; \theta)_{\mathrm{irr}}$ of Fuchsian differential equations on M defined in §1. Let $PM\colon \mathscr{E}(m; \theta)_{\mathrm{irr}} \to R(m; \theta)_{\mathrm{irr}}$ be the *projective monodromy map* which assigns to each differential equation $L \in \mathscr{E}(m; \theta)_{\mathrm{irr}}$ its projective monodromy representation class. PM is a holomorphic map and it takes each $\mathscr{E}(\mathbf{p};\theta)_{\rm irr}$ into $R(\mathbf{p};\theta)_{\rm irr}$. We obtain a commutative diagram:

$$\begin{tikzcd}
 & \mathscr{E}(m;\theta)_{\rm irr} \arrow[r, "PM"] \arrow[dl, "\pi"'] \arrow[d, "\varpi"] & R(m;\theta)_{\rm irr} \arrow[dl, "ls"] \\
X(m) \arrow[r, "\mathfrak{p}"'] & B(m) &
\end{tikzcd} \tag{8}$$

The projective monodromy map $PM\colon \mathscr{E}(m;\theta)_{\rm irr}\to R(m;\theta)_{\rm irr}$ is locally biholomorphic [I].

3. Poisson structure and the Poincaré-Lefschetz duality. The moduli space $R(m;\theta)_{\rm irr}$ of monodromy representations admits a canonical Poisson structure which arises from the Poincaré-Lefschetz duality for cohomology. To describe this Poisson structure, we have to give a cohomological description of the tangent space $T_\rho R(\mathbf{p};\theta)_{\rm irr}$ at each point $\rho\in R(\mathbf{p};\theta)$.

In this section we denote by X the space $M_{\mathbf{p}}^*$ for simplicity of notation. Let P_ρ be the flat principal G-bundle over X associated to the representation ρ. The Lie group G admits the so-called adjoint action Ad on its Lie algebra $\mathfrak{g}$. Let L_ρ be the flat $\mathfrak{g}$-bundle over X associated to P_ρ with respect to the adjoint action. Let us consider the cohomology of the pair $(X,\partial X)$ of topological spaces with coefficients in the local system L_ρ. The cohomology exact sequence for $(X,\partial X;L_\rho)$ is given as follows:

$$\begin{aligned} 0= \ & H^0(X;L_\rho) \xrightarrow{j^*} H^0(\partial X;L_\rho) \\ \xrightarrow{\delta^*} H^1(X,\partial X;L_\rho) \xrightarrow{i^*} \ & H^1(X;L_\rho) \xrightarrow{j^*} H^1(\partial X;L_\rho) \\ \xrightarrow{\delta^*} H^2(X,\partial X;L_\rho) = \ & 0. \end{aligned} \tag{9}$$

Here we obtain $H^0(X;L_\rho)=0$ from the irreducibility of the representation $\rho\in R(\mathbf{p};\theta)_{\rm irr}$ and Shur's lemma in the representation theory. We obtain $H^2(X,\partial X;L_\rho)=0$ from $H^0(X;L_\rho)=0$ and the Poincaré-Lefschetz duality which will be stated below.

We now state the Poincaré-Lefschetz duality. Since L_ρ is a flat $\mathfrak{g}$-bundle, the Killing form $B\colon \mathfrak{g}\otimes\mathfrak{g}\to\mathbb{C}$ on $\mathfrak{g}$ induces a bilinear morphism $B\colon L_\rho\otimes L_\rho\to\mathbb{C}_X$, where $\mathbb{C}_X$ is the constant system over X with fiber $\mathbb{C}$. This induces a $\mathbb{C}$-linear map

$$B^*\colon H^2(X,\partial X;L_\rho\otimes L_\rho)\to H^2(X,\partial X;\mathbb{C}_X)=\mathbb{C}.$$

On the other hand, the cup product gives a bilinear form

$$H^i(X;L_\rho)\otimes H^{2-i}(X,\partial X;L_\rho)\to H^2(X,\partial X;L_\rho\otimes L_\rho).$$

Composing the bilinear form with B^*, we obtain the Poincaré-Lefschetz bilinear form:

$$H^i(X; L_\rho) \otimes H^{2-i}(X, \partial X; L_\rho) \to \mathbb{C}. \tag{10}$$

The Poincaré-Lefschetz duality theorem asserts that this bilinear form is a perfect pairing for $i = 0, 1, 2$.

