

MULTIPLICATIVE FUNCTIONS ON FREE GROUPS AND IRREDUCIBLE REPRESENTATIONS

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Let Γ be a free group on infinitely many generators. Fix a basis for Γ and for any group element x , denote by $|x|$ its length with respect to this basis. Let e denote the group identity. A *multiplicative function* ϕ on Γ is a function satisfying the conditions $\phi(e) = 1$ and $\phi(xy) = \phi(x)\phi(y)$ whenever $|xy| = |x| + |y|$. We characterize those positive definite multiplicative functions for which the associated representation of Γ is irreducible.

0. Introduction. Fix an infinite set A^+ and let Γ be the free group on generators A^+ . Let A^- consist of the the inverse generators and let $A = A^+ \cup A^-$. Any x in Γ admits a unique shortest expression as a product of elements of A . The length of x , $|x|$, is the number of letters in this expression. A *multiplicative function* ϕ on Γ is a function satisfying the conditions

$$\begin{aligned}\phi(xy) &= \phi(x)\phi(y) && \text{when } |xy| = |x| + |y| \\ \phi(e) &= 1 .\end{aligned}$$

A multiplicative function is determined by its values on A .

Choose complex numbers $\{\phi(a)\}_{a \in A}$ such that $\phi(a^{-1}) = \overline{\phi(a)}$ and satisfying the condition $\sup_{a \in A} |\phi(a)| < 1$, and extend ϕ to a multiplicative function on Γ . For example, if $0 < r < 1$ one can choose $\phi(a) = r$ for every a . In that case $\phi(x) = r^{|x|}$ is a radial function which Haagerup [9] showed to be positive definite. DeMichele and Figà-Talamanca [4] extended Haagerup's result, showing that all ϕ constructed as above are positive definite.

To each multiplicative positive definite function ϕ one associates a unitary representation π_ϕ of Γ , specified by the property that π_ϕ has a cyclic vector for which ϕ is the matrix coefficient. When is π_ϕ

irreducible? For radial ϕ Pytlik and Szwarz [12] demonstrated irreducibility and constructed an explicit realization of π_ϕ . Following some of their ideas we prove that the condition

$$(0.1) \quad \sum_{a \in A} |\phi(a)|^2 = +\infty$$

implies irreducibility.

We show conversely that if the sum (0.1) is finite, then π_ϕ is reducible. If

$$(0.2) \quad \sum_{a \in A} |\phi(a)|^2 / (1 + |\phi(a)|^2)$$

is less than one, then ϕ is square integrable and so π_ϕ is a subrepresentation of the regular representation. According to [3] (for all infinite discrete groups) or [10, Section 1] (for the free group), such representations are never irreducible. When (0.2) is equal to one ϕ is no longer square integrable, but the associated representation is still expressible as a subrepresentation of the regular representation, and so is still reducible.

Finally, when (0.2) is greater than one, we decompose ϕ as the sum of two nonproportional positive definite functions, thereby showing that π_ϕ is reducible. Of the two positive definite summands one is square integrable, and may therefore be further decomposed, while we believe the other to be associated to an irreducible representation. Note that the arguments used in the case $\sum_{a \in A} |\phi(a)|^2$ finite may be applied, virtually unchanged, to a free group on only finitely many generators.

1. Construction of π_ϕ . From this point on we shall assume that complex numbers $\{\phi(a)\}_{a \in A^+}$ have been chosen so that

$$(1.1) \quad \Phi = \sup_{a \in A^+} |\phi(a)| < 1.$$

Let ϕ be the corresponding positive definite function as described in the introduction.

Following the ideas of [12] we proceed to construct a uniformly bounded representation having ϕ as a matrix coefficient. For any given x let $x = a_1 \dots a_{n-1} a_n$ ($a_j \in A$) be the reduced expression for x . Let δ_x denote the Kronecker δ at the point x . Let π_{reg} denote

the left regular representation of Γ on $\ell^2(\Gamma)$ so that $\pi_{\text{reg}}(x)(\delta_y) = \delta_{xy}$. Let $H \subset \ell^2(\Gamma)$ consist of all finitely supported functions. If $x \in \Gamma$ has reduced expression $a_1 \dots a_n$, and supposing $x \neq e$, define $\bar{x} = a_1 \dots a_{n-1}$. Extend this bar operation to a linear operator on H , defining $\bar{\delta}_e = 0$ and $\bar{\delta}_x = \delta_{\bar{x}}$ for x as above. Let H_x consist of the functions supported on the set $\{x, \bar{x}, \bar{\bar{x}} \dots e\}$. If the reader thinks of Γ as acting on a homogeneous (locally infinite!) tree, than H_x consists of those functions supported on the geodesic from e to x . Define a linear operator P on H by

$$\begin{aligned} P(\delta_x) &= \phi(a_n)\bar{\delta}_x & \text{for } x \neq e \\ P(\delta_e) &= 0. \end{aligned}$$

Define a representation π of Γ on H by

$$\pi(x) = (I - P)^{-1}\pi_{\text{reg}}(x)(I - P).$$

Let (\cdot, \cdot) denote the usual inner product in $\ell^2(\Gamma)$.

LEMMA 1. *The representation π extends uniquely to a uniformly bounded representation of Γ on $\ell^2(\Gamma)$ and $\phi(x) = (\pi(x)\delta_e, \delta_e)$.*

Proof. Fix $x = a_1 a_2 \dots a_n$ in Γ . Write

$$\begin{aligned} (1.2) \quad \pi(x)\pi_{\text{reg}}(x^{-1}) &= (I - P)^{-1}\pi_{\text{reg}}(x)(I - P)\pi_{\text{reg}}(x^{-1}) \\ &= I + \sum_{n=0}^{\infty} P^n (P - \pi_{\text{reg}}(x)P\pi_{\text{reg}}(x^{-1})). \end{aligned}$$

One may verify that $P - \pi_{\text{reg}}(x)P\pi_{\text{reg}}(x^{-1})$ acts as zero on H_x^\perp . Therefore $\pi(x^{-1})$ and $\pi_{\text{reg}}(x^{-1})$ coincide on H_x^\perp .

Choose $\delta_e, \dots, \delta_{\bar{x}}, \delta_x$ as a basis for H_x . With respect to this basis P acts on H_x , preserving it, by the following nilpotent upper triangular matrix

$$\mathbf{P} = \begin{pmatrix} 0 & \phi(a_1) & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \phi(a_n) \\ 0 & 0 & \dots & 0 \end{pmatrix},$$

and likewise $\pi_{\text{reg}}(x)P\pi_{\text{reg}}(x^{-1})$ is given by the conjugate transpose of \mathbf{P} . So the operator norm of P restricted to H_x is less than or

equal to Φ , while the operator norm of $(P - \pi_{\text{reg}}(x)P\pi_{\text{reg}}(x^{-1}))$ will be less than or equal to 2Φ . Therefore, for every $f \in H$ we have

$$\begin{aligned} \|\pi(x)\pi_{\text{reg}}(x^{-1})(f)\|_2 &\leq \|f\|_2 + 2\Phi \sum_{n=0}^{\infty} \Phi^n \|f\|_2 \\ &\leq \left(\frac{1+\Phi}{1-\Phi}\right) \|f\|_2 \end{aligned}$$

since we have assumed $\Phi < 1$. Hence π can be extended to a uniformly bounded representation of Γ . An elementary calculation shows that $\phi(x) = (\pi(x)\delta_e, \delta_e)$. \square

From the proof follows

COROLLARY. $\pi(x^{-1})$ and $\pi_{\text{reg}}(x^{-1})$ coincide on H_x^\perp .

