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ON ALMOST-EVERYWHERE CONVERGENCE OF INVERSE SPHERICAL TRANSFORMS

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Suppose that G/K is a rank one noncompact connected Riemannian symmetric space. We show that if f is a bi-K-invariant square integrable function on G, then its inverse spherical transform converges almost everywhere.

1. Introduction.

Recall the Carleson-Hunt theorem about almost-everywhere convergence of the partial sums of the inverse Fourier transform in one dimension. If we take $1 \leq p \leq 2$ and denote by $\widehat{f}$ the Fourier transform of a function f in $L^p(\mathbb{R})$ then for each $R > 0$ there is the partial sum

$$S_R f(x) := \int_{-R}^{R} \widehat{f}(\xi) e^{ix\xi}\, d\xi. \tag{1}$$

There is also the maximal function

$$S^* f(x) := \sup_{R>0} |S_R f(x)|\,. \tag{2}$$

The Carleson-Hunt Theorem states that if $1 < p \leq 2$ then there is a constant $c_p > 0$ such that

$$\|S^* f\|_p \leq c_p \|f\|_p, \qquad \forall f \in L^p(\mathbb{R}). \tag{3}$$

When this is combined with the fact that the inverse Fourier transform converges everywhere for elements of $C_c^\infty(\mathbb{R})$, a dense subspace of $L^p(\mathbb{R})$, then the almost-everywhere convergence of $\{S_R f(x) : R > 0\}$ follows for all $f \in L^p(\mathbb{R})$. In fact, it suffices to know that there is the weak estimate on the truncated maximal operator for all $y > 0$ and $f \in L^p(\mathbb{R})$,

$$\left|\left\{x : \sup_{R>1} |S_R f(x) - S_1 f(x)| > y\right\}\right| \leq c_p \|f\|_p^p / y^p, \tag{4}$$

and this follows from (3). The inequality (3) has been extended to Hankel transforms by Kanjin [4] and Prestini [6], for an appropriate interval of values for p. In this paper we will be concentrating on the L^2 case.

2. Bessel functions and Hankel transforms.

For $\alpha > -1/2$ and $1 \le p \le 2$ consider the weighted Lebesgue space $L_{p,\alpha}(0,\infty)$ with norm

$$\|f\|_{p,\alpha} = \left(\sigma_\alpha \int_0^\infty |f(x)|^p x^{2\alpha+1}\, dx\right)^{1/p}.$$

Here $\sigma_\alpha = 2\pi^{\alpha+1}\Gamma(\alpha+1)$. Furthermore, there is the Hankel transform

$$\tau_\alpha f(y) = \int_0^\infty f(x) \frac{J_\alpha(xy)}{(xy)^\alpha} x^{2\alpha+1}\, dx,$$

where J_α is the usual Bessel function indexed by α. The corresponding maximal function for the inversion of this transform is

$$T_\alpha^* f(x) = \sup_{R>0} \left| \int_0^R \tau_\alpha f(y) \frac{J_\alpha(xy)}{(xy)^\alpha} y^{2\alpha+1}\, dy \right|.$$

Proposition 1 (Kanjin, Prestini). *For $\alpha \ge -1/2$ and*

$$4(\alpha+1)/(2\alpha+3) < p < 4(\alpha+1)/(2\alpha+1)$$

there is a constant $c_{p,\alpha}$ such that

$$\|T_\alpha^* f\|_{p,\alpha} \le c_{p,\alpha} \|f\|_{p,\alpha}, \qquad \forall f \in L_{p,\alpha}(0,\infty).$$

Following the notation of [**9**], we set

$$\mathcal{J}_\mu(z) := 2^{\mu-1}\Gamma\left(\mu+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) z^{-\mu} J_\mu(z).$$

We will make use of the following alternative formulation of the Hankel transform for L^2 spaces. Notice that $F \in L^2(0,\infty)$ if and only if

$$\|F\|_2^2 = \int_0^\infty \left|F(\lambda)\lambda^{-\alpha-1/2}\right|^2 \lambda^{2\alpha+1}\, d\lambda < \infty.$$