Let $\imath: \partial X \hookrightarrow X$ be the inclusion map, $r = R_G(\imath): R_G(X)_{\mathrm{irr}} \to R_G(\partial X)$ the associated restriction map. It follows from the deformation theory that the tangent space $T_\rho R_G(X)_{\mathrm{irr}}$ of $R_G(X)_{\mathrm{irr}}$ at ρ is identified with the first cohomology $H^1(X; L_\rho)$. Similarly the tangent space $T_{r(\rho)} R_G(\partial X)$ is identified with $H^1(\partial X; L_\rho)$. With these identifications, the differential map $(dr)_\rho: T_\rho R_G(X)_{\mathrm{irr}} \to T_{r(\rho)} R(\partial X)$ of $r: R_G(X)_{\mathrm{irr}} \to R_G(\partial X)$ at ρ coincides with the homomorphism $j^*: H^1(X; L_\rho) \to H^1(\partial X; L_\rho)$. Since $R(\mathbf{p}; \theta)_{\mathrm{irr}}$ is the fiber of r through ρ, the tangent space $T_\rho R(\mathbf{p}; \theta)_{\mathrm{irr}}$ is identified with the kernel of $j^*: H^1(X; L_\rho) \to H^1(\partial X; L_\rho)$. The cohomology exact sequence (9) implies that the homomorphism $i^*: H^1(X, \partial X; L_\rho) \to H^1(X; L_\rho)$ induces an isomorphism

$$\begin{aligned} i^*: &H^1(X, \partial X; L_\rho)/\delta^* H^0(\partial X; L_\rho) \\ &\to \ker[j^*: H^1(X; L_\rho) \to H^1(\partial X; L_\rho)]. \end{aligned}$$

Hence we obtain the following cohomological description of the tangent space $T_\rho R(\mathbf{p}; \theta)_{\mathrm{irr}}$:

$$\begin{aligned} T_\rho R(\mathbf{p}; \theta)_{\mathrm{irr}} &= \ker[H^1(X; L_\rho) \xrightarrow{j^*} H^1(\partial X; L_\rho)] \\ &\underset{i^*}{\cong} \frac{H^1(X, \partial X; L_\rho)}{\delta^* H^0(\partial X; L_\rho)}. \end{aligned} \tag{11}$$

The subspace $\delta^* H^0(\partial X; L_\rho)$ of $H^1(X, \partial X; L_\rho)$ is the orthogonal complement of $\ker[j^*: H^1(X; L_\rho) \to H^1(\partial X; L_\rho)]$ with respect to the Poincaré-Lefschetz bilinear form (10) for $i = 1$. Hence, in view of (11), the Poincaré-Lefschetz bilinear form (10) for $i = 1$ induces a nondegenerate bilinear form on the tangent space $T_\rho R(\mathbf{p}; \theta)$. This bilinear form is skew-symmetric. Thus we have obtained an almost symplectic structure on the complex manifold $R(\mathbf{p}; \theta)$. One can show that this almost symplectic structure is integrable. Hence it is in fact a symplectic structure. We shall not prove the integrability here, because it will be simultaneously established in the course of the proof of our main theorem.

We have seen that $R(m; \theta)_{\mathrm{irr}} \to B(m)$ is a local system and each fiber $R(\mathbf{p}; \theta)_{\mathrm{irr}}$ of this local system is a symplectic manifold. There uniquely exists a Poisson structure on $R(m;\theta)_{\mathrm{irr}}$ such that every $R(\mathbf{p};\theta)_{\mathrm{irr}}$, $\mathbf{p}\in B(m)$, are symplectic leaves. We call it *the canonical Poisson structure* on $R(m;\theta)_{\mathrm{irr}}$.

Our main theorem in this paper is the following:

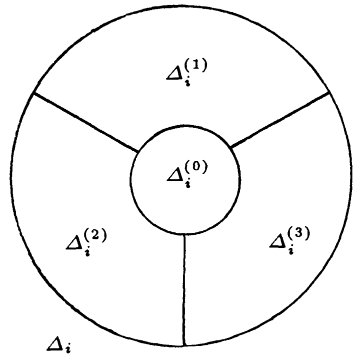
THEOREM 5 (MAIN THEOREM). *The Poisson structure on $\mathscr{E}(m;\theta)_{\mathrm{irr}}$ described in Theorem 4 coincides with the pull-back of the canonical Poisson structure on $R(m;\theta)_{\mathrm{irr}}$ by the projective monodromy map $PM\colon \mathscr{E}(m;\theta)_{\mathrm{irr}}\to R(m;\theta)_{\mathrm{irr}}$.*

4. Tangent map of the projective monodromy map. Let L be a differential operator in $\mathscr{E}(m;\theta)_{\mathrm{irr}}$. As before we always identify L with the corresponding meromorphic quadratic differential Q. Let $\rho = PM(Q)$ be the projective monodromy representation of Q. In this section we shall consider the tangent map $dPM\colon T_Q\mathscr{E}(m;\theta)_{\mathrm{irr}}\to T_\rho R(m;\theta)_{\mathrm{irr}}$ of the projective monodromy map $PM\colon \mathscr{E}(m;\theta)_{\mathrm{irr}}\to R(m;\theta)_{\mathrm{irr}}$ at Q. To give a cohomological description of this tangent map is the first step toward the proof of Theorem 5.