Although the representation π constructed above is *not* unitary, it still has some nice properties.

THEOREM 1. *Let π be as above. Suppose that*

$$\sum_{a \in A} |\phi(a)|^2 = +\infty.$$

Then $\ell^2(\Gamma)$ does not admit any closed linear subspace invariant under π .

Proof. Choose any increasing sequence of subsets A_α such that A_α is finite for every α and $\sup_\alpha \sum_{a \in A_\alpha} |\phi(a)|^2 = +\infty$. Define

$$U_\alpha = \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \sum_{a \in A_\alpha} \overline{\phi(a)} \pi(a).$$

The proof of irreducibility goes as follows:

- i) show that δ_e is a cyclic vector for π .
- ii) show that $(U_\alpha f, g)$ converges to $(f(e)\delta_e, g)$ for any f and g in H .
- iii) show that U_α are uniformly bounded as operators from $\ell^2(\Gamma)$ into $\ell^2(\Gamma)$.

Suppose now that B is any closed invariant linear subspace and let $f \in B$ be a nonzero function. Then by ii) and iii) $U_\alpha(f)$ will

approach $f(e)\delta_e$ weakly, showing that δ_e belongs to B provided that $f(e) \neq 0$. If $f(e) = 0$ let x be any shortest word in the support of f , let c be the last letter of x , and use the corollary to Lemma 1 as an aid in calculating

$$(\pi(x^{-1})f)(e) = (1 - |\phi(c)|^2)f(x) \neq 0 .$$

Then replace f by $\pi(x^{-1})f$.

Proof of i). Choose any $x \neq e$ in Γ . Let c be the last letter of x . Then

$$\delta_x = \pi(x)\delta_e - \phi(c)\pi(\bar{x})\delta_e .$$

Proof of ii). First observe that

$$U_\alpha(\delta_e) = \delta_e + \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \sum_{a \in A_\alpha} \overline{\phi(a)}\delta_a$$

and hence $(U_\alpha(\delta_e), g)$ converges to (δ_e, g) for any $g \in H$. Suppose now $x \neq e$. Then

$$U_\alpha(\delta_x) = \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \left(\sum_{a \in A_\alpha} \overline{\phi(a)}\delta_{ax} \right)$$

unless $x^{-1} \in A_\alpha$, in which case, for $x = b$, we have

$$U_\alpha(\delta_b) = \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \left(\sum_{\substack{a \in A_\alpha \\ |ab|=2}} \overline{\phi(a)}\delta_{ab} + \overline{\phi(b^{-1})}((1 - |\phi(b)|^2)\delta_e - \overline{\phi(b^{-1})}\delta_{b^{-1}}) \right)$$

showing that for every f and g in H $(U_\alpha f, g)$ converges to $(f(e)\delta_e, g) = f(e)\overline{g(e)}$.

Proof of iii). Haagerup's inequality is central for this step. Write

$$\pi(a) - \pi_{\text{reg}}(a) = (\pi(a)\pi_{\text{reg}}(a^{-1}) - I)\pi_{\text{reg}}(a) .$$

By (1.2)

$$(\pi(a)\pi_{\text{reg}}(a^{-1}) - I) = \sum_{n=0}^{+\infty} P^n (P - \pi_{\text{reg}}(a)P\pi_{\text{reg}}(a^{-1})) .$$

Remember that the right hand side annihilates H_a^\perp . On the other hand, on H_a , P and $P - \pi_{\text{reg}}(a)P\pi_{\text{reg}}(a^{-1})$ are given respectively by

$$\begin{pmatrix} 0 & \phi(a) \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \phi(a) \\ -\overline{\phi(a)} & 0 \end{pmatrix}$$

giving us

$$\|\pi(a)\pi_{\text{reg}}(a^{-1}) - I\|_{2,2} \leq \frac{2|\phi(a)|}{1 - \Phi}$$

and hence

$$(1.3) \quad \|\pi(a) - \pi_{\text{reg}}(a)\|_{2,2} \leq \frac{2|\phi(a)|}{1 - \Phi}.$$

Define now

$$S_\alpha = \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \sum_{a \in A_\alpha} \overline{\phi(a)} \pi_{\text{reg}}(a).$$

By Haagerup's inequality (see [9])

$$\|S_\alpha\|_{2,2} \leq 2 \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-\frac{1}{2}},$$

telling us that the S_α are uniformly bounded. Finally, by (1.3)

$$\begin{aligned} & \|U_\alpha - S_\alpha\|_{2,2} \\ &= \left(\sum_{a \in A_\alpha} |\phi(a)|^2 \right)^{-1} \left\| \sum_{a \in A_\alpha} \overline{\phi(a)} (\pi(a) - \pi_{\text{reg}}(a)) \right\|_{2,2} \\ &\leq \frac{2}{1 - \Phi}. \end{aligned}$$

□

We want to exhibit now a *unitary* irreducible representation having ϕ as a matrix coefficient. Let $(\pi_\phi, \mathcal{H}_\phi)$ be the representation constructed from ϕ by the Gelfand–Raikov construction. Then \mathcal{H}_ϕ contains a cyclic vector v_0 with $(\pi_\phi(x)v_0, v_0)_{\mathcal{H}_\phi} = \phi(x)$. One may obtain \mathcal{H}_ϕ as the completion of the space of finite linear combinations of translates of v_0 where the square norm of $\sum_x c_x \pi_\phi(x)v_0$ is

$$\left(\sum_x c_x \pi_\phi(x)v_0, \sum_y c_y \pi_\phi(y)v_0 \right)_{\mathcal{H}_\phi} = \sum_{x,y \in \Gamma} c_x \overline{c_y} \phi(y^{-1}x).$$

We shall prove that π_ϕ is irreducible. Recall that every finitely supported function can be written as a finite linear combination of $\{\pi(x)(\delta_e)\}_{x \in \Gamma}$. Define first $T : H \subset \ell^2(\Gamma) \rightarrow \mathcal{H}_\phi$ by letting

$$T(\pi(x)\delta_e) = \pi_\phi(x)v_0$$

and try to extend T to a (nonunitary) linear equivalence from $\ell^2(\Gamma)$ to \mathcal{H}_ϕ . If one can do that then T will intertwine π to π_ϕ and π_ϕ will also be irreducible.

Observe that at this point, T is *not even well defined*; nor does one know that the square norm from above, $\|\sum_x c_x \pi_\phi(x)v_0\|_{\mathcal{H}_\phi}^2 = (\sum_x c_x \delta_x, \sum_x c_x \delta_x)_{\mathcal{H}_\phi}$, is comparable with the ℓ^2 -square norm of $\sum_x c_x \pi(x)\delta_e$. However, we have

LEMMA 2. *For every finite linear combination $\sum_x c_x \pi(x)\delta_e$ we have*

$$\left\| \sum_x c_x \pi_\phi(x)v_0 \right\|_{\mathcal{H}_\phi}^2 \leq \left\| \sum_x c_x \pi(x)\delta_e \right\|_{\ell^2(\Gamma)}^2 \leq \frac{1}{1 - \Phi^2} \left\| \sum_x c_x \pi_\phi(x)v_0 \right\|_{\mathcal{H}_\phi}^2$$

where Φ is the constant defined in (1.1).