If $\lambda \mapsto F(\lambda)\lambda^{-\alpha-1/2}$ is in $L_{2,\alpha}(0,\infty)$ and $R > 1$ then we can take the partial Hankel transform

$$(5)\quad \int_1^R F(\lambda)\lambda^{-\alpha-1/2} \frac{J_\alpha(\lambda t)}{(\lambda t)^\alpha} \lambda^{2\alpha+1}\, d\lambda = t^{-\alpha-1/2} \int_1^R F(\lambda)\,(\lambda t)^{1/2} J_\alpha(\lambda t)\, d\lambda.$$

3. Spherical transforms.

3.1. Notation. First, G will denote a noncompact connected semisimple Lie group. Next, we fix a maximal compact subgroup K in G, and we assume that the rank of the symmetric space $K\backslash G$ is one. Furthermore, let n be the dimension of $K\backslash G$. We assume that an Iwasawa decomposition $G = ANK$ is fixed once and for all.

Let $\mathfrak{a}$ denote the Lie algebra of A inside $\mathfrak{g}$, so that $\mathfrak{a}$ is isomorphic to the real line. Following [**9**] we fix an element H_0 of $\mathfrak{a}$ so that $\mathfrak{a} = \mathbb{R}H_0$. There is the map from the real line onto A defined by $a(t) := \exp(tH_0)$, for all real numbers t. Every element of G can be written as $g = k_1 a(t) k_2$ for some k_1 and k_2 in K and $t \geq 0$. Hence, every bi-K-invariant function on G is completely determined by its restriction to the set $\{a(t) : t \geq 0\}$. There is a density D on $[0, \infty)$ which corresponds to the Haar measure on G,

$$\int_G f(x)\,dx = \int_0^\infty \int_K \int_K f(k_1 a(t) k_2) D(t)\,dk_1 dk_2 dt,$$

for all $f \in C_c(G)$. Let n be the dimension of the symmetric space $K\backslash G$, and let ρ denote the special number described in [**9**].

Lemma 1. *The density D on $[0, \infty)$ has the properties:*

$$D(t) = O(t^{n-1}) \quad \textit{as} \quad t \downarrow 0,$$

and

$$D(t) = O(e^{2\rho t}) \quad \textit{as} \quad t \to \infty.$$

3.2. Spherical Functions. To each complex number λ there is associated the spherical function φ_λ, which is a smooth bi-K-invariant function on G. If λ is real then φ_λ is bounded and there is the spherical transform

$$\mathfrak{F}f(\lambda) := \int_G f(x)\varphi_\lambda(x)\,dx$$

for all integrable functions on G. If we add the hypothesis that f is bi-K-invariant, then this reduces to a one-dimensional integral transform, namely,

$$\mathfrak{F}f(\lambda) := \int_0^\infty f(a(t))\varphi_\lambda(a(t))D(t)\,dt,$$

where D is the density used in equation (1.1) of [**9**]. It is known that there is a density $|c(\lambda)|^{-2}$ on $[0, \infty)$ so that the spherical transform extends from being a map $\mathfrak{F} : {}^K L^1(G)^K \cap L^2(G) \to C^\infty(0, \infty)$ to an isometry

$$\mathfrak{F} : {}^K L^2(G)^K \cong L^2([0, \infty), |c(\lambda)|^{-2}\,d\lambda).$$

This is the Plancherel theorem for bi-K-invariant functions [**1**]. It is also known that if $f \in {}^K C_c^\infty(G)^K$ then

$$f(x) = \lim_{R\to\infty} \int_0^R \mathfrak{F}f(\lambda)\varphi_\lambda(x)|c(\lambda)|^{-2}\, d\lambda$$

uniformly. Let $\mathcal{S}_R$ denote the partial summation operator. From the results of [**9**] about analyticity of spherical transforms, it is clear that $\mathcal{S}_R$ cannot be a bounded operator from ${}^K L^p(G)^K$ to ${}^K L^p(G)^K$, when $p < 2$. Despite this, Giulini and Mauceri have been able to treat some Riesz-Bochner means in this case, [**2**].

