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UNITARY REPRESENTATION INDUCED FROM MAXIMAL PARABOLIC SUBGROUPS FOR SPLIT F_4

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For the linear connected simple Lie group split F_4, the author determines which Langlands quotients $J(MAN, \sigma, \nu)$ are infinitesimally unitary under the condition that $\dim A = 1$.

1. Introduction and Statement of Results.

It is known that the problem of classifying irreducible unitary representations of a linear connected semisimple Lie group G comes down to deciding which Langlands quotients $J(MAN, \sigma, \nu)$ are infinitesimally unitary. Here MAN is any cuspidal parabolic subgroup of G, σ is any discrete series or nondegenerate limit of discrete series representation of M, and ν is any complex valued functional on the Lie algebra of A satisfying $\operatorname{Re}\nu > 0$ and certain symmetry properties. Using Baldoni-Silva and Knapp [**BK3**], Baldoni-Silva and Knapp [**BK1**] determined which Langlands quotients are infinitesimally unitary under the conditions that G is simple, that $\dim A = 1$ and that G is neither split F_4 nor split G_2. Recently, the related problem was discussed by D.A. Vogan [**V3**] for the simply-connected split G_2. In this note, the author determines which Langlands quotients $J(MAN, \sigma, \nu)$ are infinitesimally unitary under the conditions that $\dim A = 1$ and that G is split F_4.

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Let G be the linear connected simple Lie group split F_4. Let θ be a Cartan involution, let K be the corresponding maximal compact subgroup, and let MAN be the corresponding Langlands decomposition of a parabolic subgroup. We shall assume that $\dim A = 1$. We denote corresponding Lie algebra by lower case Italy letters. Let σ be a discrete series representation of M or a nondegenerate limit of discrete series [**KZ2**], and let ν be a complex valued functional on the Lie algebra a of A. The standard induced representation $U(MAN, \sigma, \nu)$ is defined as in [**BK1**] (cf. p. 23 in [**BK1**]). If $\operatorname{Re}\nu \geq 0$ (with positive defined relative to N) and $\nu \neq 0$, then $U(MAN, \sigma, \nu)$ has a unique irreducible quotient $J(MAN, \sigma, \nu)$, the Langlands quotient. If ν is imaginary ,then $J(MAN, \sigma, \nu)$ is trivially unitary. If $\operatorname{Re}\nu > 0$, then

$J(MAN, \sigma, \nu)$ cannot admit a nonzero invariant Hermitian form unless the Weyl group $W(A:G)$ has a nontrivial element w and w fixes the class $[\sigma]$ of σ, moreover, ν must be real. Conversely, these conditions give the existence of a nonzero invariant Hermitian form (see [**KZ1**]). Thus the problem is to decide which real parameters $\nu \geq 0$ are such that this form is semi-definite.

Clearly, $\operatorname{rank} G = \operatorname{rank} K$. Let b be a compact Cartan subalgebra of the Lie algebra g of G. We may assume that a is built by Cayley transform relative to some noncompact root α in $\Lambda = \Lambda(g^C, b^C)$. Then $b_- = \ker \alpha$ is a compact Cartan subalgebra of the Lie algebra m of M and the root system $\Lambda_- = \Lambda(m^C, b_-^C)$ is given by the members of Λ orthogonal to α. Let Λ_K and Λ_n be the subsets of compact and noncompact members of Λ. It will be convenient to identify α with its Cayley transform, so that we write ν as a multiple of α. Clearly, σ is determined by χ and a Harish-Chandra parameter (λ_0, Λ_-^+) for σ where χ has been defined in [**BK1**] (see p. 24 in [**BK1**]). Here Λ_-^+ is a positive system for Λ_- and λ_0 is dominant relative to Λ_-^+. We can introduce a positive system Λ^+ for Λ containing Λ_-^+ such that λ_0 is Λ^+ dominant and α is simple. Let $\Lambda_K^+ = \Lambda_K \cap \Lambda^+$ and $\Lambda_n^+ = \Lambda_n \cap \Lambda^+$. It is automatically true that the nontrivial element w of $W(A:G)$ exists and fixes $[\sigma]$. We can define σ to be a cotangent case or tangent case as in [**BK1**] (see p. 25 in [**BK1**]). According to [**K**], $J(MAN, \sigma, \nu)$ has one or two minimal K-types with highest weights given by the formula

$$\wedge = \lambda_0 + \delta - 2\delta_K - \frac{1}{2}(1 - \mu_\alpha)\alpha.$$

Here δ and δ_K are the half sums of positive roots for Λ^+ and Λ_K^+ and μ_α is 0 in a tangent case, and is equal to ± 1 in a cotangent case. For a given $\wedge$, let $\Lambda_{K,\perp} = \{r \in \Lambda_K \mid \langle \wedge, r \rangle = 0\}$. The special basic case associated to λ_0 is the group or root system generated by α and all simple roots of Λ^+ needed for expansion of members in $\Lambda_{K,\perp}$. This root system will be denoted by Λ_S and the component of α in Λ_S will be denoted by Λ_S^0. For a given α, let ν_0^+ and ν_0^- be the integers defined by (1.4a) and (1.4b) in [**BK1**] respectively (cf. (1.2) below). By 2.1 in [**BK1**], we may assume henceforth that $\nu_0^+ > 0$, that $\nu_0^- > 0$, and that the invariant Hermitian form on $J(MAN, \sigma, \nu)$ is positive for all ν near 0. Evidently there is nothing to prove unless $\min(\nu_0^+, \nu_0^-) > 1$ in the consideration.

Let $\Lambda(S)$ denote the subsystem of Λ generated by a subset S in Λ. A subalgebra l of g is called to be a standard subalgebra of g if there exists $S \subset \Lambda, \alpha \in S$ such that l^C is the subalgebra of g^C with root system $\Lambda(S), S \subset \Lambda$ (cf. Section 3). For convenience, let L denote the subgroup of G with Lie algebra l and let Λ_L denote the subsystem $\Lambda(S)$. A subalgebra l of g is called to be a fundamental of g if Λ_L is a subsystem generated by some simple roots and containing α. Clearly, each fundamental subalgebra of g is a standard subalgebra of g. If l is a standard (resp. fundamental) subalgebra of g, then L is called to be a standard (resp. fundamental) subgroup of G. For each standard subgroup L of G, let $\Lambda_L(u) = \{\beta \in \Lambda^+ \mid \beta \notin \Lambda_L\}$. For each fundamental subgroup L, there is a simple root system Π_L of Λ_L so that $\alpha \in \Pi_L \subset \Pi$, and let $\Lambda_{L,S}$ be the special case associated with $\lambda_{L,0}$, and let $\Lambda^0_{L,S}$ be the component of α in $\Lambda_{L,S}$. Here $\lambda_{L,0}$ is given by (3.1b) in [**BK1**]. For a fundamental subgroup L, let $\xi(\alpha, \varepsilon)$ be the sum of the simple roots strictly between α and ε in Π_L for any $\alpha, \varepsilon \in \Pi_L$. Clearly, the simple root system Π can be expressed in the form $\Pi = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where α_1 and α_2 are long, and α_3 and α_4 are short, and α_i is orthogonal to α_j if $|i-j| > 1$, $i, j = 1, 2, 3, 4$. Let Γ be the subgroup of G with $\Lambda_\Gamma = \Lambda(\alpha_1, \alpha_2)$.

In this note, we shall use the notations given by [**BK1**] directly. Now, we shall state the main results of this note.