Proof. Let $\Gamma(y)$ be the set of all words having y as their first piece and $\tilde{\Gamma}(z)$ be the set of all words having z as their last piece. So

$$\begin{aligned} \Gamma(y) &= \{x \in \Gamma : |y^{-1}x| = |x| - |y|\} \\ \tilde{\Gamma}(z) &= \{x \in \Gamma : |xz^{-1}| = |x| - |z|\} . \end{aligned}$$

Observe that

$$(\pi(x)\delta_e)(y) = \begin{cases} 0 & x \notin \Gamma(y) \\ \phi(y^{-1}x) & x \in \Gamma(y) \end{cases}$$

so that

$$\left(\sum_x c_x \pi(x)\delta_e \right) (y) = \sum_{x \in \Gamma(y)} c_x \phi(y^{-1}x)$$

and

(1.4)

$$\begin{aligned} \left\| \sum_x c_x \pi(x)\delta_e \right\|_{\ell^2(\Gamma)}^2 &= \sum_y \left| \sum_{x \in \Gamma(y)} c_x \phi(y^{-1}x) \right|^2 \\ &= \sum_y \sum_{x, z \in \Gamma(y)} c_x \overline{c_z} \phi(y^{-1}x) \overline{\phi(y^{-1}z)} . \end{aligned}$$

For $y \in \Gamma$ let $\Gamma_2(y) = \{(x, z) : x, z \in \Gamma(y) \text{ and } x \text{ and } z \text{ have exactly } |y| \text{ initial letters in common}\}$. Then $\Gamma \times \Gamma$ is the disjoint union of the sets $\Gamma_2(y)$. Observe that $\phi(z^{-1}x) = \phi(y^{-1}x)\overline{\phi(y^{-1}z)}$ for $(x, z) \in \Gamma_2(y)$. Hence

$$(1.5) \quad \left\| \sum_x c_x \pi_\phi(x) v_0 \right\|_{\mathcal{H}_\phi}^2 = \sum_{x,z} c_x \overline{c_z} \phi(z^{-1}x) \\ = \sum_y \sum_{(x,z) \in \Gamma_2(y)} c_x \overline{c_z} \phi(y^{-1}x) \overline{\phi(y^{-1}z)}.$$

Subtracting (1.5) from (1.4) we get

$$\left\| \sum_x c_x \pi(x) \delta_e \right\|_{\ell^2(\Gamma)}^2 - \left\| \sum_x c_x \pi_\phi(x) v_0 \right\|_{\mathcal{H}_\phi}^2 \\ = \sum_y \sum_{\substack{x,z \in \Gamma(y) \\ (x,z) \notin \Gamma_2(y)}} c_x \overline{c_z} \phi(y^{-1}x) \overline{\phi(y^{-1}z)} \\ = \sum_y \sum_{\substack{|a|=1 \\ |ya|=|y|+1}} \sum_{x,z \in \Gamma(ya)} c_x \overline{c_z} \phi(y^{-1}x) \overline{\phi(y^{-1}z)} \\ = \sum_{y,a} \left| \sum_{x \in \Gamma(ya)} c_x \phi(y^{-1}x) \right|^2,$$

that is

$$0 \leq \left\| \sum_x c_x \pi(x) \delta_e \right\|_{\ell^2(\Gamma)}^2 - \left\| \sum_x c_x \pi_\phi(x) v_0 \right\|_{\mathcal{H}_\phi}^2 \\ = \sum_{y,a} |\phi(a)|^2 \left| \sum_{x \in \Gamma(ya)} c_x \phi(a^{-1}y^{-1}x) \right|^2$$

and letting $y' = ya$

$$= \sum_a |\phi(a)|^2 \sum_{y' \in \tilde{\Gamma}(a)} \left| \sum_{x \in \Gamma(y')} c_x \phi(y'^{-1}x) \right|^2 \\ \leq \Phi^2 \sum_{y'} \left| \sum_{x \in \Gamma(y')} c_x \phi(y'^{-1}x) \right|^2 \leq \Phi^2 \left\| \sum_x c_x \pi(x) \delta_e \right\|_{\ell^2(\Gamma)}^2.$$

□

As a corollary of Lemma 2 we can state

THEOREM 2. *Suppose that $\sum_{a \in A} |\phi(a)|^2 = +\infty$ and let π_ϕ be as above. Then π_ϕ is irreducible. Moreover there exists an inner product $(\cdot, \cdot)_T$ on $\ell^2(\Gamma)$ so that $(\cdot, \cdot)_T$ is equivalent to the usual ℓ^2 inner product and the original π is unitary with respect to $(\cdot, \cdot)_T$.*

Proof. Lemma 2 ensures that T extends uniquely to a linear equivalence between $\ell^2(\Gamma)$ and \mathcal{H}_ϕ . Since T intertwines π to π_ϕ , any nontrivial closed invariant subspace of \mathcal{H}_ϕ comes from a nontrivial closed invariant subspace of $\ell^2(\Gamma)$ and the first assertion is a consequence of Theorem 1.

To construct $(\cdot, \cdot)_T$ set

$$(f, g)_T = (T^*Tf, g)$$

where T^* is the adjoint of T and (\cdot, \cdot) is the ℓ^2 inner product.

$$\begin{aligned} (\pi(x)f, \pi(x)g)_T &= (T^*T\pi(x)f, \pi(x)g) = (T\pi(x)f, T\pi(x)g)_{\mathcal{H}_\phi} \\ &= (\pi_\phi(x)Tf, \pi_\phi(x)Tg)_{\mathcal{H}_\phi} = (Tf, Tg)_{\mathcal{H}_\phi} \\ &= (T^*Tf, g) = (f, g)_T . \end{aligned}$$

□

2. The case $\sum_{a \in A} |\phi(a)|^2 < +\infty$. Throughout this section we shall assume that $\sum_{a \in A} |\phi(a)|^2$ is finite. We observe that all the constructions of the previous section are still valid, but in this case the representation π will not be irreducible. Using the arguments of [8, Lemma II.1.5] and the monotonic convergence theorem it is easy to show that ϕ will be in $\ell^2(\Gamma)$ if and only if $\sum_{a \in A} \frac{|\phi(a)|^2}{1+|\phi(a)|^2} < 1$. Since no ℓ^2 function can be the coefficient of an irreducible representation we are really interested in the case

$$\sum_{a \in A} \frac{|\phi(a)|^2}{1 + |\phi(a)|^2} \geq 1 .$$

Although the case $\sum_{a \in A} \frac{|\phi(a)|^2}{1+|\phi(a)|^2} = 1$ requires more work, we shall see that in that case too ϕ is a coefficient of the regular representa-

tion. First let us consider the opposite case:

$$(2.1) \quad \sum_{a \in A} \frac{|\phi(a)|^2}{1 + |\phi(a)|^2} > 1 .$$

Given (2.1) it is possible to choose a large finite subset A_n of A and a positive ϵ so that

$$\sum_{a \in A_n} \frac{\exp(-2\epsilon)|\phi(a)|^2}{1 + \exp(-2\epsilon)|\phi(a)|^2} > 1$$

telling us that the function $\exp(-\epsilon|x|)\phi(x)$ does not belong to $\ell^2(\Gamma_n)$ where Γ_n is the subgroup generated by A_n . According to Haagerup's result ϕ is not a coefficient of a representation weakly contained in the regular. We shall prove the following