The analogue of the maximal function (2) is

$$\mathfrak{M}f(a(t)) := \sup_{R>0} \left| \int_0^R \mathfrak{F}f(\lambda)\varphi_\lambda(a(t))|c(\lambda)|^{-2}\, d\lambda \right| = \sup_{R>0} |\mathcal{S}_R f(a(t))|\,.$$

As we remarked above, to prove almost everywhere convergence, it is enough to consider the truncated version of this maximal function,

$$\mathfrak{M}^* f(a(t)) := \sup_{R>1} \left| \int_1^R \mathfrak{F}f(\lambda)\varphi_\lambda(a(t))|c(\lambda)|^{-2}\, d\lambda \right|. \tag{6}$$

We wish to understand the L^2 mapping properties of $\mathfrak{M}^*$. This will involve estimates on $\varphi_\lambda(a(t))$ for all $t > 0$ and large λ. These asymptotic results were found by Stanton and Tomas [**9**]. In [**5**] we use the results of Schindler[**8**] and direct estimates on the Dirichlet kernel to treat the case when $G = SL(2, \mathbb{R})$ and $K = SO(2)$. There we show that $\mathfrak{M}^*$ is bounded from ${}^K L^p(SL(2,\mathbb{R}))^K$ to $L^2 + L^p$, when $4/3 < p \le 2$.

4. Asymptotic results.

Theorem 2.1 of [**9**] gives the asymptotics of $\varphi_\lambda(a(t))$ for small values of t. In this case $\varphi_\lambda(a(t))$ behaves like a combination of Bessel functions.

Theorem 2. *There exist $B_0 > 1$ and $B_1 > 1$ such that for all $0 \le t \le B_0$,*

$$\begin{aligned}\varphi_\lambda(a(t)) = c_0 \left(\frac{t^{n-1}}{D(t)}\right)^{1/2} & \mathcal{J}_{(n-2)/2}(\lambda t) \\ & + c_0 \left(\frac{t^{n-1}}{D(t)}\right)^{1/2} t^2 a_1(t) \mathcal{J}_{n/2}(\lambda t) + E_2(\lambda, t)\end{aligned} \tag{7}$$

with $|a_1(t)| \le cB_1^{-1}$, for all $0 \le t \le B_0$, and

$$|E_2(\lambda,t)| \le \begin{cases} c_2 t^4 & \text{if } |\lambda t| \le 1 \\ c_2 t^4\, (\lambda t)^{-((n-1)/2+2)} & \text{if } |\lambda t| > 1. \end{cases}$$

Similarly, they have the case for large t. Following Harish-Chandra [**3**], they write

$$\varphi_\lambda(a(t)) = c(\lambda)e^{(i\lambda-\rho)t}\phi_\lambda(t) + c(-\lambda)e^{(-i\lambda-\rho)t}\phi_{-\lambda}(t)$$

so that

$$\varphi_\lambda(a(t)) = c(\lambda)e^{i\lambda t}e^{-\rho t} + c(-\lambda)e^{-i\lambda t}e^{-\rho t} + \text{error terms.}$$

Corollary 3.9 of [**9**] then describes the asymptotics of the functions ϕ_λ.

Proposition 3. *For integers $M > 0$ and $m \geq 0$, real numbers $t \geq B_0$, and real λ, there exist functions $\Lambda_m(\lambda, t)$ and $\mathcal{E}_{M+1}(\lambda, t)$ and a constant $A > 0$ such that*

$$\phi_\lambda(t) = \Lambda_0(t) + \sum_{m=1}^{\infty}\Lambda_m(\lambda,t)e^{-2mt} = \Lambda_0(t) + \sum_{m=1}^{M}\Lambda_m(\lambda,t)e^{-2mt} + \mathcal{E}_{M+1}(\lambda,t),$$

where $\Lambda_0(t) \leq AG_0(t)$,

$$|D_\lambda^\alpha \Lambda_m(\lambda,t)| \leq A\rho^m e^{2m}|\lambda|^{-(m+\alpha)}2^\alpha G_0(t)$$
$$|\mathcal{E}_{M+1}(\lambda,t)| \leq A\rho^{M+1}e^{2(M+1)}|\lambda|^{-(M+1)}G_0(t).$$

Here $G_0(t) = \sum_{j=0}^{\infty} e^{2j(1-t)}$.

The material at the top of page 260 in [**9**] shows that the $m = 0$ term in this expansion is independent of λ since the factors γ_0^k used there are constant in λ. Also notice that

$$G_0(t) = \sum_{j=0}^{\infty} e^{2j(1-t)} = \frac{1}{1-e^{2-2t}}, \qquad \forall t > 1. \tag{8}$$

In particular, G_0 is uniformly bounded on $[B_0, \infty)$.