Let $\Delta(M)=\{\Delta_k\,;\,k\in K\}$ be a cell decomposition of the Riemann surface M, where Δ_k are closed 2-cells with piecewise smooth boundary. We provide each 2-cell Δ_k with the orientation induced from that of the Riemann surface M. Assume that this cell decomposition is sufficiently fine, so that there exists a projective coordinate system $\mathscr{U}=\{(U_k\,,\,x_k)\,;\,k\in K\}$ such that $\Delta_k\subset U_k$ for $k\in K$. Moreover we assume that the index set K contains $\{1\,,2\,,\dots\,,m+n\}$. For $i=1\,,\dots\,,m$, we subdivide the cell Δ_i into four smaller cells $\Delta_i^{(a)}$, $a=0\,,1\,,2\,,3$, in a manner indicated in Figure 1 (see next page).

Put $X=M\setminus\bigcup_{i=1}^m \operatorname{Int}\Delta_i^{(0)}$. Then $\Delta(X)=\{\Delta_i^{(a)}\,;\,i=1\,,\dots\,,m\,,a=1\,,2\,,3\}\cup\{\Delta_k\,;\,k\in K\setminus\{1\,,\dots\,,m\}\}$ is a cell decomposition of X. For $j=1\,,\dots\,,n$, we take a simply connected open set V_j such that $\operatorname{Cl}V_j\subset\operatorname{Int}\Delta_{m+j}$, where Cl and Int mean closure and interior, respectively. We put $W=\prod_{i=1}^m\operatorname{Int}\Delta_i^{(0)}\times\prod_{j=1}^n V_j$. We denote by $\mathscr{E}(m;\theta)_{\mathrm{irr}}|W$ the inverse image of W by the projection $\pi\colon\mathscr{E}(m;\theta)_{\mathrm{irr}}\to X(m)$. From now on we shall consider differential operators L in $\mathscr{E}(m;\theta)_{\mathrm{irr}}|W$. We denote by d and δ the exterior differentials on the Riemann surface M and on the moduli space $\mathscr{E}(m;\theta)_{\mathrm{irr}}$, respectively.

Recall that L is a differential operator $L\colon\mathscr{M}(\xi)\to M(\xi\otimes\kappa^{\otimes 2})$ (see §1). Let (ξ_{jk}) be the transition function of the line bundle ξ with respect to the covering $\mathscr{U}$. Let $L_k=-D_k^2+Q_k$ be the local expression in U_k of the differential operator L, where $D_k=d/dx_k$.



Δ_i : 2-cells$(i = 1, \dots, m)$.

FIGURE 1

Let us consider the differential equation

$$(12) \qquad L_k u_k = 0 \quad \text{in } U_k .$$

We choose the following fundamental system $h_k = (f_k, g_k)$ of solutions of (12) for each k.

(i) For $k = 1, \dots, m$; (12) has a regular singular point at p_k with characteristic exponents $\frac{1}{2}(1 \pm \theta_k)$. Since we assume that θ_k is not an integer, there uniquely exists the fundamental system $h_k = (f_k, g_k)$ of solutions of (12) in U_k such that

$$f_k = \frac{1}{\sqrt{\theta_k}}(x_k - t_k)^{(1-\theta_k)/2}\{1 + O(x_k - t_k)\},$$
$$g_k = \frac{1}{\sqrt{\theta_k}}(x_k - t_k)^{(1+\theta_k)/2}\{1 + O(x_k - t_k)\} \quad \text{as } x_k \to t_k .$$

Here we put the constant functor $1/\sqrt{\theta_k}$ so that the Wronskian $W(f_k, g_k) = 1$. Note that h_k is multivalued and holomorphic in (x_k, Q). The multivaluedness is given by

$$h_k(t_k + \exp(2\pi\sqrt{-1})(x_k - t_k)) = h_k(x_k)C_k ,$$

where C_k is the diagonal matrix $\mathrm{diag}(\exp(-\pi\sqrt{-1}\theta_i), \exp(\pi\sqrt{-1}\theta_i))$. Fix a branch of h_k, then h_k determines single valued fundamental systems $h_k^{(a)}$ of solutions of (12) on $\Delta_k^{(a)}$, $a = 1, 2, 3$. They satisfy the transition relations:

$$(13)\qquad \begin{aligned} h_k^{(a+1)} &= h_k^{(a)} \quad \text{on } \Delta_k^{(a+1)} \cap \Delta_k^{(a)} \text{ for } a = 1, 2, \\ h_k^{(1)} &= h_k^{(3)} C_k \quad \text{on } \Delta_k^{(1)} \cap \Delta_k^{(3)}. \end{aligned}$$

(ii) For $k \in K \backslash \{1, 2, \ldots, m\}$; fix a base point $x_k = s_k$ in U_k. If $k = m+1, \ldots, m+n$, then we take s_k such that $s_k \notin V_{k-m}$. Let $h_k = (f_k, g_k)$ be the fundamental system of solutions of (12) which satisfy the initial conditions $f_k(s_k) = 1$, $D_k f_k(s_k) = 0$, $g_k(s_k) = 0$ and $D_k g_k(s_k) = 1$. Note that h_k is holomorphic in (x_k, Q). If $k = m+1, \ldots, m+n$, then h_k is double-valued:

$$h_k(\lambda_k + \exp(2\pi\sqrt{-1})(x_k - \lambda_k)) = -h_k(x_k);$$

otherwise h_k is single-valued.