THEOREM 3. *Suppose that (2.1) holds. Then there exists a multiplicative positive definite function $\tilde{\phi}$ and a real number $\kappa > 1$ so that $\phi - \frac{\tilde{\phi}}{\kappa}$ is positive definite and $\tilde{\phi}$ is in $\ell^2(\Gamma)$.*

We were led to this idea by the following facts: when ϕ is radial and the set A is finite the representation π (to be precise we need to conjugate π with a bounded operator to make it unitary) breaks up into two pieces as proved in [14]. One piece is just a subrepresentation of the regular representation and the other gives a representation which is equivalent to one of the complementary spherical series representations of Figà-Talamanca–Picardello [5] as shown in [11]. It is natural to think that the present case will be analogous to the radial case: the spherical series will be replaced by the *anisotropic series* of Figà-Talamanca–Steger [8]. Since ϕ is not weakly contained in the regular representation we expect to see a complementary anisotropic series representation. Both the complementary and the principal anisotropic series can be realized on some suitable space of functions defined on Ω , the boundary of the tree associated with Γ . Let $\mathbf{1}$ be the function identically one on Ω . For any π in either of the anisotropic series there exists a probability measure μ on A and a real number λ_0 so that

$$\pi(\mu)\mathbf{1} = \lambda_0\mathbf{1} .$$

The matrix coefficient $\langle \pi(x)\mathbf{1}, \mathbf{1} \rangle$ is proportional to the difference of two functions g_π and \tilde{g}_π which are each Green functions for μ and λ_0 . That is, g_π and \tilde{g}_π both satisfy

$$g * (\lambda_0 \delta_e - \mu) = \delta_e .$$

Each of the Green functions is a constant times a multiplicative function.

Keeping this in mind let us return to our case. First we shall look for a measure μ supported on A and for a real λ_0 so that ϕ will satisfy the resolvent condition with respect to μ and λ_0 . Second, we shall look for another function $\tilde{\phi}$ satisfying the same condition. (2.1) will imply that $\tilde{\phi}$ exists and is in $\ell^2(\Gamma)$.

Any complex multiplicative ϕ can be written as the product of a unitary character χ with $|\phi|$. If ϕ is never zero, then χ is determined by $\chi(x) = \phi(x)/|\phi(x)|$, and is a character because we assume $\phi(x^{-1}) = \overline{\phi(x)}$. If $\phi(a) = 0$ for some $a \in A$, then $\chi(a)$ can be chosen arbitrarily. From this point on we will assume ϕ is nonnegative.

First of all we need μ . We want μ and ϕ to satisfy

$$(2.2) \quad \begin{aligned} \phi * \mu(x) &= \lambda_0 \phi(x) && \text{for } x \neq e \\ \phi * \mu(e) &= \lambda_0 \phi(e) - 1 \end{aligned}$$

for some positive constant λ_0 . Letting $x = b$ ($b \in A$) in (2.2) we get

$$\mu(b)(\phi(b)^{-1} - \phi(b^{-1})) + \sum_{a \in A} \phi(a^{-1})\mu(a) = \lambda_0 .$$

Consequently we set

$$(2.3) \quad \mu(b) = \frac{1}{\phi(b)^{-1} - \phi(b^{-1})} = \frac{\phi(b)}{1 - \phi(b)^2}$$

$$(2.4) \quad \lambda_0 = 1 + \sum_{a \in A} \frac{\phi(a)^2}{1 - \phi(a)^2} .$$

Since ϕ is nonnegative, μ is a positive measure. If μ were in $\ell^1(\Gamma)$ one might wish to normalize it to be a probability measure. In general, however, $\mu \notin \ell^1(\Gamma)$. On the other hand

$$\sum_{b \in A} |\mu(b)|^2 = \sum_{b \in A} \frac{\phi(b)^2}{(1 - \phi(b)^2)^2} \leq \frac{1}{(1 - \Phi^2)^2} \sum_{b \in A} \phi(b)^2 < +\infty .$$

As μ is supported on A , Haagerup's inequality says that μ is a bounded convolver of $\ell^2(\Gamma)$. A similar calculation shows that the convolution $\phi * \mu$ exists pointwise.

Look now for another real multiplicative function $\tilde{\phi}$ satisfying $\tilde{\phi}(a) = \tilde{\phi}(a^{-1})$ and

$$(2.2') \quad \begin{aligned} \tilde{\phi} * \mu(x) &= \lambda_0 \tilde{\phi}(x) && \text{for } x \neq e \\ \tilde{\phi} * \mu(e) &= \lambda_0 \tilde{\phi}(e) - \kappa \end{aligned}$$

with μ and λ_0 given by (2.3) and (2.4) and for a suitable constant κ to be found. We must solve

$$\begin{aligned} \mu(b)(\tilde{\phi}(b)^{-1} - \tilde{\phi}(b^{-1})) + \sum_{a \in A} \mu(a) \tilde{\phi}(a^{-1}) &= \lambda_0 \\ \sum_{a \in A} \mu(a) \tilde{\phi}(a^{-1}) &= \lambda_0 - \kappa \end{aligned}$$

giving

$$\kappa = \mu(b) \left(\tilde{\phi}(b)^{-1} - \tilde{\phi}(b^{-1}) \right)$$

for all $b \in A$. If such a solution exists, it must be given by

$$(2.5) \quad \tilde{\phi}_\kappa(a) = \frac{-\kappa(\phi(a)^{-1} - \phi(a)) + \sqrt{\kappa^2(\phi(a)^{-1} - \phi(a))^2 + 4}}{2}$$

and it must be that

$$(2.6) \quad \begin{aligned} \lambda_0 &= \kappa + \sum_{a \in A} \mu(a) \tilde{\phi}_\kappa(a) \\ &= \kappa + \sum_{a \in A} \frac{1}{2} \left(-\kappa + \sqrt{\kappa^2 + \frac{4}{(\phi(a)^{-1} - \phi(a))^2}} \right) \\ &= \kappa + \sum_{a \in A} 2 \left(\frac{\phi(a)}{1 - \phi(a)^2} \right)^2 \frac{1}{\kappa + \sqrt{\kappa^2 + \frac{4}{(\phi(a)^{-1} - \phi(a^{-1}))^2}}} . \end{aligned}$$

If we think of the right hand side of (2.6) as a function $\lambda = \lambda(\kappa)$, we recognize $\lambda(\kappa)$ as the convex function used by Akemann and Ostrand [2] who obtained the convolver norm of μ as the minimum

of $\lambda(\kappa)$ ($\kappa > 0$). One may verify that $\lambda(\kappa)$ and $\lambda'(\kappa)$ exist for $\kappa > 0$. We must now guarantee that (2.6) has a solution for $\lambda_0 = 1 + \sum_{a \in A} \frac{\phi(a)^2}{1 - \phi(a)^2}$.