We conclude this section by pointing out the long range behaviour of the c-functions, see Lemma 4.2 in [**9**].

Proposition 4. *For real λ and integers $\alpha \geq 0$,*

$$\left|D_\lambda^\alpha |c(\lambda)|^{-2}\right| \leq c_\alpha\,(1+|\lambda|)^{n-1-\alpha}.$$

In particular,

$$|c(\lambda)|^{-1} = O(\lambda^{(n-1)/2}), \quad \text{for large } \lambda. \tag{9}$$

Also note that

$$c(-\lambda) = \overline{c(\lambda)}, \qquad \forall \lambda \in \mathbb{R}.$$

This means that $c(\lambda)/|c(\lambda)|$ and $c(-\lambda)/|c(\lambda)|$ both have absolute value one.

5. The Main Theorem.

Theorem 1. *Suppose that G is a non-compact, connected, semisimple Lie group with finite centre and real rank one, with maximal compact subgroup K. For every bi-K-invariant square-integrable function f on G, the partial sums of the inverse spherical transform converge almost-everywhere on G.*

5.1. Transplanting to one dimension. To prove this result we transplant the problem to one about Hankel and Fourier transforms. This follows an idea found in Schindler's paper [**8**]. If f is a square-integrable bi-K-invariant function on G, set

$$\mathfrak{R}f(t) := (D(t))^{1/2} f(a(t)), \qquad \forall t > 0.$$

Immediately we see that $\mathfrak{R}f \in L^2(0,\infty)$ and

$$\|\mathfrak{R}f\|_{L^2(0,\infty)} = \|f\|_{L^2(G)}, \qquad \forall f \in {}^K L^2(G)^K. \tag{10}$$

For real numbers λ and $t > 0$, set

$$\psi_\lambda(t) := |c(\lambda)|^{-1} (D(t))^{1/2} \varphi_\lambda(a(t)),$$

and define an integral transform on functions on $(0,\infty)$ by

$$\mathcal{K}F(\lambda) := \int_0^\infty F(t)\psi_\lambda(t)\,dt, \qquad \forall \lambda > 0.$$

This has the properties that it is an isometry from $L^2(0,\infty)$ to itself and that

$$\mathcal{K}(\mathfrak{R}f)(\lambda) = |c(\lambda)|^{-1} \mathfrak{F}f(\lambda), \qquad \forall f \in {}^K L^2(G)^K. \tag{11}$$

Finally, notice that the maximal function we are interested in has the description as

$$\mathfrak{M}^* f(a(t)) = (D(t))^{-1/2} \sup_{R>1} \left| \int_1^R \mathcal{K}(\mathfrak{R}f)(\lambda)\psi_\lambda(t)\,d\lambda \right|. \tag{12}$$

We wish to prove that if $\mathfrak{R}f \in L^2(0,\infty)$ then $t \mapsto (D(t))^{1/2}\mathfrak{M}^* f(a(t))$ is in $L^2(0,\infty)$, which is the same as asking that

$$t \mapsto \sup_{R>1} \left| \int_1^R \mathcal{K}(\mathfrak{R}f)(\lambda)\psi_\lambda(t)\,d\lambda \right|$$

be square integrable on $[0, \infty)$. We also need to estimate the norm of this in terms of the norm of $\mathfrak{R}f$.

For the moment, replace $\mathcal{K}(\mathfrak{R}f)$ by an arbitrary $F \in L^2(0,\infty)$, with the same L^2-norm. Notice that F may be thought of as the restriction to $(0,\infty)$ of the Fourier transform of an element of $L^2(\mathbb{R})$. The results of Stanton and Tomas show that we can write $\psi_\lambda(t)$ in different ways, depending on the size of t. For $0 \le t \le B_0$ the expansion in Proposition 4 means that we have three pieces:

$$
\begin{aligned}
\text{(13)} \qquad \psi_\lambda(t) = {} & c_0 |c(\lambda)|^{-1} t^{(n-1)/2} \mathcal{J}_{(n-2)/2}(\lambda t) \\
& + c_0 |c(\lambda)|^{-1} t^{2+(n-1)/2} a_1(t) \mathcal{J}_{n/2}(\lambda t) \\
& + |c(\lambda)|^{-1} D(t)^{1/2} E_2(\lambda, t).
\end{aligned}
$$