For a fundamental system $h = (f, g)$ of solutions of Q, we denote by $\mathbf{h} = [h] = [f, g]$ its projectivization, i.e., ratio of f and g. We call such an $\mathbf{h}$ a projective solution of Q. The differential equation Q determines a flat principal $\mathrm{PSL}_2(\mathbb{C})$-bundle $P = P_Q$ over X whose horizontal section on each 2-cell Δ_k is a projective solution of Q. Note that $\mathbf{h}_i^{(a)} = [h_i^{(a)}]$ is a horizontal section of P on $\Delta_i^{(a)}$ for $i = 1, \ldots, m$, $a = 1, 2, 3$. Similarly, h_k is double-valued for $k = m+1, \ldots, m+n$, $\mathbf{h}_k$ is well-defined.

The Lie group $\mathrm{PSL}_2(\mathbb{C})$ admits the adjoint action Ad on its Lie algebra $\mathrm{sl}_2(\mathbb{C})$. Let $E = P \times_{\mathrm{Ad}} \mathrm{sl}_2(\mathbb{C})$ be the associated flat bundle with fiber $\mathrm{sl}_2(\mathbb{C})$. Note that E is isomorphic as a flat bundle to L_ρ considered in §3, where $\rho = PM(Q)$. We denote by $\langle \mathbf{h}, F \rangle$ the element of E determined by $(\mathbf{h}, F) \in P \times \mathrm{sl}_2(\mathbb{C})$. Let ∇ be the covariant differential on E associated to its flat structure. If $\mathbf{h}$ is horizontal, then $\nabla \langle \mathbf{h}, F \rangle = \langle \mathbf{h}, dF \rangle$.

It is sometimes better to rename the 2-cells $\Delta(X) = \{\Delta_i^{(a)}; i = 1, \ldots, m\} \cup \{\Delta_k; k \in K \backslash \{1, \ldots, m\}\}$ as $\Delta(X) = \{\Delta'_\alpha; \alpha \in A\}$. If $\Delta'_\alpha = \Delta_i^{(a)}$ (resp. Δ_k), then we put $\mathbf{h}'_\alpha = \mathbf{h}_i^{(a)}$ (resp. $\mathbf{h}_k$).

There exists a matrix $C_{\alpha\beta} \in \mathrm{PSL}_2(\mathbb{C})$ such that $\mathbf{h}'_\alpha = \mathbf{h}'_\beta C_{\alpha\beta}$ on $\Delta'_\alpha \cap \Delta'_\beta$, if $\Delta'_\alpha \cap \Delta'_\beta$ is nonempty. $C_{\alpha\beta}$ is holomorphic in Q. Put $F_{\alpha\beta} = C_{\alpha\beta}^{-1} \delta C_{\alpha\beta}$. This is an $\mathrm{sl}_2(\mathbb{C})$-valued 1-form on $\mathscr{E}(m; \theta)_{\mathrm{irr}} | W$. Recall that a cochain c with values in the flat bundle E assigns to each cycle σ in X a horizontal section $c(\sigma) \in \Gamma_h(\sigma; E)$ of E, where $\Gamma_h(\cdot)$ means the set of horizontal sections. Now we shall define an E-valued 1-cocycle $c = c_Q$ in the following manner: Let σ be a 1-cell in X given by $\sigma = \Delta'_\alpha \cap \Delta'_\beta$ and assume that σ and $\partial \Delta'_\alpha$ have the same

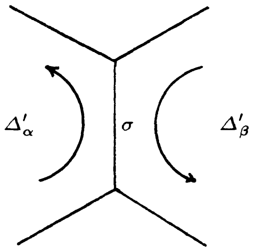


FIGURE 2

orientation. We call such a σ the 1-cell determined by the (ordered) pair $(\Delta'_\alpha, \Delta'_\beta)$ of 2-cells. See Figure 2.

We define $c(\sigma)$ by $\langle \mathbf{h}'_\beta|_\sigma, F_{\alpha\beta}\rangle \in \Gamma_h(\sigma; E)$, where $\mathbf{h}'_\beta|_\alpha$ is the restriction of $\mathbf{h}'_\beta$ to σ. We note that if $(\Delta'_\alpha, \Delta'_\beta) = (\Delta_i^{(2)}, \Delta_i^{(1)})$, $(\Delta_i^{(3)}, \Delta_i^{(2)})$, or $(\Delta_i^{(1)}, \Delta_i^{(3)})$, then (13) implies that $F_{\alpha\beta} = 0$ and hence $c(\sigma) = 0$. We put $c(\sigma) = 0$ for 1-cycles σ on the boundary ∂X.