Since this value is attained for $\kappa = 1$, since $\lambda(\kappa)$ is continuous, and since $\lambda(\kappa)$ approaches $+\infty$ as κ approaches $+\infty$, we only need to check the derivative of λ at the point $\kappa = 1$. A direct calculation using the identity

$$\sqrt{\kappa^2 + \frac{4}{(\phi(a)^{-1} - \phi(a^{-1}))^2}} = (\tilde{\phi}_\kappa(a) + \tilde{\phi}_\kappa(a)^{-1})\mu(a)$$

gives the following formula for $\lambda'(\kappa)$:

$$\lambda'(\kappa) = 1 - \sum_{a \in A} \frac{\tilde{\phi}_\kappa(a)^2}{1 + \tilde{\phi}_\kappa(a)^2}.$$

At $\kappa = 1$, $\tilde{\phi}_\kappa = \phi$, so

$$\lambda'(1) = 1 - \sum_{a \in A} \frac{\phi(a)^2}{1 + \phi(a)^2} < 0.$$

Hence ϕ is the solution of (2.6) which corresponds to a point with negative derivative. Therefore we may be sure that there exists $\kappa_0 > 1$ so that $\lambda(\kappa_0)$ is equal to the value λ_0 given in (2.4).

Define κ_0 to be the unique solution bigger than 1 of (2.6). Let $\tilde{\phi} = \tilde{\phi}_{\kappa_0}$ be the multiplicative function defined by (2.5) and see that $\tilde{\phi}$ satisfies (2.2'). By convexity of $\lambda(\kappa)$

$$\left(1 - \sum_{a \in A} \frac{\tilde{\phi}(a)^2}{1 + \tilde{\phi}(a)^2}\right) = \lambda'(\kappa_0) > 0$$

and so $\tilde{\phi} \in \ell^2(\Gamma)$.

Proof of Theorem 3. Let

$$\eta = \frac{\kappa_0 \phi - \tilde{\phi}}{\kappa_0 - 1}.$$

We will show that η is positive definite. For $a \in A$, $\phi(a) = 0$ if and only if $\mu(a) = 0$ if and only if $\tilde{\phi}(a) = 0$. Therefore η is supported on the subgroup generated by those a such that $\phi(a) \neq 0$. It is enough

to prove η positive definite on that subgroup. Therefore, passing to the subgroup, we may assume that $\phi(a)$ is never 0. The proof proceeds in three steps.

Step 1: Construct an algebraic (untopologized) representation α of Γ which has η as a matrix coefficient.

Step 2: Construct an inner product on the representation space which is preserved by the α -action of Γ .

Step 3: Demonstrate that the inner product is positive semidefinite.

Our algebraic representation will act on a certain subspace $\mathcal{H}^\infty \subseteq C(\Omega)$ where Ω is the boundary of Γ . Although μ is not necessarily a finite measure on A and although Ω is not even locally compact, the main steps of the construction of the complementary series given in [8] are still valid.

Step 1. To construct Ω think of Γ as acting on a homogeneous (locally infinite) tree \mathcal{T} . Identify Γ , as a set, with the set of vertices of \mathcal{T} , the action being given by left multiplication. Let Ω consist of all equivalence classes of half infinite geodesics (two geodesics are said to be equivalent if, up to a shift, they coincide). Any equivalence class has exactly one representative starting at the identity e . Denote by $[e, \omega]$ such a geodesic. Choosing $[e, \omega]_{\omega \in \Omega}$ as a set of representatives, we may also identify Ω with the set of all infinite reduced words. Refer to [7] for unexplained details. Fix $x \in \Gamma$. Let

$$\Omega(x) = \{\omega \in \Omega : [e, \omega] \text{ contains } x\} .$$

$\Omega(x)$ can also be thought of as the set of all infinite reduced words starting with x . Let $\{\Omega(x)\}_{x \in \Gamma}$ constitute a basis for the topology of Ω . Define a probability measure ν on the sets $\Omega(x)$ by letting

$$\nu(\Omega(a)) = \nu_a = \frac{\phi(a)\tilde{\phi}(a)}{1 + \phi(a)\tilde{\phi}(a)}$$

and

$$(2.7) \quad \nu(\Omega(xa)) = \phi(x)\tilde{\phi}(x)\nu_a .$$

Since $\Omega(x)$ is the disjoint union of the sets $(\Omega(xa))_{|xa|=|x|+1}$ we need

$$(2.8) \quad \nu(\Omega(x)) = \sum_{\substack{a \in A \\ |xa|=|x|+1}} \nu(\Omega(xa)) .$$

To verify this we only need to show that

$$(2.9) \quad \sum_{a \in A} \frac{\phi(a)\tilde{\phi}(a)}{1 + \phi(a)\tilde{\phi}(a)} = 1 .$$

Now

$$(2.10) \quad \sum_{a \in A} \frac{\phi(a)\tilde{\phi}(a)}{1 + \phi(a)\tilde{\phi}(a)} = \sum_{a \in A} \frac{\phi(a) - \tilde{\phi}(a)}{(\tilde{\phi}(a)^{-1} - \tilde{\phi}(a)) - (\phi(a)^{-1} - \phi(a))} ,$$

$$(2.11) \quad \mu(a)(\phi(a)^{-1} - \phi(a)) = 1 \quad \text{and} \quad \mu(a)(\tilde{\phi}(a)^{-1} - \tilde{\phi}(a)) = \kappa_0 .$$

Applying (2.11) to the right hand side of (2.10) we get

$$\frac{\sum_{a \in A} \mu(a)(\phi(a) - \tilde{\phi}(a))}{\kappa_0 - 1} = \frac{(\lambda_0 - 1) - (\lambda_0 - \kappa_0)}{\kappa_0 - 1} = 1$$

since ϕ and $\tilde{\phi}$ satisfy respectively (2.2) and (2.2'). That ν is a probability measure also follows from (2.9).

If $f(\omega)$ is a bounded complex valued function depending only on the first n letters of ω , define

$$\int_{\Omega} f(\omega) d\nu(\omega) = \sum_{|x|=n} f(\Omega(x))\nu(\Omega(x)) .$$

This is a proper definition in the sense that two values of n lead always to the same value for the integral of f . Indeed, one may use (2.8) to verify that n and $n + 1$ give the same integral. It is possible, although not necessary in what follows, to extend ν to the σ -algebra generated by the sets $\{\Omega(x)\}_{x \in \Gamma}$.