For every $B_2 > B_0$ and $B_0 < t < B_2$ the expansion in Proposition 4 means that we can write $\psi_\lambda(t)$ as

$$
\begin{aligned}
\text{(14)} \qquad \psi_\lambda(t) = \sum_{\epsilon=\pm 1} \Bigg\{ & \frac{c(\epsilon\lambda)}{|c(\lambda)|} e^{\epsilon i\lambda t} e^{-\rho t} D(t)^{1/2} \Lambda_0(t) \\
& + \frac{c(\epsilon\lambda)}{|c(\lambda)|} e^{\epsilon i\lambda t} e^{-\rho t - 2t} D(t)^{1/2} \Lambda_1(\epsilon\lambda, t) \\
& + \frac{c(\epsilon\lambda)}{|c(\lambda)|} e^{\epsilon i\lambda t} e^{-\rho t} D(t)^{1/2} \mathcal{E}_2(\epsilon\lambda, t) \Bigg\}.
\end{aligned}
$$

The remaining case, when $t > B_2 > B_0$ is

$$
\begin{aligned}
\text{(15)} \quad \psi_\lambda(t) = {} & \frac{c(\lambda)}{|c(\lambda)|} e^{i\lambda t} e^{-\rho t} D(t)^{1/2} \Lambda_0(t) + \frac{c(-\lambda)}{|c(\lambda)|} e^{-i\lambda t} e^{-\rho t} D(t)^{1/2} \Lambda_0(t) \\
& + \frac{c(\lambda)}{|c(\lambda)|} e^{i\lambda t} e^{-\rho t} D(t)^{1/2} \sum_{m=1}^{\infty} \Lambda_m(\lambda, t) e^{-2mt} \\
& + \frac{c(-\lambda)}{|c(\lambda)|} e^{-i\lambda t} e^{-\rho t} D(t)^{1/2} \sum_{m=1}^{\infty} \Lambda_m(-\lambda, t) e^{-2mt}.
\end{aligned}
$$

Later we will fix one value for B_2 depending on the values of B_0, n, and ρ.

• **Case of small t, first piece.** Here we must estimate

$$
T_1(t) = \sup_{R>1} \left| \int_1^R F(\lambda) |c(\lambda)|^{-1} t^{(n-1)/2} J_{(n-2)/2}(\lambda t)(\lambda t)^{-(n-2)/2} \, d\lambda \right|
$$

with $0 \le t \le B_0$. Notice that $|c(\lambda)|^{-1} \le \text{const.}(1 + |\lambda|)^{(n-1)/2}$ and so the function

$$
F_1(\lambda) = F(\lambda) |c(\lambda)|^{-1} \lambda^{-(n-1)/2}
$$

is in $L^2(1,\infty)$ and $\|F_1\|_2 \le \text{const.}\|F\|_2$. Then we must estimate

$$\sup_{R>1}\left|\int_1^R F_1(\lambda)(\lambda t)^{1/2}J_{(n-2)/2}(\lambda t)\,d\lambda\right|.$$

See equation (5). The Kanjin-Prestini theorem implies that

$$t \mapsto \sup_{R>1}\left|\int_1^R F_1(\lambda)(\lambda t)^{1/2}J_{(n-2)/2}(\lambda t)\,d\lambda\right|.t^{-(n-2)/2-1/2}$$

is in $L_{2,(n-1)/2}(0,\infty)$ with norm less than or equal to a constant multiple of $\|F_1\|_2$, where the constant depends only on $K\backslash G$. But this means that

$$\left(\int_0^\infty |T_1(t)|^2\,dt\right)^{1/2} \le \text{const.}\|F\|_2.$$

This completes the necessary estimate on the first part.