The 1-cocycle c_Q determines a cohomology class

$$[c_Q] \in H^1(X, \partial X; E) \simeq H^1(X, \partial X; L_\rho).$$

Recall that $T_\rho R(m; \theta)_{\mathrm{irr}}$ is naturally isomorphic to

$$H^1(X, \partial X; L_\rho)/\delta^* H^0(\partial X; L_\rho).$$

See (11). Hence $[c_Q]$ determines an element of $T_\rho R(m; \theta)_{\mathrm{irr}}$. More precisely, $[c_Q]$ is in $T_Q^*\mathscr{E}(m; \theta)_{\mathrm{irr}} \otimes T_\rho R(m; \theta)_{\mathrm{irr}}$ and gives the differential map dPM of the projective monodromy map PM at $Q \in \mathscr{E}(m; \theta)_{\mathrm{irr}}|W$.

We shall give another representation of the 1-cocycle c. Let $\Phi_k \in \mathrm{SL}_2(\mathbb{C})$ be a matrix defined by

$$\Phi_k = \begin{pmatrix} f_k & g_k \\ D_k f_k & D_k g_k \end{pmatrix}. \tag{14}$$

Put $G_k = \Phi_k^{-1} \cdot \delta\Phi_k \in \mathrm{sl}_2(\mathbb{C})$. For $i = 1, \ldots, m$, Φ_i has branch point at p_i whose circuit matrix is given by

$$C_i = \mathrm{diag}(\exp(-\pi\sqrt{-1}\theta_i), \exp(\pi\sqrt{-1}\theta_i)).$$

Hence we have

$$G_i(t_i + \exp(2\pi\sqrt{-1})(x_i - t_i)) = \mathrm{Ad}(C_i)^{-1} G_i(x_i).$$

Fix a branch of G_i; then G_i determines single-valued holomorphic functions $G_i^{(a)}$ on the 2-cell $\Delta_i^{(a)}$ for $a = 1, 2, 3$. For $j = 1, \ldots, n$,

Φ_{m+j} has branch point at q_j whose circuit matrix is given by $-I$. Hence G_{m+j} is single-valued meromorphic with pole at q_j. For $k \in K\backslash\{1, \dots, m+n\}$, Φ_k and G_k are single-valued holomorphic. When we express the 2-cells by $\Delta(X) = \{\Delta'_\alpha ; \alpha \in A\}$, we put $G'_\alpha = G_i^{(a)}$ (resp. G_k) if $\Delta'_\alpha = \Delta_i^{(a)}$ (resp. Δ_k). Let $\mathscr{M}(E)$ be the sheaf of meromorphic sections of E on X. We shall define an $\mathscr{M}(E)$-valued function u on the 2-cells $\Delta(X)$ by $u(\Delta'_\alpha) = \langle \mathbf{h}'_\alpha, G'_\alpha \rangle \in \Gamma(\Delta'_\alpha ; \mathscr{M}(E))$. We can easily see that if c is regarded as an $\mathscr{M}(E)$-valued 1-cochain, then

$$c(\sigma) = u(\Delta'_\beta)|_\sigma - u(\Delta'_\alpha)|_\sigma , \tag{15}$$

where σ is the 1-cycle in X determined by $(\Delta'_\alpha, \Delta'_\beta)$. In particular, since the 1-cocycle c vanishes in a neighbourhood of ∂X, u determines a C^∞-section of E on ∂X, which we shall denote by $\chi \in \Gamma(\partial X ; \mathscr{E}^\infty(E))$. Notice that $\partial X = -\sum_{i=1}^m \gamma_i$ with $\gamma_i = \partial\Delta_i^{(0)}$ and χ is given by

$$\chi|_{\gamma_i} = \langle \mathbf{h}_i , G_i \rangle|_{\gamma_i} \quad \text{for } i = 1, \dots, m.$$

The right-hand side is well-defined as a section of $\mathscr{E}^\infty(E)|_{\gamma_i}$.

5. Reduction to a residue calculus. In this section we shall compute the Poincaré-Lefschetz duality pairing (10) explicitly. We first express the above duality in the framework of the de Rham cohomology and then reduce the problem to a residue calculus around the singular points of differential operator $Q \in \mathscr{E}(m; \theta)_{\text{irr}}$. The notation will be the same as that in §4.

Let $\mathscr{E}_0^\infty(E)$ be the sheaf of C^∞-sections of E which vanish on ∂X. Since $\mathscr{E}_0^\infty(E)$ is a soft sheaf, there exist a $\mathscr{E}_0^\infty(E)$-valued function v on the 2-cells $\Delta(X)$ such that

$$c(\sigma) = v(\Delta'_\beta)|_\sigma - v(\Delta'_\alpha)|_\sigma , \tag{16}$$

where σ is the 1-cycle in X determined by $(\Delta'_\alpha, \Delta'_\beta)$. By (16) we can define a $C_0^\infty(E)$-valued closed 1-form ϕ on X by $\phi|_{\Delta'_\alpha} = \nabla v(\Delta'_\alpha)$. Notice that ϕ vanishes on ∂X. This closed 1-form represents the de Rham class corresponding to $[c] \in H^1(X, \partial X; E)$.