Let $\mathbf{1}$ denote the function identically one on Ω and let $\mathbf{1}_x$ denote the characteristic function of the set $\Omega(x)$. Let \mathcal{H}^∞ consist of the linear span of the $\mathbf{1}_x$. For F and $G \in \mathcal{H}^\infty$, define

$$\langle F, G \rangle = \int_{\Omega} F(\omega)\overline{G(\omega)} d\nu(\omega) .$$

For $a \in A$ let

$$(2.12) \quad P(a, \omega) = \begin{cases} \phi(a)^{-1} & \text{if } \omega \in \Omega(a) \\ \phi(a) & \text{if } \omega \notin \Omega(a) . \end{cases}$$

Observe that $P(a^{-1}, \omega)P(a, a\omega) = 1$. Since the free group is generated by the $a \in A$ with no relations except $a^{-1}a = e$, it is possible to extend $P(a, \omega)$ to a function $P(x, \omega)$ on $\Gamma \times \Omega$ satisfying the cocycle identity:

$$(2.13) \quad P(xy, \omega) = P(x, \omega)P(y, x^{-1}\omega).$$

The outcome is that if $x = a_1 \dots a_n \in \Gamma$ and $\omega \in \Omega$ have k and only k first letters in common, then

$$(2.14) \quad P(x, \omega) = \phi(a_1 \dots a_k)^{-1} \phi(a_{k+1} \dots a_n).$$

Observe that $P(x, \omega)$ depends only on the first $|x|$ letters of ω .

For $F \in \mathcal{H}^\infty$ define

$$(\alpha(x)F)(\omega) = P(x, \omega)F(x^{-1}\omega)$$

and verify that it gives a representation of Γ on \mathcal{H}^∞ . We claim that

$$(2.15) \quad \eta(x) = \langle \alpha(x)\mathbf{1}, \mathbf{1} \rangle = \int_{\Omega} P(x, \omega) d\nu.$$

(2.15) is obviously true when $x = e$. Suppose now that it holds for $|x| \leq n$ and let $y = xa$ with $a \in A$ and $|y| = |x| + 1$. By (2.12), the definition of $P(a, \omega)$, and (2.13), the cocycle identity,

$$P(xa, \omega) = \begin{cases} \phi(a)^{-1}P(x, \omega) & \text{if } \omega \in \Omega(xa) \\ \phi(a)P(x, \omega) & \text{if } \omega \in \Omega \setminus \Omega(xa). \end{cases}$$

So

$$\begin{aligned} & \int_{\Omega} P(xa, \omega) d\nu \\ &= \phi(a)^{-1} \int_{\Omega(xa)} P(x, \omega) d\nu + \phi(a) \int_{\Omega \setminus \Omega(xa)} P(x, \omega) d\nu \\ & \quad + \phi(a) \int_{\Omega(xa)} P(x, \omega) d\nu - \phi(a) \int_{\Omega(xa)} P(x, \omega) d\nu \\ &= \phi(a) \frac{\kappa_0 \phi(x) - \tilde{\phi}(x)}{\kappa_0 - 1} + (\phi(a)^{-1} - \phi(a)) \int_{\Omega(xa)} P(x, \omega) d\nu \\ &= \phi(a) \frac{\kappa_0 \phi(x) - \tilde{\phi}(x)}{\kappa_0 - 1} \\ & \quad + (\phi(a)^{-1} - \phi(a)) \phi(x)^{-1} \phi(x) \tilde{\phi}(x) \frac{\phi(a) \tilde{\phi}(a)}{1 + \phi(a) \tilde{\phi}(a)} \\ &= \phi(a) \frac{\kappa_0 \phi(x) - \tilde{\phi}(x)}{\kappa_0 - 1} + (\phi(a)^{-1} - \phi(a)) \tilde{\phi}(x) \frac{(\phi(a) - \tilde{\phi}(a)) \mu(a)}{\kappa_0 - 1} \end{aligned}$$

where for the last equality we wrote $\frac{\phi(a)\tilde{\phi}(a)}{1+\phi(a)\tilde{\phi}(a)}$ as in (2.10) and used (2.11). Therefore

$$\begin{aligned} \int_{\Omega} P(xa, \omega) d\nu &= \phi(a) \frac{\kappa_0 \phi(x) - \tilde{\phi}(x)}{\kappa_0 - 1} + \tilde{\phi}(x) \frac{\phi(a) - \tilde{\phi}(a)}{\kappa_0 - 1} \\ &= \frac{\kappa_0 \phi(xa) - \tilde{\phi}(xa)}{\kappa_0 - 1} \end{aligned}$$

as desired.

Step 2. If $\alpha(x)$ preserved the inner product on \mathcal{H}^∞ , then (2.15) would imply immediately that η was positive definite. Our actual next step is to construct another representation $\tilde{\alpha}$ so that

$$(2.16) \quad \langle \alpha(x)F, \tilde{\alpha}(x)G \rangle = \langle F, G \rangle .$$

Set

$$\tilde{P}(a, \omega) = \begin{cases} \tilde{\phi}(a)^{-1} & \text{if } \omega \in \Omega(a) \\ \tilde{\phi}(a) & \text{if } \omega \notin \Omega(a) . \end{cases}$$

Extend this to a definition of $\tilde{P}(x, \omega)$ using the cocycle identity (2.13). Define

$$(\tilde{\alpha}(x)F)(\omega) = \tilde{P}(x, \omega)F(x^{-1}\omega).$$

We compute the Radon–Nikodym derivative $d\nu(x^{-1}\omega)/d\nu(\omega)$:

$$\frac{d\nu(x^{-1}\omega)}{d\nu(\omega)} = P(x, \omega)\tilde{P}(x, \omega) .$$

For $x = a \in A$, this follows from (2.7). To extend to all x , use the cocycle identity, which applies to the Radon–Nikodym derivative as well as to P and \tilde{P} . Verify (2.16) by

$$\begin{aligned} \int_{\Omega} F(\omega)\overline{G(\omega)} d\nu(\omega) &= \int_{\Omega} F(x^{-1}\omega)\overline{G(x^{-1}\omega)}P(x, \omega)\tilde{P}(x, \omega) d\nu(\omega) \\ &= \int_{\Omega} (\alpha(x)F)(\omega)\overline{(\tilde{\alpha}(x)G)(\omega)} d\nu(\omega) . \end{aligned}$$

From (2.15) follows

(2.15')

$$\langle \tilde{\alpha}(x)\mathbf{1}, \mathbf{1} \rangle = \langle \mathbf{1}, \tilde{\alpha}(x) \rangle^- = \langle \alpha(x^{-1})\mathbf{1}, \mathbf{1} \rangle^- = \bar{\eta}(x^{-1}) = \eta(x) .$$

Comparison of (2.15) and (2.15') suggests the construction of an intertwining map I from \mathcal{H}^∞ to \mathcal{H}^∞ satisfying

$$(2.17) \quad I(\mathbf{1}) = \mathbf{1}$$

$$(2.18) \quad I\alpha(x) = \tilde{\alpha}(x)I.$$

For $|xa| = |x| + 1$ a direct computation using (2.11) shows that

$$(2.19) \quad \begin{aligned} \mathbf{1}_{xa} &= \mu(a)\phi(x)(P(xa, \cdot) - \phi(a)P(x, \cdot)) \\ &= \mu(a)\phi(x)(\alpha(xa)\mathbf{1} - \phi(a)\alpha(x)\mathbf{1}) \\ \mathbf{1}_{xa} &= \frac{1}{\kappa_0}\mu(a)\tilde{\phi}(x)\left(\tilde{P}(xa, \cdot) - \tilde{\phi}(a)\tilde{P}(x, \cdot)\right) \\ &= \frac{1}{\kappa_0}\mu(a)\tilde{\phi}(x)(\tilde{\alpha}(xa)\mathbf{1} - \tilde{\phi}(a)\tilde{\alpha}(x)\mathbf{1}). \end{aligned}$$