Theorem 1 (Main Theorem). *For $c > 0$, then, $J(MAN, \sigma, \frac{1}{2}c\alpha)$ with three exceptions is infinitesimally unitary exactly when*

$$0 < c \le \min(\nu_0^+, \nu_0^-) = c_0,$$

the exceptions occur when there exists a fundamental subgroup L of G which is of one of the following form:

(A.1) *$L \cong \mathrm{SO}(4,3)$ and $\Lambda^0_{L,S} = \Lambda_\Gamma$ with α long, and there is a basic short root ε in Π_L.*

(i) *Suppose that $\nu^\Upsilon_{0,L} \le 1$. Then $J(MAN.\sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$0 < c \le \min(\nu^+_{0,L}, \nu^-_{0,L}) = c_0.$$

(ii) *Suppose that $\nu^\Upsilon_{0,L} > 1$. Then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$0 < c \le c_0' = \min(\nu^+_{0,L} - d, \nu^-_{0,L} - d'), \quad \text{or} \quad c = c_0 = \min(\nu^+_{0,L}, \nu^-_{0,L}).$$

Here $d = 0, d' = 1$ and $\Upsilon = -$ if $\xi(\alpha, \varepsilon)$ is noncompact, $d = 1, d' = 0$ and $\Upsilon = +$ if $\xi(\alpha, \varepsilon)$ is compact or zero.

(A.2) *$L \cong \mathrm{SO}(5,2)$ and $\Lambda_\Gamma \subset \Lambda^0_{L,S} = \Lambda_L$ with α long and α is the middle of three simple root in Π_L. Suppose that there is a unique positive noncompact root β_0 in Λ_L such that β_0 is orthogonal to α and is conjugate to $-\alpha$ (resp. α) by K within L. Then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary when*

$$0 < c \le c_0 = \min(\nu^+_{0,L}, \nu^-_{0,\Gamma} + 1), \ (\textit{resp. } c_0 = \min(\nu^+_{0,\Gamma} + 1, \nu^-_{0,L})).$$

(B.1) *$L \cong \mathrm{Sp}(2,1)$ and $\Lambda^0_{L,S} = \Lambda_L$ with α short. Suppose that there is a long compact root of Π_L next α , and $\mu_\alpha = 0$. Then $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when*

$$0 < c \leq \min(\nu_0^+, \nu_0^-) - 2 = c_0' \quad or \;\; c = c_0 = \min(\nu_0^+, \nu_0^-).$$

Remark. For the case (A.1), (ii), or for the case (B.1), $J(MAN, \sigma, \frac{1}{2}c_0\alpha)$ is an isolated unitary representation and there is a gap (c_0', c_0) if $c_0' < c_0$. The situations for (A.1), (ii) (resp. (B.1)) is a similar fashion as in situations for (iii) (resp. for (i)) in Theorem 1.1 of [**BK1**].

For each $r \in \Lambda$, let g_r^C be the root space corresponding to r. Then g^C has the following decomposition:

$$g^C = b^C + \sum_{r \in \Lambda} g_r^C.$$

Let θ denote the Cartan involution for the Lie algebra g of G. Then $g_u = g_+ + ig_-, i = \sqrt{-1}$ is a compact Lie algebra where $g_\pm = \{X \in g \mid \theta(X) = \pm X\}$. For each $r \in \Lambda$, let $u_r = \frac{1}{2}(e_r + e_{-r})$ and $v_r = \frac{1}{2i}(e_r - e_{-r})$. Here $e_{\pm r} \in g_\pm^C, e_{\pm r} \neq 0$ with $(e_r, e_{-r}) = 1$. Then g_u can be interpreted as a vector space generated by $\{u_r, v_r \mid r \in \Lambda\}$ over real number field $\mathbf{R}$. Let θ denote the extension of θ to g^C also. Clearly, $\theta(e_{\pm r}) = e_{\pm r}$ if $r \in \Lambda_K$, $\theta(e_{\pm r}) = -e_{\pm r}$ if $r \in \Lambda_n$.

If g is n-dimensional vector space over $\mathbf{R}$, then g^C can be interpreted as a 2n-dimensional vector space over $\mathbf{R}$. If $Z = X + iY \in g^C, X, Y \in g$ then we denote by $\bar{Z}$ the element $X - iY$ in g^C.

Lemma 1.1. *For each $r \in \Lambda$, $\bar{e}_r = e_{-r}$ or $\bar{e}_r = -e_{-r}$.*

Proof. If $r \in \Lambda_K$, then $u_r, v_r \in g_+ \subset g$, so $e_r = u_r + iv_r$ and $e_{-r} = u_r - iv_r$. Thus for each $r \in \Lambda_K$ we have $\bar{e}_r = e_{-r}$. If $r \in \Lambda_n$, then $iu_r, iv_r \in g_- \subset g$, so, $e_r = iv_r - i(iu_r)$ and $-e_{-r} = iv_r + i(iu_r)$. Thus for each $r \in \Lambda_n$, we have $\bar{e}_r = -e_{-r}$.

In order to describe the root system Λ,it is convenient to use an orthonormal base e_1, e_2, e_3, e_4 of a Euclidean space E_R of dimension 4. Clearly, we have

$$\Lambda = \left\{ \pm e_i \pm e_j, \pm e_i, \; 1 \leq i, \; j \leq 4, \; i \neq j, \; \frac{1}{2}(\pm e_1 \pm e_2 \pm e_3 \pm e_4) \right\}.$$

Let

$$\Pi = \left\{ \alpha_1 = e_2 - e_3, \alpha_2 = e_3 - e_4, \alpha_3 = e_4, \alpha_4 = -\frac{1}{2}(e_1 + e_2 + e_3 + e_4) \right\}.$$

Clearly, Π is a simple root system of Λ. For the simple root system Π, the positive root system Λ^+ can be expressed in the form:

$$\Lambda^+ = \left\{-e_1 \pm e_i, e_i \pm e_j, e_j, 2 \leq i < j \leq 4, -e_1, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

For convenience,the coordinate of the element $x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4$ of E_R can be written as

$$(x_1, x_2, x_3, x_4). \tag{1.1}$$

It is clear that $2\delta = (-11, 5, 3, 1)$.

Hereafter, we shall fix the root system Λ, positive root system Λ^+ and the simple root system Π in the consideration. We shall assume that λ_0 is Λ^+ dominant, $\Lambda_K^+ \subset \Lambda^+$, and $\alpha \in \Pi$. Let Π_K be the simple root system of Λ_K associated with Λ_K^+ and let $\Pi_K^0 = \Pi \cap \Lambda_K^+$.

For two ordered elements (x, y) in $E_R \otimes E_R$, let $(x, y) = 2\langle x, y\rangle/\langle y, y\rangle$.

A Dynkin diagram of Π is called an explicit diagram if every simple root of Π is either white or black. For an explicit diagram of Π, let Π_0 be the set of the white roots in the explicit diagram of Π, and let (Π, Π_0) denote the explicit diagram of Π. Clearly, the explicit diagrams are ones in Table 1.1 and Table 1.2. (See the end of this note.)

In [**G**], $\theta_C = (c_1, c_2, c_3, c_4), c_i = 0, \pm 1, \pm 2$, $i = 1, 2, 3, 4$, denotes a canonical involutive automorphism of g_u for Λ given above. Let c be the element of E_R with coordinate (c_1, c_2, c_3, c_4) in (1.1). Then for any $r \in \Lambda$, $\theta_C(e_r) = e_r$ or $= -e_r$ according to $k_{c,r}$ is even or odd where $k_{c,r} = \langle c, r\rangle$. The canonical involution θ_C of g_u determines a maximal compact subalgebra C of g_u. Here $C = \{X \in g_u \mid \theta_C(X) = X\}$. In fact, C is the subalgebra of g_u generated by the elements in the set $\{u_r, v_r, r \in \Lambda \mid k_{c,r} \in 2\mathbf{Z}\}$. Clearly, $\Lambda_C = \{r \in \Lambda \mid k_{c,r} \in 2\mathbf{Z}\}$ is a root system of C. It is clear that $\Lambda_C \cap \Lambda^+ = \Lambda_C^+$ is a positive root system of Λ_C. Let Π_C be the simple root system of C associated with Λ_C^+ and $\Pi_C^0 = \Pi \cap \Lambda_C^+$. In fact, $\Pi_C^0 = \{r \in \Pi \mid k_{c,r} \in 2\mathbf{Z}\}$. Then, for each involution θ_C, there is an explicit diagram (Π, Π_0) such that $\Pi_0 = \Pi_C^0$. Conversely, for each explicit diagrqam (Π, Π_0), there is an involution θ_C corresponding to (Π, Π_0) such that $\Pi_C^0 = \Pi_0$. Since the Lie algebra $k = g_+$ of K is isomorphic to the semisimple Lie algebra $A_1 + C_3$ by [**G**] (or by [**Y**]), for any involution θ_C, the root system Λ_C of C is a compact root system Λ_K for G if and only if $C \cong A_1 + C_3$. By [**G**], it is easily verified that for every involutions θ_C corresponding to the explicit diagrams given by Table 1.1 (resp. Table 1.2), we have $C \cong A_1 + C_3$ (resp. $C \ncong A_1 + C_3$). □