The Killing form $B(\cdot, \cdot)$ on the Lie algebra $\mathrm{sl}_2(\mathbb{C})$ induces a horizontal symmetric bilinear form $B(\cdot, \cdot)$ on the flat bundle E. Explicitly, it is given as follows: For local sections $s_\nu = \langle \mathbf{h}_\nu , F_\nu \rangle$, $\nu = 1, 2$ with $\mathbf{h}_\nu$ horizontal,

$$B(s_1, s_2) = \frac{1}{2\pi\sqrt{-1}} \operatorname{trace}(F_1 F_2).$$

We extend it to the bilinear form

$$B\colon \left(E\otimes\bigwedge^{p}T^{*}X\right)\otimes\left(E\otimes\bigwedge^{q}T^{*}X\right)\to\bigwedge^{p+q}T^{*}X$$

in an obvious manner. Hereafter we write $u_\alpha = u(\Delta'_\alpha)$ and $v_\alpha = v(\Delta'_\alpha)$ for simplicity of notation. The fundamental 2-form Ω on $\mathscr{E}(m\,;\theta)_{\mathrm{irr}}$ associated to the Poincaré-Lefschetz duality is then represented by the integral

$$\Omega = \int_X B(\phi\,,\phi)\,.$$

We have

$$\begin{aligned}\Omega &= \sum_\alpha \int_{\Delta'_\alpha} B(\phi\,,\phi) = \sum_\alpha \int_{\Delta'_\alpha} B(\nabla v_\alpha\,,\phi)\\ &= \sum_\alpha \int_{\Delta'_\alpha} dB(v_\alpha\,,\phi) = \sum_\alpha \int_{\partial\Delta'_\alpha} B(v_\alpha\,,\phi)\,,\end{aligned}$$

where the summation is taken over all $\alpha\in A$. Put $w_\alpha = v_\alpha - u_\alpha$. Then (15) and (16) imply that $w_\alpha = w_\beta$ on $\Delta'_\alpha\cap\Delta'_\beta$. Namely w defined by $w|_{\Delta'_\alpha} = w_\alpha$ is a global section on X. We note that $w|_{\partial X} = -\chi$, where $\chi\in\Gamma(\partial X\,;\mathscr{E}^\infty(E))$ is defined in the end of §4. Hence we have

$$\Omega = \sum_\alpha \int_{\partial\Delta'_\alpha} B(u_\alpha\,,\phi) + \sum_\alpha \int_{\partial\Delta'_\alpha} B(w\,,\phi)\,.$$

The second term on the right-hand side equals $\int_{\partial X} B(w\,,\phi)$. Since $B(w\,,\phi) = 0$ on ∂X, this term equals zero. Hence we have

$$\Omega = \sum_\alpha \int_{\partial\Delta'_\alpha} B(u_\alpha\,,\nabla u_\alpha) + \sum_\alpha \int_{\partial\Delta'_\alpha} B(u_\alpha\,,\nabla w)\,.$$

We denote by $(*)$ and $(**)$ the first and the second terms on the right-hand side, respectively. First we shall compute $(*)$. If $\Delta'_\alpha = \Delta_{m+j}$ with $j = 1\,,\dots\,,n$, then $B(u_\alpha\,,\nabla u_\alpha)$ is meromorphic in Δ'_α with pole at q_j. If Δ'_α is either $\Delta_i^{(a)}$ with $i = 1\,,\dots\,,m$, $a = 1\,,2\,,3$ or Δ_k with $k\in K\backslash\{1\,,\dots\,,m+n\}$, then $B(u_\alpha\,,\nabla u_\alpha)$ is holomorphic in Δ'_α. Hence the residue theorem implies

$$\begin{aligned}(*) &= 2\pi\sqrt{-1}\sum_{j=1}^{n}\mathrm{Res}_{q_j} B(u(\Delta_{m+j})\,,\nabla u(\Delta_{m+j}))\\ &= \sum_{j=1}^{n}\mathrm{Res}_{q_j}\mathrm{trace}(G_{m+j}\cdot dG_{m+j})\,,\end{aligned}$$

where $\mathrm{Res}(\cdot)$ denotes the residue at a point $p\in M$.

Next we shall compute $(**)$. By (15) we can define an $\mathscr{M}(E)$-valued 1-form ψ by letting $\psi|_{\Delta'_\alpha} = \nabla u_\alpha$. Note that $\psi|_{\partial X} = \nabla\chi$. We have

$$(**) = \sum_\alpha \int_{\partial\Delta'_\alpha} dB(u_\alpha, w) - \sum_\alpha \int_{\partial\Delta'_\alpha} B(\psi, w).$$

Evidently the first term on the right-hand side vanishes. The second term equals $-\int_{\partial X} B(\psi, w) = \int_{\partial X} B(\nabla\chi, \chi)$. Hence we have

$$\begin{aligned}(**) &= \int_{\partial X} B(\nabla\chi, \chi) = -\sum_{i=1}^m \int_{\gamma_i} B(\nabla\chi, \chi)|_{\gamma_i} \\ &= -\frac{1}{2\pi\sqrt{-1}} \sum_{i=1}^m \int_{\gamma_i} \operatorname{trace}(dG_i \cdot G_i).\end{aligned}$$