Therefore $\mathbf{1}$ is (algebraically) cyclic for α and for $\tilde{\alpha}$. Since we wish $I : \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ to satisfy (2.17) and (2.18), let us try to define it by

$$(2.20) \quad I(\alpha(f)\mathbf{1}) = \tilde{\alpha}(f)\mathbf{1} \quad \text{for any finitely supported function } f \text{ on } \Gamma.$$

If $\alpha(f)\mathbf{1} = 0$, then for any $x \in \Gamma$

$$\begin{aligned} 0 &= \langle \alpha(f)\mathbf{1}, \tilde{\alpha}(\delta_x)\mathbf{1} \rangle = \langle \alpha(\delta_x^{-1} * f)\mathbf{1}, \mathbf{1} \rangle \\ &= \langle \tilde{\alpha}(\delta_x^{-1} * f)\mathbf{1}, \mathbf{1} \rangle = \langle \tilde{\alpha}(f)\mathbf{1}, \alpha(\delta_x)\mathbf{1} \rangle \end{aligned}$$

using the fact that $\langle \alpha(x)\mathbf{1}, \mathbf{1} \rangle = \eta(x) = \langle \tilde{\alpha}(x)\mathbf{1}, \mathbf{1} \rangle$. Since $\mathbf{1}$ is cyclic for α , it follows that $\tilde{\alpha}(f)\mathbf{1} = 0$. Therefore (2.20) is a meaningful definition on its domain. It is also clear that that domain is all of \mathcal{H}^∞ and that conditions (2.17) and (2.18) follow from definition (2.20).

Let f and g be two finitely supported functions on Γ . The following calculation establishes that I is self-adjoint.

$$\begin{aligned} \langle \alpha(f)\mathbf{1}, I\alpha(g)\mathbf{1} \rangle &= \langle \alpha(f)\mathbf{1}, \tilde{\alpha}(g)\mathbf{1} \rangle \\ &= \langle \alpha(g^* * f)\mathbf{1}, \mathbf{1} \rangle = \langle \tilde{\alpha}(g^* * f)\mathbf{1}, \mathbf{1} \rangle \\ &= \langle \tilde{\alpha}(f)\mathbf{1}, \alpha(g)\mathbf{1} \rangle = \langle I\alpha(f)\mathbf{1}, \alpha(g)\mathbf{1} \rangle. \end{aligned}$$

Define

$$(2.21) \quad \langle F, G \rangle_I = \langle F, IG \rangle.$$

From (2.16) and (2.18) it follows that $\alpha(x)$ preserves this new inner product. From (2.15) and (2.17) follows $\langle \alpha(x)\mathbf{1}, \mathbf{1} \rangle_I = \langle \alpha(x)\mathbf{1}, \mathbf{1} \rangle = \eta(x)$.

Step 3. It remains to show that $\langle \cdot, \cdot \rangle_I$ is positive semidefinite. We proceed to diagonalize I . Set

$$(2.22) \quad \begin{aligned} F_e &= \mathbf{1} \\ F_{xa} &= \frac{1}{\nu(\Omega(xa))} \mathbf{1}_{xa} - \frac{1}{\nu(\Omega(x))} \mathbf{1}_x \quad \text{for } |xa| = |x| + 1. \end{aligned}$$

We shall see that each F_y is an eigenfunction for I .

Consider F_{xa} with $|x| = n$. The integral of F_{xa} over $\Omega(w)$ is zero for any w of length n . For any z with $|z| \leq n$ the value of $\tilde{P}(z, \omega)$ depends only on the first n letters of ω . Therefore

$$\begin{aligned} 0 &= \int_{\Omega} F_{xa}(\omega) \tilde{P}(z, \omega) d\nu(\omega) = \langle F_{xa}, \tilde{\alpha}(z)\mathbf{1} \rangle \\ &= \langle F_{xa}, I\alpha(z)\mathbf{1} \rangle = \langle IF_{xa}, \alpha(z)\mathbf{1} \rangle. \end{aligned}$$

Using (2.19), this implies in turn that

$$(2.23) \quad 0 = \langle IF_{xa}, \mathbf{1}_w \rangle = \int_{\Omega(w)} IF_{xa}(\omega) d\nu(\omega)$$

for any w of length n .

Suppose that $n = |x| > 0$ and let $x = yb$ where b is the final letter of x . According to (2.19) F_{xa} is a linear combination of $\alpha(y)\mathbf{1}$, $\alpha(x)\mathbf{1}$, and $\alpha(xa)\mathbf{1}$. Consequently IF_{xa} is a linear combination of $\tilde{\alpha}(y)\mathbf{1}$, $\tilde{\alpha}(x)\mathbf{1}$, and $\tilde{\alpha}(xa)\mathbf{1}$, that is $\tilde{P}(y, \cdot)$, $\tilde{P}(x, \cdot)$, and $\tilde{P}(xa, \cdot)$. The form of $\tilde{P}(z, \omega)$ is analogous to that given by (2.14) for $P(z, \omega)$. It follows that IF_{xa} is constant on $\Omega(w)$ for any $w \neq x$ of length n , constant on $\Omega(x) \setminus \Omega(xa)$, and constant on $\Omega(xa)$. Together with (2.23), this implies that IF_{xa} equals some multiple of F_{xa} . If $n = 0$, $x = e$, an almost identical argument leads to the same conclusion.

For what eigenvalue is F_{xa} an eigenvector?

$$\langle F_{xa}, \alpha(xa)\mathbf{1} \rangle = \int_{\Omega} F_{xa}(\omega) P(xa, \omega) d\nu(\omega)$$

$$\begin{aligned} &= \nu(\Omega(xa)) \left(\frac{1}{\nu(\Omega(xa))} - \frac{1}{\nu(\Omega(x))} \right) \phi(x)^{-1} \phi(a)^{-1} \\ &\quad - (\nu(\Omega(x)) - \nu(\Omega(xa))) \frac{1}{\nu(\Omega(x))} \phi(x)^{-1} \phi(a) \\ &= \left(1 - \frac{\nu(\Omega(xa))}{\nu(\Omega(x))} \right) \phi(x)^{-1} (\phi(a)^{-1} - \phi(a)) , \end{aligned}$$

a direct calculation using (2.14) and (2.22). Similarly

$$\begin{aligned} \langle IF_{xa}, \alpha(xa)\mathbf{1} \rangle &= \langle F_{xa}, I\alpha(xa)\mathbf{1} \rangle = \langle F_{xa}, \tilde{\alpha}(xa)\mathbf{1} \rangle \\ &= \left(1 - \frac{\nu(\Omega(xa))}{\nu(\Omega(x))} \right) \tilde{\phi}(x)^{-1} (\tilde{\phi}(a)^{-1} - \tilde{\phi}(a)) . \end{aligned}$$

It follows, making use of (2.11), that $IF_{xa} = c_x F_{xa}$ with

$$c_x = \frac{\tilde{\phi}(x)^{-1} (\tilde{\phi}(a)^{-1} - \tilde{\phi}(a))}{\phi(x)^{-1} (\phi(a)^{-1} - \phi(a))} = \kappa_0 \frac{\phi(x)}{\tilde{\phi}(x)} .$$