Therefore, we have

Lemma 1.2. *There is a one to one correspondence between the explicit diagrams* (Π, Π_0) *given in Table* 1.1 *and the positive compact root systems* Λ_K^+ *satisfying* $\Lambda_K^+ \subset \Lambda^+$ *such that* $\Pi_0 = \Pi_K^0 = \Pi \cap \Lambda_K^+$. *Moreover, for each explicit diagram* (Π, Π_0) *in Table* 1.1, *there is an involution* θ_C *of* g_u *such that* $\Pi_0 = \Pi_C^0 = \Pi \cap \Lambda_C^+$ *and* $\Lambda_C^+ = \Lambda_C \cap \Lambda^+$ *is the positive compact root system corresponding to the explicit diagram* (Π, Π_0) *mentioned above.*

For any finite set Y, *let* $\#(Y)$ *denote the number of the elements of* Y. *For a given* $\alpha \in \Pi \cap \Lambda_n^+$, *let*

$$\Phi^{\pm} = \{\beta \in \Lambda_n^+ \mid \beta \pm \alpha \in \Lambda\};$$
$$\Phi_\alpha^{\pm} = \{\beta \pm \alpha \mid \beta \in \Phi^{\pm}\};$$
$$\Psi_0^{\pm} = \{\beta \in \Phi_\alpha^{\pm} \mid 2\langle \wedge, \beta\rangle = 0\};$$
$$\Psi_1^{\pm} = \{\beta \in \Phi_\alpha^{\pm} \mid |\beta| < |\alpha|, 2\langle \wedge, \beta\rangle / \langle \beta, \beta\rangle = (\wedge, \beta) = 1\}.$$

We shall give an explicit formula for ν_0^+ *and* ν_0^-:

$$\nu_0^{\pm} = 1 \pm \mu_\alpha + 2\#(\Psi_0^{\mp}) + \#(\Psi_1^{\mp}). \tag{1.2}$$

For convenience, function $f(\mu_\alpha), \mu_\alpha = 0, \pm 1$ *will be expressed in the form* $f = [f(-1), f(0), f(1)]$.

2. The proof of Theorem 1.

For given $\wedge, \alpha$ and Λ_K^+, $(\wedge + \zeta\alpha)^\vee, \zeta = \pm$ was defined in [**BK1**]. Clearly it can be expressed in the form

$$(\wedge + \zeta\alpha)^\vee = \wedge + \omega^\zeta,$$

where ω^ζ is a root in Λ which can be expressed in the form uniquely

$$\omega^\zeta = \zeta\alpha + m_1 r_1 + m_2 r_2 + \cdots + m_q r_q \tag{2.1}$$

where $m_1, m_2, ..., m_q \in \mathbf{Z}$ and $r_1, r_2, ..., r_q \in \Pi_K$.

Let $\Lambda(\omega, \zeta) = \{-m_1 r_1, -m_2 r_2, ..., -m_q r_q\}$. Let δ^+ and δ^- be the results of making α and $-\alpha$, respectively, dominant for $\Lambda_{K,\perp}$. Clearly $\delta^{\pm} = \omega^{\pm}$ if and only if $\wedge \pm \delta^{\pm}$ is Λ_K^+ dominant (cf. p. 34 in [**BK1**]).

Lemma 2.1. *Suppose that* $\zeta 2\alpha$ *is not a sum of some roots in* $\Lambda(\omega, \zeta) \cup \Lambda_K^+$. *Then* (b) *holds in Theorem* 3.2 *of* [**BK1**] *if* $\zeta = +$, *in Theorem* 3.2′ *of* [**BK1**] *if* $\zeta = -$.

Proof. Assume that (b) dose not hold. Then by the properties of highest weight and (2.1) we have.

$$\omega^{\pm} \pm \alpha = \wedge \pm \omega^{\pm} - (\wedge \mp \alpha) = \sum_{r \in \Lambda_K^+} k_r r,$$

$$\omega^{\pm} \mp \alpha = \wedge \pm \omega^{\pm} - (\wedge \pm \alpha) = \sum_{r\in\Lambda(\omega,\pm)} m_r r.$$

Here k_r are nonegative integer. Thus $\zeta 2\alpha$ is a sum of some roots in $\Lambda(\omega, \zeta) \cup \Lambda_K^+$. Hence, the lemma follows. □

2.1. The proof of Theorem 1. By similar methods used in Sections 3–7 of [**BK1**], we shall determine a least positive integer c_0 such that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c_0 < c$. By similar methods used in Sections 8–11 of [**BK1**], we shall determine a greatest postive integer c_0' such that $U(MAN, \sigma, \frac{1}{2}\alpha)$ is irreducible for $0 < c < c_0'$. It follows from a general continuity argument (cf. [**KS**], Sect. 14) that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly for $0 < c \leq c_0$ if $c_0' = c_0$. If $c_0' < c_0$, then by the methods mentioned above, we don't know whether $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for $c_0' < c \leq c_0$ and we say that there is a "gap" (c_0', c_0). If $c_0' < c_0$ and by the methods given by D.A. Vogan [**V1**] we can finally show that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly for $0 < c \leq c_0'$ or $c = c_0$, then the "gap" (c_0', c_0) is a gap mentioned in the Remark of Theorem 1.

In fact, for short α, integer c_0 was determined by 6.1 of [**BK1**] (cf. pp. 45–49 in [**BK1**]), so, we shall only need to determine integer c_0' for this case.

By Lemma 1.2, in order to prove Theorem 1, it is sufficient to prove Theorem 1 for the cases (1)–(6) and (1)′–(6)′ given in the Table 1.1. Now we shall first determine the positive integers c_0' and c_0 case by case.

(1) $\theta_C = (1, -1, 0, 0)$: It is easy to see that

$$\Lambda_K^+ = \left\{e_3, e_4, -e_1 \pm e_2, e_3 \pm e_4, -\frac{1}{2}(e_1 + e_2 \pm e_3 \pm e_4)\right\},$$

$$\Lambda_n^+ = \left\{-e_1, e_2, -e_1 \pm e_3, -e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4, -\frac{1}{2}(e_1 - e_2 \pm e_3 \pm e_4)\right\},$$

$$\Pi_K = \{\alpha_2, \alpha_3, \alpha_4, -e_1 + e_2\}, \quad 2\delta_K = -4e_1 - 2e_2 + 3e_3 + e_4.$$

(1.A) Let $\alpha = \alpha_1$. Clearly, we have

$$\Phi^- = \left\{e_2, -e_1 - e_3, e_2 \pm e_4, -\frac{1}{2}(e_1 - e_2 + e_3 \pm e_4)\right\},$$

$$\Phi_\alpha^- = \left\{e_3, -e_1 - e_2, e_3 \pm e_4, -\frac{1}{2}(e_1 + e_2 - e_3 \pm e_4)\right\};$$

$$\Phi^+ = \{-e_1 + e_3\}, \qquad \Phi_\alpha^+ = \{-e_1 + e_2\}.$$

By Table 2.1 of [**BK1**], the following formula are easily verified

$$(2.1.1) \qquad \langle \Lambda, \alpha_1 \rangle \geq 5 + \mu_\alpha, \ \langle \Lambda, \alpha_i \rangle \geq 0, \quad i = 2, 3, 4.$$