Since $\operatorname{trace}(dG_i \cdot G_i)$ is a single-valued meromorphic function in $\Delta_i^{(0)}$ with pole at p_i, the residue theorem implies

$$(**) = \sum_{i=1}^m \operatorname{Res}_{p_i} \operatorname{trace}(G_i \cdot dG_i).$$

We have thus obtained the following:

LEMMA 6.

$$\Omega = \sum_{j=1}^n \operatorname{Res}_{q_j} \operatorname{trace}(G_{m+j} \cdot dG_{m+j}) + \sum_{i=1}^m \operatorname{Res}_{p_i} \operatorname{trace}(G_i \cdot dG_i).$$

6. Proof of Main Theorem. In this section we shall complete the proof of our main theorem. We put $p_k = q_{k-m}$ for $k = m+1, \ldots, n$. By Lemma 6 we have only to compute the residue of $\operatorname{trace}(G_k \cdot dG_k)$ at the singular point p_k for $k = 1, \ldots, m+n$. We shall derive a more explicit formula for $\operatorname{trace}(G_k \cdot dG_k)$.

Recall that $G_k = \Phi_k^{-1}\delta\Phi_k$, where Φ_k is defined by (14). Note that Φ_k satisfies the differential equation $D_k\Phi_k = P_k\Phi_k$, where

$$P_k = \begin{pmatrix} 0 & 1 \\ Q_k & 0 \end{pmatrix}.$$

An easy calculation shows $D_kG_k = -\Phi_k^{-1}\delta P\Delta_k$. This can be rewritten as $D_kG_k = \Psi_k\delta Q_k$, where

$$\Psi_k = \begin{pmatrix} f_kg_k & g_k^2 \\ -f_k^2 & -f_kg_k \end{pmatrix}.$$

Fix a base point s_k in $U_k\backslash\{p_k\}$. Integrating the above formula with respect to x_k, we obtain

$$G_k(x_k) = G_k(s_k) + \int_{s_k}^{x_k} \Psi_k(t)\delta Q_k(t)\,dt.$$

Hence we obtain $\operatorname{trace}(G_k \cdot dG_k) = R_k \wedge \delta Q_k \cdot dx_k$, where

$$\begin{aligned} (17) \qquad R_k(x_k) &= \operatorname{trace}(G_k(s_k)\Psi_k(x_k)) \\ &\quad + \int_{s_k}^{x_k} \operatorname{trace}(\Psi_k(t)\Psi_k(x_k))\delta Q_k(t)\,dt. \end{aligned}$$

Since $\operatorname{trace}(G_k \cdot dG_k)$ is a single-valued meromorphic function in U_k with pole at p_k, so is R_k.

We associate to $L_k = -D_k^2 + Q_k$ the third order differential operator

$$A_k = -\tfrac{1}{2}D_k^2 + 2Q_kD_k + D_kQ_k.$$

It is well known that if (f_k, g_k) is a fundamental system of solutions of $L_kf = 0$ then (f_k^2, f_kg_k, g_k^2) is a fundamental system of solutions of (A_k): $A_kv = 0$. The entries of the matrix Ψ_k are solutions of (A_k). Hence the first term on the right-hand side of (17) is a solution of (A_k). Let us consider the second term. Put $Z_k(x_k, t) = \operatorname{trace}(\Psi_k(t)\Psi_k(x_k))$. By a direct calculation we have $Z_k(x_k, t) = -\{f_k(t)g_k(x_k) - f_k(x_k)g_k(t)\}^2$. This implies that $Z_k(x_k, t)$ is a fundamental solution of (A_k). Namely $Z_k(x_k, t)$ is a solution of (A_k) with respect to x_k and satisfies the initial condition: $D_k^\nu Z_k(x_k, t)|_{x_k=t} = 0$ for $\nu = 0, 1$ and $D_k^2 Z_k(x_k, t)|_{x_k=t} = -2$. This means that the function v defined by $v(x_k) = \int_{s_k}^{x_k} Z_k(x_k, t)q(t)\,dt$ is a solution of $A_kv = q$. Hence the second term on the right-hand side of (17) is a solution of $A_kv = \delta Q_k$. Therefore we conclude that R_k is a solution of $A_kv = \delta Q_k$. We have thus obtained the following:

LEMMA 7.

$$\operatorname{trace}(G_k \cdot dG_k) = R_k \wedge \delta Q_k \cdot dx_k,$$

where R_k is a single-valued meromorphic function in U_k with pole at p_k and satisfies $A_kR_k = \delta Q_k$ for $k = 1, \dots, m+n$.