• **Case of small t, second piece.** Next, set $T_2(t)$ to be equal to

$$\sup_{R>1}\left|\int_1^R F(\lambda)|c(\lambda)|^{-1}t^{2+(n-1)/2}a_1(t)J_{n/2}(\lambda t)(\lambda t)^{-n/2}\,d\lambda\right|.$$

This can be rearranged to become

$$ta_1(t)\sup_{R>1}\left|\int_1^R F(\lambda)\lambda^{-1}|c(\lambda)|^{-1}\lambda^{-(n-1)/2}J_{n/2}(\lambda t)(\lambda t)^{1/2}\,d\lambda\right|.$$

But $F_2(\lambda) = F(\lambda)\lambda^{-1}|c(\lambda)|^{-1}\lambda^{-(n-1)/2}$ is in $L^2(1,\infty)$ and

$$\|F_2\|_{L^2(1,\infty)} \le \text{const.}\|F\|_2.$$

Now apply the Kanjin-Prestini theorem to

$$\sup_{R>1}\left|\int_1^R F_2(\lambda)(t\lambda)^{1/2}J_{n/2}(\lambda t)\,d\lambda\right|.$$

We also know that a_1 is bounded on $[0,B_0]$. We have proved that

$$\left(\int_0^{B_0} |T_2(t)|^2\,dt\right)^{1/2} \le \text{const.}\|F\|_2.$$

• **Case of small t, third piece.** Set

$$T_3(t) = D(t)^{1/2}\sup_{R>1}\left|\int_1^R F(\lambda)|c(\lambda)|^{-1}E_2(\lambda,t)\,d\lambda\right|$$

for all $0 \le t \le B_0$. From the estimates for the error term described in Proposition 4 we see that $T_3(t)$ is less than or equal to

$$(16)\quad \text{const.}D(t)^{1/2}t^4 \int_1^{1/t} |F(\lambda)||c(\lambda)|^{-1}\, d\lambda + \\ \text{const.}D(t)^{1/2}t^{2-(n-1)/2} \int_{1/t}^{R} |F(\lambda)||c(\lambda)|^{-1}\lambda^{-(2+(n-1)/2)}\, d\lambda.$$

The first term is dominated by

$$\text{const.}D(t)^{1/2}t^4\|F\|_2 \left(\int_1^{1/t} \lambda^{n-1}\, d\lambda\right)^{1/2} \le \text{const.}D(t)^{1/2}t^4\|F\|_2(1-t^{-n})^{1/2}.$$

Recalling that $D(t) = O(t^{(n-1)})$ as $t \to 0$, we see that this is square integrable over $[0, B_0]$.

For the second term, use the fact that it is dominated by

$$(17)\quad \text{const.}D(t)^{1/2}t^{2-(n-1)/2} \int_{1/t}^{R} |F(\lambda)|\lambda^{-2}\, d\lambda \\ \le \text{const.}D(t)^{1/2}t^{2-(n-1)/2}\|F\|_2 \left(t^3 - \frac{1}{R^3}\right)^{1/2}.$$

This shows that

$$\left(\int_0^{B_0} |T_3(t)|^2\, dt\right)^{1/2} \le \text{const.}\|F\|_2.$$

• **Small t, summary.** So far, we have shown that there is a $B_0 > 1$ and a constant $c > 0$, depending on $K\backslash G$, such that for all f in ${}^K L^2(G)^K$,

$$(18)\qquad \left(\int_0^{B_0} |\mathfrak{M}^* f(a(t))|^2 D(t)\, dt\right)^{1/2} \le c\|f\|_2.$$

• **Case of medium size t** Using the results of Proposition 4 we see that if $B_0 < t < B_2$, then we need to estimate terms of the form

$$(19)\qquad t \mapsto \sup_{R>1} \left| \int_1^R F(\lambda)\frac{c(\lambda)}{|c(\lambda)|} e^{(i\lambda-\rho)t}(D(t))^{1/2}\Lambda_0(t)\, d\lambda \right|,$$

$$(20)\qquad t \mapsto \sup_{R>1} \left| \int_1^R F(\lambda)\frac{c(\lambda)}{|c(\lambda)|} e^{(i\lambda-\rho-2)t}(D(t))^{1/2}\Lambda_1(\lambda, t)\, d\lambda \right|,$$

and

$$t \mapsto \sup_{R>1} \left| \int_1^R F(\lambda) \frac{c(\lambda)}{|c(\lambda)|} e^{(i\lambda-\rho)t} (D(t))^{1/2} \mathcal{E}_2(\lambda, t)\, d\lambda \right|. \tag{21}$$