(Equality holds if α_i, $i = 1, 2, 3, 4$ is basic.) It follows that

$$\begin{aligned} &\langle \Lambda, e_3 \rangle \geq 0, \left\langle \Lambda, -\frac{1}{2}(e_1 + e_2 - e_3 \pm e_4) \right\rangle \geq 0 \\ &\langle \Lambda, e_3 \pm e_4 \rangle \geq 0, \langle \Lambda, -e_1 - e_2 \rangle \geq 0; \\ (2.1.2) \qquad &\langle \Lambda, -e_1 + e_2 \rangle \geq 2(5 + \mu_\alpha). \end{aligned}$$

It follows from (2.1.2) that

$$\begin{aligned} &\#(\Psi_0^-) \geq [6, 6, 6] = 6, \quad \#(\Psi_1^-) = [0, 0, 0] = 0; \\ &\#(\Psi_0^+) = \#(\Psi_1^+) = [0, 0, 0] = 0. \end{aligned}$$

Thus, $\min(\nu_0^+, \nu_0^-) \leq [2, 1, 0]$. (Equality holds if α_i,1=1,2,3,4, are basic.)

By (2.1.1), $\langle \Lambda, -e_1 + e_2 \rangle > 0$, hence $-e_1 + e_2 \notin \Lambda_{K,\perp}$. Therefore, it is easily shown that $-\alpha$ is $\Lambda_{K,\perp}^+$ dominant, so $\delta^- = -\alpha$. Clearly, $\Lambda' = (\Lambda - \alpha)^\vee = \Lambda - \alpha$ is dominant for Λ_K^+. For this case $\omega^- = \delta^- = -\alpha$ and $\Lambda(\omega, -)$ is empty. Thus by the Remark of 7.2 (or 3.1) in [**BK1**], (a) holds in 3.2′ in [**BK1**]. By computing, it is easy to see that -2α is not a sum of some roots in $\Lambda(\omega, -) \cup \Lambda_K^+$, so, by Lemma 2.1, (b) holds in 3.2′ of [**BK1**]. Since $\Lambda' - \Lambda = -\alpha$, (c) holds in 3.2′ of [**BK1**]. Thus, by 3.2′ in [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > \min(\nu_0^+, \nu_0^-) = c_0$.

We will consider the irreduciblility of $U(MAN, \sigma, \frac{1}{2}c\alpha)$.

(1) Suppose $\mu_\alpha = -1$. Let $\Lambda_L = \Lambda(\alpha_1, \alpha_2, \alpha_3)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{SO}(5, 2)$.

Obviously, there is a unique positive noncompact root $\beta_0 = e_2 + e_3$ in Λ_L such that β_0 is conjugate to α by K within L. We shall consider the condition:

(2.1.q): $\quad \Lambda_\Gamma \subset \Lambda_{L,S}^0 = \Lambda_L$.

(i) Suppose that (2.1.q) does not hold. Then $\langle \Lambda, \alpha_2 \rangle > 0$. Therefore, it follows that $\#(\Psi_0^-) = 0$. Then we have $\min(\nu_0^+, \nu_0^-) = \nu_0^+ \leq 1$. For this case, $\omega^+ = \delta^+ = -e_1 + e_3$ and $\Lambda(\omega, +) = \{-\alpha_2, -\alpha_3, -\alpha_4\}$. A similar argument shows that (a), (b) and (c) hold in 3.2 of [**BK1**]. Thus, by 3.2 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = \nu_0^+ = 1$. By 8.3 of [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0 = 1$.

(ii) Suppose that (2.1.q) holds. Then $\min(\nu_{0,\Gamma}^+ + 1, \nu_{0,L}^-) = 2$ since $\nu_{0,L}^- = 2$ and $\nu_{0,\Gamma}^+ + 1 = 3$. Thus by 11.2 in [**BK1**], $U^L(M_L A N_L, \sigma_L, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 = \nu_0^- = 2$. By Table 1.2 in [**BK1**], we have

$$(2.1.3)\qquad \begin{aligned} &\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_1 \right\rangle \geq 1, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_2 \right\rangle \geq -\frac{1}{2} \\ &\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_3 \right\rangle \geq \frac{1}{2}, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_4 \right\rangle \geq \frac{1}{2}. \end{aligned}$$

(Equality holds if α_i, $i = 1, 2, 3, 4$, are basic.) By (2.1.3), $\langle \lambda_0 + \frac{1}{2}\alpha, \beta \rangle \geq 0$ for all $\beta \in \Lambda_L(u)$. Thus, by 8.2 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 \leq 2$.

(2) Suppose $\mu_\alpha \neq -1$. By 8.3 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 \leq 1$.

Summarizing the results of (1) and (2), $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0$. Therefore, by the continuity argument, (cf. [**KS**]), for case **(1)**, Theorem 1 is proved since $\Pi \cap \Lambda_n^+ = \{\alpha_1\}$.

(2) $\theta_C = (0, 1, -1, 0)$. It is easy to see that

$$\Lambda_K^+ = \left\{ -e_1, e_4, -e_1 \pm e_4, e_2 \pm e_3, -\frac{1}{2}(e_1 + ze_2 + ze_3 \pm e_4), z = \pm 1 \right\},$$

$$\begin{aligned} \Lambda_n^+ = \Big\{ & e_2, e_3, -e_1 \pm e_2, -e_1 \pm e_3, e_2 \pm e_4, e_3 \pm e_4, \\ & -\frac{1}{2}(e_1 + ze_2 - ze_3 \pm e_4), z = \pm 1 \Big\}, \end{aligned}$$

$$\Pi_K = \{\alpha_3, \alpha_4, e_2 + e_3, \alpha_1\}, \quad 2\delta_K = -5e_1 + 2e_2 + e_4.$$

(2.A) Let $\alpha = \alpha_2$. Clearly, we have

$$\begin{aligned} \Phi^- &= \left\{ -e_1 + e_3, e_2 - e_4, e_3, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4) \right\}, \\ \Phi_\alpha^- &= \left\{ -e_1 + e_4, e_2 - e_3, e_4, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4) \right\}; \\ \Phi^+ &= \left\{ -e_1 - e_3, e_2 + e_4, -\frac{1}{2}(e_1 - e_2 + e_3 - e_4) \right\}, \\ \Phi_\alpha^+ &= \left\{ -e_1 - e_4, e_2 + e_3, -\frac{1}{2}(e_1 - e_2 - e_3 + e_4) \right\}. \end{aligned}$$

By Table 1.2 in [**BK1**], the following formula are easily verified

$$\langle \Lambda, \alpha_1 \rangle \geq 0, \qquad\qquad \langle \Lambda, \alpha_2 \rangle \geq 1 + \mu_\alpha,$$

$$(2.2.1) \quad \langle\Lambda, \alpha_3\rangle \geq \frac{1}{2}\left(\left|\mu_\alpha + \frac{1}{2}\right| - \frac{1}{2}\right) - \frac{1}{2}\mu_\alpha, \qquad \langle\Lambda, \alpha_4\rangle \geq 0.$$

It follows from (2.2.1) that

$$\#(\Psi_0^-) \leq [1,3,3], \quad \#(\Psi_1^-) \leq [2,0,0],$$
$$\#(\Psi_0^+) = \#(\Psi_1^+) = 0.$$

Thus, $\min(\nu_0^+, \nu_0^-) \leq [2,1,0]$.