We shall find lower order terms of the Laurent expansion of R_k at p_k. Recall that Q_k admits the following Laurent expansion at p_k:

$$Q_k = \frac{a}{(x-t)^2} + \frac{b}{x-t} + c + e(x-t) + O((x-\lambda)^2).$$

In the notation of §1 we have $x = x_k$, $t = t_k$, $a = \frac{1}{4}(\theta_k^2 - 1)$, $b = H_k$ for $k = 1, \dots, m$ and $x = y_{k-m}$, $t = \lambda_{k-m}$, $a = \frac{3}{4}$, $b = -\mu_{k-m}$, $c = \mu_{k-m}^2$ for $k = m+1, \dots, m+n$, respectively. Since a is independent of Q, we obtain

$$\delta Q_k = \frac{2a\delta t}{(x-t)^3} + \frac{b\delta t}{(x-t)^2} + \frac{\delta b}{x-t} + 2b\delta b - e\delta t + O(x-t).$$

Since $(t_1, \dots, t_m; \lambda_1, \dots, \lambda_n; \mu_1, \dots, \mu_n)$ is a local coordinate of the moduli space $\mathscr{E}(m; \theta)_{\mathrm{irr}}$, δt is nowhere vanishing. Moreover $a \neq 0$. Hence the pole of Q_k at p_k is precisely of order 3. By Lemma 7, R_k is a single-valued meromorphic function in U_k with pole at p_k. We assume that R_k admits the following Laurent expansion at p_k:

$$\begin{aligned} R_k = {} & \alpha(x-t)^N + \beta(x-t)^{N+1} + \gamma(x-t)^{N+2} \\ & + \varepsilon(x-t)^{N+3} + O((x-t)^{N+4}), \end{aligned}$$

where N is an integer and $\alpha \neq 0$.

We shall determine the order N of R_k at p_k. We see that $A_k R_k = \tau(x-t)^{N-3} + O((x-t)^{N-2})$, where $\tau = -\frac{1}{2}\alpha(N-1)(N^2 - 2N - 4a)$. By Lemma 7, since $A_k R_k = \delta Q_k$ holds, $A_k R_k$ must have a pole of order 3 at p_k. For this it is necessary that N is a non-positive integer. If $\tau \neq 0$, then we have $N = 0$. Let us consider the alternative case $\tau = 0$ which happens if and only if $N = 1$ or $1 \pm \sqrt{4a+1}$. For $k = 1, \dots, m$, $1 \pm \sqrt{4a+1} = 1 \pm \theta_k$. Hence we have $N = 1$ or $1 \pm \theta_k$. These are however inadequate because $N = 1$ is positive and $N = 1 \pm \theta_k$ are not integers. For $k = m+1, \dots, m+n$, $1 \pm \sqrt{4a+1} = -1, 3$. Since N is non-positive, we have $N = -1$. Thus we have only to consider the following two cases: (Case 1) $N = 0$, (Case 2) $k = m+1, \dots, m+n$ and $N = -1$.

First we shall consider (Case 1). In this case a direct calculation shows

$$A_k R_k = -\frac{2a\alpha}{(x-t)^3} - \frac{b\alpha}{(x-t)^2} + \frac{b\beta + 2a\gamma}{x-t} + O(1).$$

Comparing the Laurent coefficients of the equation $A_k R_k = \delta Q_k$, we obtain $\alpha = -\delta t$ and $b\beta + 2a\gamma = \delta b$. On the other hand, we have $\mathrm{Res}_{p_k} R_k \wedge Q_k \cdot dx_k = \alpha \wedge \delta b + (b\beta + 2a\gamma) \wedge \delta t$. Therefore we obtain

$$\mathrm{Res}_{p_k} R_k \wedge Q_k \cdot dx_k = 2\delta b \wedge \delta t. \tag{18}$$

Next we shall consider (Case 2). In this case a direct calculation shows

$$A_k R_k = -\frac{3(2b\alpha + \beta)}{2(x-t)^3} - \frac{b(2b\alpha + \beta)}{(x-t)^2} + \frac{-2e\alpha + 2b\gamma + 3\varepsilon}{2(x-t)} + O(1).$$

Comparing the Laurent coefficients of the equation $A_k R_k = \delta Q_k$, we obtain $-(2b\alpha+\beta) = \delta t$ and $-e\alpha + b\gamma + \frac{2}{3}\varepsilon = \delta b$. On the other hand, we have $\mathrm{Res}_{p_k}(R_k \wedge \delta Q_k \cdot dx_k) = (2b\alpha+\beta)\wedge\delta b + (-e\alpha+b\gamma+\frac{3}{2}\varepsilon)\wedge dt$. Therefore, also in this case, we obtain the same result as (18). We have thus obtained the following:

LEMMA 8.

$$\mathrm{Res}_{p_k}(R_k \wedge \delta Q_k \cdot dx_k) = \begin{cases} 2\delta H_k \wedge \delta t_k & \text{for } k = 1, \dots, m, \\ -2\delta \mu_j \wedge \delta \lambda_j & \text{for } j = 1, \dots, n, \end{cases}$$

By Lemma 6–Lemma 8, we obtain

$$\Omega = -2\sum_{j=1}^{n} \delta\mu_j \wedge \delta\lambda_j + 2\sum_{i=1}^{m} \delta H_i \wedge \delta t_i .$$

Hence Ω coincides with the fundamental 2-form on $\mathscr{E}(m;\theta)$ defined by (4) up to the constant multiple -2. This establishes Theorem 5.

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