We will describe the cases with $\lambda > 0$, the cases where λ is replaced by $-\lambda$ are handled in the same manner. For the term (19) note that $\lambda \mapsto c(\lambda)/|c(\lambda)|$ is a multiplier of L^2. The Carleson-Hunt theorem states that

$$t \mapsto \sup_{R>1} \left| \int_1^R F(\lambda) \frac{c(\lambda)}{|c(\lambda)|} e^{i\lambda t}\, d\lambda \right|$$

is in $L^2(0,\infty)$ and the norm is less than or equal to const.$\|F\|_2$. Recall that Λ_0 is bounded on $[B_0,\infty)$ and take into account the factor of $t \mapsto e^{-\rho t} D(t)^{1/2}$, which is also bounded on $[B_0,\infty)$.

For the term (20) we can use integration by parts, since F is locally integrable. That is, write

$$\begin{aligned} \int_1^R F(\lambda) \frac{c(\lambda)}{|c(\lambda)|} e^{(i\lambda-\rho-2)t} D(t)^{1/2} \Lambda_1(\lambda, t)\, d\lambda = \\ D(t)^{1/2} e^{(-\rho-2)t} \Lambda_1(R,t) \int_1^R F(\lambda) \frac{c(\lambda)}{|c(\lambda)|} e^{i\lambda t}\, d\lambda \\ - D(t)^{1/2} e^{(-\rho-2)t} \int_1^R \left(\int_1^s F(\lambda) \frac{c(\lambda)}{|c(\lambda)|} e^{i\lambda t}\, d\lambda \right) \frac{\partial}{\partial s} \Lambda_1(s,t)\, ds. \end{aligned} \tag{22}$$

The absolute value of these terms are less than or equal to

$$\text{const.}\, (D(t))^{1/2} e^{(-\rho-2)t}\, S^*h(t) G_0(t) \left(\frac{1}{R} + \int_1^R \frac{ds}{s^2} \right)$$

where S^*h is the Carleson-Hunt maximal operator applied to the function $h \in L^2(\mathbb{R})$ with $\widehat{h}(\lambda) = F(\lambda)c(\lambda)|c(\lambda)|^{-1}$, if $\lambda \geq 1$, and zero elsewhere. We know that $\|S^*h\|_2 \leq$ const.$\|F\|_2$. Recalling that there is a factor of $e^{-\rho t}(D(t))^{1/2}$ to take into account, we then see that the term (20) is in $L^2([B_0, B_2], D(t)dt)$ and the norm is dominated by a constant multiple of $\|F\|_2$, with the constant depending on G, B_0, and B_2.

Now we concentrate on (21). The estimates in Proposition 4 show that this is dominated by

$$\text{const.} D(t)^{1/2} \int_1^\infty |F(\lambda)| e^{-\rho t} G_0(t) \lambda^{-2}\, d\lambda \leq \text{const.} D(t)^{1/2} e^{-\rho t} G_0(t) \|F\|_2.$$

This is clearly square integrable on intervals of the form $[B_0, B_2]$.

• **Medium t, summary.** Now we have shown that for $B_2 > B_0 > 1$ there is a constant $c > 0$, depending on $K\backslash G$, such that for all f in ${}^K L^2(G)^K$,

$$\left(\int_{B_0}^{B_2} |\mathfrak{M}^* f(a(t))|^2 D(t)\,dt\right)^{1/2} \le c\|f\|_2. \tag{23}$$

• **Case of large** t Here we know that

$$\phi_\lambda(t) = \Lambda_0(t) + \sum_{m=1}^{\infty} \Lambda_m(\lambda,t)e^{-2mt}$$

with $|\Lambda_m(\lambda,t)| \le A\rho^m e^{2m}|\lambda|^{-m} G_0(t)$ and

$$\left|\frac{\partial}{\partial\lambda}\Lambda_m(\lambda,t)\right| \le A\rho^m e^{2m}|\lambda|^{-1-m} 2G_0(t).$$

If $t > B_0 + 2 + \log(\rho)$, then the series above converges absolutely uniformly on intervals of the form $[B_0 + 2 + \log(\rho) + \delta, \infty)$ with $\delta > 0$. We have set

$$\psi_\lambda(t) = \frac{c(\lambda)}{|c(\lambda)|} D(t)^{1/2} e^{-\rho t} e^{i\lambda t} \phi_\lambda(t) + \frac{c(-\lambda)}{|c(\lambda)|} D(t)^{1/2} e^{-\rho t} e^{-i\lambda t} \phi_{-\lambda}(t).$$