By (2.2.1), $\langle\Lambda, e_2+e_3\rangle > 0$. Therefore, $e_2+e_3 \notin \Lambda_{K,\perp}$. Therefore, it is easily verified that $-\alpha$ is $\Lambda_{K,\perp}^+$ dominant and $\Lambda-\alpha$ is Λ_K^+ dominant. Thus, it follows that $\omega^- = \delta^- = -\alpha$. A similar argument used in case **(1)** shows that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = \min(\nu_0^+, \nu_0^-)$.

We shall consider the irreducibility of $U(MAN, \sigma, \frac{1}{2}c\alpha)$.

Let $\Lambda_L = \Lambda(\alpha_1, \alpha_2, \alpha_3)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{SO}(4,3)$. Let $\varepsilon = \alpha_3$. So ε is short and $\xi(\alpha, \varepsilon) = 0$. Then it is easy to see that $\min(\nu_{0,L}^+ - 1, \nu_{0,L}^-) \leq 2$ since $\nu_{0,L}^+ \leq 3$ and $\nu_{0,L}^- \leq 2$. We shall consider the condition

(2.2.q): $\quad \Lambda_{L,S}^0 = \Lambda_\Gamma$.

(1) Suppose that $\mu_\alpha = -1$.

(i) Suppose that (2.2.q) holds.

(a) Suppose that $\alpha_3 = e_4 \notin \Psi_1^-$. Then we have $\nu_{0,L}^+ - 1 = 1$ and $\nu_{0,L}^- = 2$. Therefore $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for $0 < c \leq c_0' = 1$ and there is a gap $(c_0', c_0) = (1,2)$. We shall show that for this case, (A.2), (ii) holds in Theorem 1. By (iii) in Theorem 1.1 of [**BK1**], it is easily shown that $J^L(M_LAN_L, \sigma_L, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when

(2.2.2)
$$0 < c \leq c_0' = \min(\nu_{0,L}^+, \nu_{0,L}^- - 1) = 1, \quad c = \min(\nu_{0,L}^+, \nu_{0,L}^-) = 2.$$

By Table 2.1 of [**BK1**], we have $\lambda_{0,b} = (-1,0,0,0)$ and

$$\Lambda_b = \lambda_{0,b} + \delta - 2\delta_K - \frac{2}{2}\alpha = \frac{1}{2}(-3,1,1,1).$$

Clearly, $(\Lambda_b, \alpha_3) = 1$. Clearly, if $\alpha_3 \notin \Psi_1^-$, then $(\Lambda, \alpha_3) > 1$, so it follows that $\Lambda \neq \Lambda_b$ and $\lambda_0 \neq \lambda_{0,b}$. Hence, it is easy to see that if (2.2.q) holds and $\alpha_3 \notin \Psi_1^-$, then λ_0 must be

$$(2.2.3) \qquad \lambda_{0,b} + \frac{1}{2}(-3,1,1,1) = \frac{1}{2}(-5,1,1,1).$$

Clearly, $(\lambda_0, \alpha_i) = (\lambda_{0,b}, \alpha_i)$ for $i = 1, 2, 4$ and $(\lambda_0, \alpha_3) = (\lambda_{0,b}, \alpha_3)$ $+1$. Set $\gamma(z) = \lambda_0 + (1 - z)\alpha$, $0 \leq z < \frac{1}{2}$. By (2.2.3) we have

$$\gamma(z) = \frac{1}{2}(-5, 1, 3 - 2z, -1 + 2z). \tag{2.2.4}$$

By (2.2.4), for all $\beta \in \Lambda_L(u)$ $\langle \gamma(z), \beta\rangle > 0$ if $z > 0$, $\langle \gamma(z), \beta\rangle \geq 0$ if $z = 0$.

Thus, by Theorem 1.3a of D.A. Vogan [**V1**] (or Theorem 5.11 of D.A. Vogan [**V3**]), it follows from (2.2.2) that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when $0 < c \leq 1$ or $c = 2$. Hence, (A.1), (ii) holds in Theorem 1.

(b) Suppose that $\alpha_3 = e_4 \in \Psi_1^-$. Then it is easy to see that $\nu_{0,L}^+ - 1 = 2$ and $\nu_{0,L}^- = 2$. Therefore, by 11.2 in [**BK1**], $U^L(M_LAN_L, \sigma_L, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0, c_0' = c_0 \leq 2$. By Table 1.2 in [**BK1**], we have

$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_1 \right\rangle \geq -\frac{1}{2}, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_2 \right\rangle \geq 1,$$
$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_3 \right\rangle \geq -\frac{1}{2}, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_4 \right\rangle \geq \frac{1}{2}. \tag{2.2.5}$$

It follows from (2.2.5) that $\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle \geq 0$ for all β in $\Lambda_L(u)$. Then by 8.2 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 = 2$. By continuity argument (cf. [**KS**]), Theorem 1 holds.

(ii) Suppose that (2.2.q) does not hold. Then $\alpha_1 \notin \Lambda_{L,\perp}$. Therefore, it is easy to see that $\langle \lambda_0, \alpha_1\rangle > 0$. Let $\Lambda_L = \Lambda(\alpha_2, \alpha_3, \alpha_4)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{Sp}(3, \mathbf{R})$. It is easily verified that $\min(\nu_{0,L}^+, \nu_{0,L}^-) = \min(\nu_0^+, \nu_0^-)$ since $e_4, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4) \in \Lambda_L$.

(a) Suppose that $\#(\Psi_1^-) < 2$. Then $\min(\nu_0^+, \nu_0^-) = \nu_0^+ \leq 1$. Then for this case, we have $\omega^+ = \delta^+ = -e_1 + e_2$ and similar arguments as used in the case **(1.A)**,(1),(i) show that Theorem 1 holds with $c_0' = c_0 = \nu_0^+ = 1$.

(b) Suppose that $\#(\Psi_1^-) = 2$. Then by 11.2 of [**BK1**], $U^L(M_LAN_L, \sigma_L, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0 = 2$. Here $c_0 = \min(\nu_{0,L}^+, \nu_{0,L}^-)$. Since $\langle \lambda_0, \alpha_1\rangle > 0$, it follows from (2.2.5) that $\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle \geq 0$ for all $\beta \in \Lambda_L(u)$. Then by 8.2 of [**BK1**], it is easy to see that $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0 \leq 2$. By contiunity argument (cf. [**KS**]), Theorem 1 holds.

(iii) Suppose $\mu_\alpha \neq -1$. By 8.3 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 = 1$. By continuity argument (cf. [**KS**]), Theorem 1 holds.

Summarizing the results of (1) and (2), for case **(2)**, Theorem 1 is proved since $\Pi \cap \Lambda_n^+ = \{\alpha_2\}$.

(3) $\theta_C = (1, 0, 0, -1)$. It is easy to see that

$$\Lambda_K^+ = \left\{e_2, e_3, -e_1 \pm e_4, e_2 \pm e_3, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 + e_4)\right\};$$

$$\Lambda_n^+ = \left\{-e_1, e_4, -e_1 \pm e_2, -e_1 \pm e_3, e_2 \pm e_4, e_3 \pm e_4, \right.$$
$$\left. -\frac{1}{2}(e_1 \pm e_3 \pm e_3 - e_4)\right\};$$

$$\Pi_K = \{\alpha_1, e_3, \alpha_4, -e_1 + e_4\}, \quad 2\delta_K = -4e_1 + 3e_2 + e_3 - 2e_4.$$

(3.A) Let $\alpha = \alpha_2$. It is clear that

$$\Phi^- = \{-e_1 + e_3, e_2 - e_4\},$$
$$\Phi_\alpha^- = \{-e_1 + e_4, e_2 - e_3\};$$
$$\Phi^+ = \left\{e_4, -e_1 - e_3, e_2 + e_4, -\frac{1}{2}(e_1 \pm e_2 + e_3 - e_4)\right\},$$
$$\Phi_\alpha^+ = \left\{e_3, -e_1 - e_4, e_2 + e_3, -\frac{1}{2}(e_1 \pm e_2 - e_3 + e_4)\right\}.$$

By the Table 1.2 in [**BK1**], the following formulas are easily verified

(2.3.1)
$$\langle \Lambda, \alpha_1 \rangle \geq 0, \quad \langle \Lambda, \alpha_2 \rangle \geq -3 + \mu_\alpha,$$
$$\langle \Lambda, \alpha_3 \rangle \geq \frac{1}{2}\left(\left|\mu_\alpha - \frac{1}{2}\right| - \frac{1}{2}\right) + 2\frac{1}{2} + \frac{1}{2}(1 - \mu_\alpha), \quad \langle \Lambda, \alpha_4 \rangle \geq 0.$$

It follows from (2.3.1) that

$$\#(\Psi_0^-) \leq 1,\ \#(\Psi_1^-) = 0,\ \#(\Psi_0^+) \leq [5, 5, 0],\ \#(\Psi_1^+) \leq [0, 0, 2].$$

Therefore, we have $\min(\nu_0^+, \nu_0^-) \leq [2, 3, 2]$.