The positivity of the c_x gives us what we want. Fix any $G \in \mathcal{H}^\infty$. We can write

$$G = G_0 + \sum_{x \in \Gamma} G_x$$

where G_0 is a constant and each G_x is a linear combination of the F_{xa} for $a \in A$ such that $|xa| = |x| + 1$. Indeed, for $G = \mathbf{1}_z$ this decomposition is easily accomplished by induction on $|z|$. These terms are orthogonal to one another, and are, moreover, eigenvectors of I . Therefore

$$\langle G, G \rangle_I = \langle G, IG \rangle = \langle G_0, G_0 \rangle + \sum_{x \in \Gamma} c_x \langle G_x, G_x \rangle \geq 0 . \quad \square$$

In [1] Alesina and De Michele showed that under our hypotheses ϕ does not belong to the orthogonal complement $A^\perp(\Gamma)$ of the Fourier algebra $A(\Gamma)$ in the Fourier–Stieltjes algebra $B(\Gamma)$. Theorem 3 gives more, namely that the representation π_ϕ canonically associated with ϕ has a subrepresentation contained in the regular.

THEOREM 4. *Suppose that*

$$\sum_{a \in A} \frac{\phi(a)^2}{1 + \phi(a)^2} = 1 .$$

Then there exists a positive definite ψ in $\ell^2(\Gamma)$ so that $\phi = \psi * \psi$.

Proof of Theorem 3. We may, as in the proof of Theorem 2, multiply ϕ by a unitary character and so make it nonnegative. Let, as in (2.4), $\lambda_0 = 1 + \sum_{a \in A} \frac{\phi(a)^2}{1 - \phi(a)^2}$. For any $\kappa > 0$, construct $\tilde{\phi}_\kappa$ according to (2.5). Let

$$\begin{aligned} \lambda(\kappa) &= \kappa + \sum_{a \in A} \mu(a) \tilde{\phi}_\kappa(a) \\ &= \kappa + \sum_{a \in A} \frac{1}{2} \left(-\kappa + \sqrt{\kappa^2 + \frac{4}{(\phi(a)^{-1} - \phi(a))^2}} \right), \end{aligned}$$

the right hand side of (2.6). Then, by its construction, $\tilde{\phi}_\kappa$ satisfies the following version of (2.2').

$$(2.24) \quad \begin{aligned} \tilde{\phi}_\kappa * \mu(x) &= \lambda(\kappa) \tilde{\phi}_\kappa(x) \quad \text{for } x \neq e \\ \tilde{\phi}_\kappa * \mu(e) &= \lambda(\kappa) \tilde{\phi}_\kappa(e) - \kappa. \end{aligned}$$

As in the proof of Theorem 2 the function $\lambda(\kappa)$ is defined, differentiable, and strictly convex for $\kappa > 0$ with derivative

$$\lambda'(\kappa) = 1 - \sum_{a \in A} \frac{\tilde{\phi}_\kappa(a)^2}{1 + \tilde{\phi}_\kappa(a)^2}.$$

This derivative is zero by hypothesis for $\kappa = 1$, and so by convexity it is positive for $\kappa > 1$. According to Lemmas II.1.5–6 of [8], which extend easily to the case of infinite A , $\sum_{a \in A} \frac{\tilde{\phi}_\kappa(a)^2}{1 + \tilde{\phi}_\kappa(a)^2} < 1$ implies that $\tilde{\phi}_\kappa$ is a bounded convolver of $\ell^2(\Gamma)$. Equations (2.24) establish then that

$$\tilde{\phi}_\kappa = \kappa(\lambda(\kappa) - \mu)^{-1}$$

in the algebra of bounded convolvers of $\ell^2(\Gamma)$.

Denote by $\text{spec}(\mu)$ the spectrum of μ as a left convolution operator. By the above $\text{spec}(\mu) \subseteq (-\infty, \lambda_0]$. (It is easy to see but irrelevant to the proof that $\lambda_0 \in \text{spec}(\mu)$.) Let E be the resolution of the identity for μ on $\text{spec}(\mu)$. Let $d\sigma = dE_{\delta_e, \delta_e}$. By the spectral theorem (see Chapter 12 of [13])

$$\int_{\text{spec}(\mu)} \frac{1}{\lambda(\kappa) - t} d\sigma(t) = \left(\frac{\tilde{\phi}_\kappa}{\kappa} * \delta_e, \delta_e \right) = \frac{\tilde{\phi}_\kappa(e)}{\kappa} = \frac{1}{\kappa} \leq 1$$

for $\kappa > 1$. By the monotone convergence theorem

$$(2.25) \quad \int_{\text{spec}(\mu)} \frac{1}{\lambda_0 - t} d\sigma(t) \leq 1 .$$

Since μ acts as a left convolution operator, so does any polynomial in μ and so does any element of the C^* -algebra generated by μ . It follows that

$$dE_{\delta_x, \delta_x} = dE_{\delta_e, \delta_e} = d\sigma$$

for any $x \in \Gamma$. By the Cauchy–Schwarz inequality and the spectral theorem

$$\begin{aligned} & \left| \int_{\text{spec}(\mu)} f(t) dE_{\delta_e, \delta_x}(t) \right| \\ & \leq \left(\int_{\text{spec}(\mu)} |f(t)| dE_{\delta_e, \delta_e}(t) \right)^{1/2} \left(\int_{\text{spec}(\mu)} |f(t)| dE_{\delta_x, \delta_x}(t) \right)^{1/2} \\ & = \int_{\text{spec}(\mu)} |f(t)| d\sigma(t) \end{aligned}$$

for any continuous $f(t)$. It follows that $|dE_{\delta_e, \delta_x}| \leq d\sigma$. Therefore, using (2.25) and the dominated convergence theorem we may take the limit in

$$\int_{\text{spec}(\mu)} \frac{1}{\lambda(\kappa) - t} dE_{\delta_e, \delta_x}(t) = \left(\frac{\tilde{\phi}_\kappa}{\kappa} * \delta_e, \delta_x \right) = \frac{\tilde{\phi}_\kappa(x)}{\kappa}$$

and obtain $\int_{\text{spec}(\mu)} \frac{1}{\lambda_0 - t} dE_{\delta_e, \delta_x}(t) = \phi(x)$.

Let

$$h_N(t) = \begin{cases} (\lambda_0 - t)^{-1/2} & \text{if } \lambda_0 - t \geq 1/N \\ 0 & \text{if } \lambda_0 - t < 1/N. \end{cases}$$

The spectral theorem establishes that $\psi = \lim_{N \rightarrow \infty} h_N(\mu)\delta_e$ exists as a limit in $\ell^2(\Gamma)$. Moreover

$$\begin{aligned} \psi * \psi(x) &= \lim_{N \rightarrow \infty} \langle h_N^2(x)\delta_e, \delta_x \rangle \\ &= \lim_{N \rightarrow \infty} \int_{\text{spec}(\mu) \cap \{\lambda_0 - t \geq 1/N\}} \frac{1}{\lambda_0 - t} dE_{\delta_e, \delta_x}(t) = \phi(x) . \end{aligned}$$

□

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