Take $F \in L^2(0,\infty)$, then to each $R > 1$,

$$\int_1^R \frac{c(\lambda)}{|c(\lambda)|} F(\lambda)\, D(t)^{1/2} e^{-\rho t} e^{i\lambda t} \phi_\lambda(t)\, d\lambda$$

is equal to the sum

$$\begin{aligned} D(t)^{1/2} e^{-\rho t}\Lambda_0(t) &\int_1^R \widehat{h_1}(\lambda) e^{i\lambda t}\, d\lambda \\ &+ \sum_{m=1}^{\infty} D(t)^{1/2} e^{-2mt-\rho t} \int_1^R \widehat{h_1}(\lambda)\Lambda_m(\lambda,t) e^{i\lambda t}\, d\lambda, \end{aligned} \tag{24}$$

where $h_1 \in L^2(\mathbb{R})$ has $\widehat{h_1}(\lambda) = c(\lambda)|c(\lambda)|^{-1} F(\lambda)$ for $\lambda > 1$, and similarly for the $\phi_{-\lambda}$ term. The Lebesgue dominated convergence theorem justifies the interchange of integration and summation. The first part is handled directly by the Carleson-Hunt theorem. On the second part, use integration by parts on each of the summands. Since $\widehat{h_1}$ is locally integrable, we see that

$$\int_1^R \widehat{h_1}(\lambda)\Lambda_m(\lambda,t) e^{i\lambda t}\, d\lambda$$

is equal to

$$-\int_1^R \left(\int_1^s \widehat{h_1}(\lambda) e^{i\lambda t}\, d\lambda\right) \frac{\partial}{\partial s}\Lambda_m(s,t)\, ds + \Lambda_m(R,t) \int_1^R \widehat{h_1}(\lambda) e^{i\lambda t}\, d\lambda.$$

Taking absolute values we see that

$$\left|\int_1^R \widehat{h_1}(\lambda)\Lambda_m(\lambda,t)e^{i\lambda t}\,d\lambda\right| \tag{25}$$
$$\leq 2AS^*h_1(t)G_0(t)\rho^m e^{2m}\int_1^R s^{-1-m}\,ds$$
$$+ AS^*h_1(t)G_0(t)\rho^m e^{2m}R^{-m},$$

and this is less than or equal to

$$4AS^*h_1(t)G_0(t)\rho^m e^{2m}$$

for all $R > 1$. From this it follows that

$$\left|\int_1^R \frac{c(\lambda)}{|c(\lambda)|}F(\lambda)\,D(t)^{1/2}e^{-\rho t}e^{i\lambda t}\phi_\lambda(t)\,d\lambda\right| \tag{26}$$
$$\leq D(t)^{1/2}e^{-\rho t}\,AG_0(t)\,S^*h_1(t)$$
$$+ 4AS^*h_1(t)G_0(t)D(t)^{1/2}e^{-\rho t}\sum_{m=1}^{\infty} e^{-2mt+m\log(\rho)+2m}$$

We are free to take $B_2 > B_0 + \log(\rho) + 2$ so that the sum on the right hand side is uniformly bounded for all $t > B_2$. The Carleson-Hunt theorem shows that

$$\|S^*h_1\|_2 \leq c\|h_1\|_2 \leq c'\|F\|_2.$$

• **Summary of the large t case.** Now we have shown that there exists $B_2 > B_0 > 1$ and a constant $c > 0$, depending on $K\backslash G$, such that for all f in ${}^K L^2(G)^K$,

$$\left(\int_{B_2}^{\infty} |\mathfrak{M}^*f(a(t))|^2 D(t)\,dt\right)^{1/2} \leq c\|f\|_2. \tag{27}$$

This completes the proof of the theorem. Notice that we frequently move from one L^2 function to another using the Plancherel theorem for Fourier and Hankel transforms, and we use the fact that bounded functions are multipliers of L^2. These devices are not available to us for other L^p spaces, so that this method can only be expected to apply to the setting of L^2.

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