Suppose $\mu_\alpha \neq 1$. It is easy to see that $\delta^+ = e_2 - e_4$ and $\Lambda' = (\Lambda + \alpha)^\vee = \Lambda + e_2 - e_4$ is Λ_K^+ dominant. For this case, we have $\omega^+ = \delta^+ = e_2 - e_4$ and $\Lambda(\omega, +) = \{-(e_2 - e_3)\}$.

By Remark of 7.2 (or 3.1) in [**BK1**], (a) holds in 3.2 of [**BK1**]. It is clear that 2α is not a sum of some compact roots in $\Lambda(\omega, +) \cup \Lambda_K^+$. Thus, by Lemma 2.1, (b) holds in 3.2 of [**BK1**]. Clearly, $\Lambda' - \Lambda = e_2 - e_4$, so, (c) holds in 3.2 of [**BK1**]. Therefore, 3.2 of [**BK1**] shows that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > \nu_0^+ = c_0$.

We shall consider the irreducibility of $U(MAN,\sigma,\frac{1}{2}c\alpha)$ for $\mu_\alpha \neq 1$.

Let $\Lambda_L = \Lambda(\alpha_1,\alpha_2,\alpha_3)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{SO}(5,2)$. Clearly there is a unique positive noncompact root $\beta_0 = e_3+e_4$ such that β_0 is conjugate to $-\alpha = e_3 - e_4$ by K within L.

In the following, we shall consider the condition:

(2.3.q): $\quad \Lambda_\Gamma \subset \Lambda^0_{L,S} = \Lambda_L$.

Clearly, if (2.3.q) holds, then by (ii) in 11.2 of [**BK1**], $U^L(M_LAN_L,\sigma_L,\frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = 2$ since $\nu^+_{0,L} = 2$ and $\nu^-_{0,\Gamma} = 2$ if $\mu_\alpha = -1$,$\nu^+_{0,L} = 3$ and $\nu^-_{0,\Gamma} = 1$ if $\mu_\alpha = 0$. Clearly, by Table 2.1 of [**BK1**], we have

$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_1 \right\rangle \geq \left[-\frac{1}{2}, 0, \frac{1}{2}\right], \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_2 \right\rangle \geq 1,$$

$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_3 \right\rangle \geq \left[0, -\frac{1}{2}, -\frac{1}{2}\right], \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_4 \right\rangle \geq \frac{1}{2}. \tag{2.3.2}$$

Thus $\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle \geq 0$ for all β in $\Lambda_L(u)$, hence, by 8.2 of [**BK1**] it is easy to see that $U(MAN,\sigma,\frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = 2$ if (3.2.q) holds.

(1) Suppose that $\mu_\alpha = 0$.

(i) Suppose that (2.3.q) does not holds. Then $\alpha_1 \notin \Lambda_{K,\perp}$. Thus, it is easy to see that $\langle \wedge, \alpha_1\rangle > 0$, it follows that $\alpha \notin \Psi_0^-$. Therefore, for this case, $\#(\Psi_0^-) = 0$. Hence $\min(\nu_0^+,\nu_0^-) = \nu_0^+ = c_0 = 1$. It is shown that $J(MAN,\sigma,\frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = 1$. For this case, by 8.3 of [**BK1**], $U(MAN,\sigma,\frac{1}{2}\alpha)$ is irreducible for $0 < c < c_0' = c_0 = 1$, hence, Theorem 1 holds with $c_0' = c_0 = 1$ by continuity argument (cf. [**KS**]).

(ii) Suppose that (2.3.q) holds. Then $U(MAN,\sigma,\frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = 2$. Clearly we have $c_0' = 2$ and $c_0 = 3$, so, there is a "gap" $(c_0',c_0) = (2,3)$ that is called the gap (A.2). But $(c_0',c_0) = (2,3)$ is not a true gap since we shall show that $J(MAN,\sigma,\frac{1}{2}c\alpha)$ is infinitesimally unitary exactly when $0 < c < c_0' = c_0 \leq 2$ in Section 4 (see Proposition 4.1). Thus for this case, (A.2) holds in Theorem 1 by continuty argument (cf. [**KS**], Sect. 14).

(2) Suppose $\mu_\alpha = -1$. If (3.2.q) does not hold, then a similar argument as used in (1), (i) mentioned above shows that $\min(\nu_0^+,\nu_0^-) = \nu_0^+ = 1$ and Theorem 1 holds for this case with $c_0' = c_0 = 1$. If (3.2.q) holds, then $c_0' = c_0 = 2$, so, by contiunity argument (cf. [**KS**]), (A.2) holds in Theorem 1.

(3) Suppose $\mu_\alpha = 1$. Then $\min(\nu_0^+,\nu_0^-) = \nu_0^- \leq 2$.

Clearly, $\Lambda^+_{K,\perp} = \{\alpha_1,\alpha_4\}$. Thus, $\delta^- = -\alpha$ is $\Lambda^+_{K,\perp}$ dominant but $\wedge - \alpha$ is not Λ^+_K dominant. Clearly, $\gamma = e_3$ is a short compact root satisfying the conditions for producing ω^- in Section 3 of [**BK1**]. It is easy to see that $-\alpha+\gamma = e_4$ is not $\Lambda^+_{K,\perp}$ dominant since $\langle e_4, \alpha_4\rangle < 0$. Thus it is clear that $\wedge' = (\wedge - \alpha)^\vee = \wedge - \omega^-$. Here $\omega^- = -\frac{1}{2}(e_1+e_2+e_3-e_4)$ is the image of $-\alpha+\gamma = e_4$ under the reflection in α_4. For this case, $\omega^- \neq \delta^-$ and $\Lambda(\omega,-) = \{-\alpha_4, -e_3\}$.

Since α_4 is the only short simple root in $\Lambda^+_{K,\perp}$ but is not strongly orthogonal to β, 3.1 in [**BK1**] shows that (a) holds in 3.2′ in [**BK1**].

It is clear that -2α is not a sum of some compact roots in $\Lambda(\omega,-)\cup\Lambda^+_K$. Thus by Lemma 2.1, (b) holds in 3.2′ of [**BK1**].

Clearly $\wedge' - \wedge = \omega^- = \alpha_4+\alpha_3$, hence, we can prove that (c') holds in 3.2′ of [**BK1**]. Therefore, by 3.2′ in [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = \min(\nu_0^+, \nu_0^-) = \nu_0^- \leq 2$ since $\nu_0^+ \geq 2$.

Let $\Lambda_L = \Lambda(\alpha_2, \alpha_3, \alpha_4)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{Sp}(3, \mathbf{R})$. Clearly $e_3, -\frac{1}{2}(e_1+e_2-e_3+e_4) \in \Lambda_L$, so, $\nu^-_{0,L} = \nu_0^- \leq 2$ and $\nu^+_{0,L} = \nu_0^+ \geq 2$. By 11.2 in [**BK1**], $U^L(M_LAN_L, \sigma_L, \frac{1}{2}\alpha)$ is irreducible for $0 < c < c_0', c_0' = \nu^-_{0,L} \leq 2$.

By (2.3.2) $\langle \lambda_0 + \frac{1}{2}\alpha, \beta\rangle \geq 0$ for all β in $\Lambda_L(u)$. It follows from 8.2 in [**BK1**] that $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0 = \nu_0^- \leq 2$. Therefore, by continuity argument (cf. [**KS**]), Theorem 1 follows for this case.

Summarizing the results of (1) and (2), Theorem 1 follows for $\alpha = \alpha_2$.

(3.B) Let $\alpha = \alpha_3$. It is clear that

$$\begin{aligned}
\Phi^- &= \left\{-e_1, e_2+e_4, e_3+e_4, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 - e_4)\right\},\\
\Phi^-_\alpha &= \left\{-e_1-e_4, e_2, e_3, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 + e_4)\right\},\\
\Phi^+ &= \{-e_1, e_2-e_4, e_3-e_4\},\\
\Phi^+_\alpha &= \{-e_1+e_4, e_2, e_3\}.
\end{aligned}$$

By Table 1.2 in [**BK1**], the following formulas are easily verified

$$\begin{aligned}
&\langle \wedge, \alpha_1\rangle \geq 0, && \langle \wedge, \alpha_2\rangle \geq [-1,-1,-2],\\
&\langle \wedge, \alpha_3\rangle \geq \left[\frac{3}{2}, 2, \frac{5}{2}\right], && \langle \wedge, \alpha_4\rangle \geq 0.
\end{aligned}$$

Thus it follows that $\#(\Psi_0^-) = 1$ and $\#(\Psi_0^+) = 0$. We have $\min(\nu_0^+, \nu_0^-) = c_0 \leq [2,1,0]$. By 6.1 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0$. By 8.3 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0', c_0' = c_0$. Therefore, Theorem 1 follows for $\alpha = \alpha_3$.

It follows from **(3.A)** and **(3.B)** that Theorem 1 is proved for case **(3)** since $\Pi \cap \Lambda_n^+ = \{\alpha_2, \alpha_3\}$.

(4) $\theta_C = (0,1,0,-1)$. It is easy to see that

$$\Lambda_K^+ = \left\{-e_1, e_3, -e_1 \pm e_3, e_2 \pm e_4, -\frac{1}{2}(e_1 + ze_2 \pm e_3 + ze_4), z = \pm 1\right\},$$

$$\Lambda_n^+ = \Big\{e_2, e_4, -e_1 \pm e_2, -e_1 \pm e_4, e_2 \pm e_3, e_3 \pm e_4,$$
$$-\frac{1}{2}(e_1 + ze_2 \pm e_3 - ze_4), z = \pm 1\Big\};$$

$$\Pi_K = \{e_3, \alpha_4, e_2 \pm e_4\}, \quad 2\delta_K = -5e_1 + 2e_2 + e_3.$$

(4.A.a) Let $\alpha = \alpha_1$. It is clear that

$$\Phi^- = \left\{e_2, -e_1 + e_2, -\frac{1}{2}(e_1 - e_2 + e_3 + e_4)\right\},$$
$$\Phi_\alpha^- = \left\{e_3, -e_1 + e_3, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4)\right\};$$
$$\Phi^+ = \left\{e_3 \pm e_4, -e_1 - e_2, -\frac{1}{2}(e_1 + e_2 - e_3 - e_4)\right\},$$
$$\Phi_\alpha^+ = \left\{e_2 \pm e_4, -e_1 - e_3, -\frac{1}{2}(e_1 - e_2 + e_3 - e_4)\right\}.$$

By Table 1.2 in [**BK1**], the following formulas are easily verified

$$\begin{aligned} &\langle \Lambda, \alpha_1 \rangle \geq -1 + \mu_\alpha, \quad &&\langle \Lambda, \alpha_2 \rangle \geq 1 - \mu_\alpha, \\ &\langle \Lambda, \alpha_3 \rangle \geq \frac{1}{2}, &&\langle \Lambda, \alpha_4 \rangle \geq 0. \end{aligned} \tag{2.4.1}$$

It follows from (2.4.1) that

$$\#(\Psi_0^-) \leq 0, \quad \#(\Psi_1^-) \leq [0,0,2], \quad \#(\Psi_0^+) = 1, \quad \#(\Psi_1^+) = 0.$$

Therefore, we have $\min(\nu_0^+, \nu_0^-) \leq [0.1.2]$.

(1) Suppose that $\mu_\alpha = 1$.

Clearly, $\langle \Lambda, e_3 \rangle, \langle \Lambda, e_2 + e_4 \rangle > 0$. Thus, it is easy to see that $\delta^- = e_3 - e_4$ and $\Lambda' = (\Lambda - \alpha)^\vee = \Lambda + e_3 - e_4$ is Λ_K^+ dominant. So, $\omega^- = \delta^- = e_3 - e_4$, and $\Lambda(\omega, -) = \{-(e_2 - e_4)\}$.

Clearly, α_4 is orthogonal but not strongly to $e_3 - e_4$. Thus by 3.1 of [**BK1**], (a) holds in 3.2′. It is easy to see that -2α is not a sum of some compact roots in $\Lambda(\omega, -) \cup \Lambda_K^+$. Hence, by Lemma 2.1, (b) holds in 3.2, of [**BK1**]. Clearly, $\Lambda' - \Lambda = e_3 - e_4 = \alpha_2$, so, (c) holds in 3.2′ of [**BK1**]. Therefore, 3.2′ of [**BK1**] shows that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = \nu_0^+ \leq 2$.

Let $\Lambda_L = \Lambda(\alpha_1, \alpha_2, \alpha_3)$. Then L is a fundamental subgroup of G and $L \cong \mathrm{SO}(4.3)$. Let $\varepsilon = \alpha_3$. Then ε is short and $\xi(\varepsilon, \alpha) = \alpha_2$ is noncompact. It is easy to see that $\nu_{0,L}^- \leq 2$ and $\nu_{0,L}^+ \leq 3$. In the following, we shall consider the condition:

(2.4.q): $\Lambda_{L,S}^0 = \Lambda_\Gamma$.

(i) Suppose that (2.4.q) does not hold. Then $\alpha_2 \notin \Lambda_{K,\perp}$. Thus, it is easy to see that $\langle \wedge, \alpha_2 \rangle > 0$. Therefore, it follows that $\alpha_1 + \alpha_2 \notin \Psi_0^+$. So, $\#(\Psi_0^+) = 0$. Hence, we have $\min(\nu_0^+, \nu_0^-) = \nu_0^- = c_0 = 1$. It follows that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimslly unitary exactly for $0 < c \leq c_0' = c_0 = 1$. Consequently, Theorem 1 holds for this case.

(ii) Suppose that (2.4.q) holds. Clearly, $\nu_0^- = \nu_{0,L}^- = 2$ since the root $e_2 - e_4$ that is in Ψ_0^+ is also in Λ_L. Therefore, we have $\nu_{0,L}^- - 1 = 1$. It has been shown that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = 2$ and is infinitesimally unitary for $0 < c \leq c_0' = 1$ by continuity argument (cf. [**KS**]).

Since $c_0 = 2 > c_0' = 1$, there is a gap $(c_0', c_0) = (1, 2)$, (cf. the Remark of Theorem 1). Consequently, for this case, (A.1), (ii) holds in Theorem 1 by Proposition 4.1. The gap $(c_0', c_0) = (1, 2)$ is called the gap (A.1).

(2) Suppose that $\mu_\alpha \neq 1$. Then $\min(\nu_0^+, \nu_0^-) = \nu_0^+$.

By (2.4.1), $\langle \wedge, e_3 \rangle > 0$, and $\langle \wedge, e_2 + e_4 \rangle > 0$. Thus, $\delta^+ = \alpha$ is $\Lambda_{K,\perp}^+$ dominant. It is easily verified that $\wedge' = (\wedge + \alpha)^\vee = \wedge + \alpha$ is dominant for Λ_K^+. For this case, $\omega^+ = \delta^+ = \alpha$ and $\Lambda(\omega^+)$ is empty.

Clearly, by 7.2 (or 3.1) in [**BK1**], (a) holds in 3.2 of [**BK1**]. It is easily verified that 2α is not a sum of some compact roots in $\Lambda(\omega^+) \cup \Lambda_K^+$, so, by Lemma 2.1, (b) holds in 3.2 of [**BK1**]. Clearly, $\wedge' - \wedge = \alpha$, hence, (c) holds in 3.2 of [**BK1**]. Thus 3.2 in [**BK1**] shows that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0$ where $c_0 = \min(\nu_0^+, \nu_0^-) = 1$.

By 8.3 of [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = 1 = c_0$. Thus by continuity argument (cf. [**KS**]) $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary exactly for $0 < c \leq c_0' = c_0 = 1$.

Summarizing the results of (1) and (2), Theorem 1 follows for $\alpha = \alpha_1$.

(4.A.b) Let $\alpha = \alpha_2$. It is clear that

$$\Phi^- = \left\{ -e_1 - e_4, e_2 + e_3, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4) \right\},$$

$$\Phi_\alpha^- = \left\{ -e_1 - e_3, e_2 + e_4, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4) \right\},$$

$$\Phi^+ = \left\{ e_4, -e_1 + e_4, e_2 + e_3, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4) \right\},$$

$$\Phi_\alpha^+ = \left\{ e_3, -e_1 + e_3, e_2 - e_4, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4) \right\}.$$

By Table 1.2 of [**BK1**], the following formulas are easily verified

$$\langle \Lambda, \alpha_1 \rangle \geq \left[1, \frac{1}{2}, 0\right], \quad \langle \Lambda, \alpha_2 \rangle \geq \left[-1, -\frac{1}{2}, 0\right],$$

$$\langle \Lambda, \alpha_3 \rangle \geq \frac{1}{2}[3, 2, 1], \quad \langle \Lambda, \alpha_4 \rangle \geq [0, 0, 0]. \tag{2.4.2}$$

It follows from (2.4.2) that

$$\#(\Psi_0^-) \leq [0, 0, 0] = 0, \quad \#(\Psi_1^-) \leq [0, 0, 1],$$
$$\#(\Psi_0^+) \leq [1, 1, 1] = 1, \quad \#(\Psi_1^+) \leq [0, 0, 2].$$

Thus $\min(\nu_0^+, \nu_0^-) \leq [0, 1, 2]$. In fact, under the reflection in α, then μ_α and (0,1,0,-1) are replaced by $-\mu_\alpha$ and (0,1,-1,0) respectively, moreover, δ^+ and δ^- are replaced by δ^- and δ^+ respectively. Under the reflection in α, the data of case **(4.A.b)** with $\alpha = \alpha_2$ are replaced by the data of case **(2.A)** with $\alpha = \alpha_2$. For example, if $\min(\nu_0^+, \nu_0^-) = [m_{-1}, m_0, m_1]$ for case **(4.A.b)**, then $\min(\nu_0^+, \nu_0^-) = [m_1, m_0, m_{-1}]$ for case **(2.A)**. Therefore, by a similar argument used in **(2.A)**, Theorem 1 can be shown for this case. Similarly, if (2.2.q) holds, $\mu_\alpha = 1$ and $e_3 \notin \Psi_1^+$, then there is gap $(c_0', c_0) = (1, 2)$ that is the gap (A.1).

Remark. The details of the device, called *reflection* in α were given in [**BK1**] and [**BK3**] (cf. pp. 31, 35, 39 in [**BK1**] and p. 190 in [**BK3**]).

(4.B) Let $\alpha = \alpha_3$. It is easy to see that $\#(\Psi_0^+) = [2, 0, 0]$ and $\#(\Psi_0^-) = [1, 1, 2]$, hence $\min(\nu_0^+, \nu_0^-) = c_0 \leq [2, 1, 0]$. By 6.1 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0$. By 8.3 in [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0$. Therefore, Theorem 1 follows for $\alpha = \alpha_3$.

It follows from **(4.A.a)**, **(4.A.b)** and **(4.B)** that Theorem is proved for case **(4)** since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_2, \alpha_3\}$.

(5) $\theta_C = (1, 0, -1, 0)$.

(5.A.a) Let $\alpha = \alpha_1$. Under the reflection in α, then the data of the case **(5.A.a)** with $\alpha = \alpha_1$ are replaced by the data of the case **(1.A)** with $\alpha = \alpha_1$. Thus, a similar argument as in **(1.A)** shows that Theorem 1 holds for this case.

(5.A.b) Let $\alpha = \alpha_2$. Under the reflection in α, then the data of case **(5.A.b)** with $\alpha = \alpha_2$ are replace by the data of the case **(3.A)** with $\alpha = \alpha_2$. Thus, a similar argument as in **(3.A)** shows that Theorem 1 holds for this case.

It follows from **(5.A.a)** and **(5.A.b)** that for case **(5)**, Theorem 1 follows since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_2\}$.

(6) $\theta_C = (0, 0, 1, -1)$.

(6.A) Let $\alpha = \alpha_1$. Under the reflection in α, then the data of case **(6.A)** with $\alpha = \alpha_1$ are replaced by the data of the case of **(4.A.a)** with $\alpha = \alpha_1$. Therefore, by a similar argument used in **(4.A.a)**, Theorem 1 can be shown for this case. Similarly, if (2.4.q) holds and $\mu_\alpha = -1$, then there is a gap $(c_0', c_0) = (1, 2)$ that is the gap (A.1).

(6.B) Let $\alpha = \alpha_3$. It is easy to see that

$$\Lambda_K = \left\{-e_1, e_2, -e_1 \pm e_2, e_3 \pm e_4, -\frac{1}{2}(e_1 \pm e_2 + ze_3 + ze_4), z = \pm 1\right\}.$$

$$\Lambda_n = \Big\{e_3, e_4, -e_1 \pm e_3, -e_1 \pm e_4, e_2 \pm e_3, e_2 \pm e_4,$$
$$-\frac{1}{2}(e_1 \pm e_2 + ze_3 - ze_4), z = \pm 1\Big\}.$$

$$\Pi_K = \left\{e_2, -\frac{1}{2}(e_1 + e_2 + e_3 + e_4), e_3 \pm e_4\right\}, \quad 2\delta_K = -5e_1 + e_2 + 2e_3.$$

It is clear that

$$\Phi^- = \left\{e_3, -e_1 + e_4, e_2 + e_4, -\frac{1}{2}(e_1 \pm e_2 + e_3 - e_4)\right\},$$

$$\Phi_\alpha^- = \left\{e_3 - e_4, -e_1, e_2, -\frac{1}{2}(e_1 \pm e_2 + e_3 + e_4)\right\},$$

$$\Phi^+ = \left\{e_3, -e_1 - e_4, e_2 - e_4, -\frac{1}{2}(e_1 \pm e_2 - e_3 + e_4)\right\},$$

$$\Phi_\alpha^+ = \left\{e_3 + e_4, -e_1, e_2, -\frac{1}{2}(e_1 \pm e_2 - e_3 - e_4)\right\}.$$

By Table 1.2 in [**BK1**], the following formulas are easily verified

$$\langle \Lambda, \alpha_1 \rangle \geq 2, \qquad \langle \Lambda, \alpha_2 \rangle \geq [1, 0, 0],$$
$$\langle \Lambda, \alpha_3 \rangle \geq \left[-\frac{1}{2}, 0, \frac{1}{2}\right], \quad \langle \Lambda, \alpha_4 \rangle \geq 0.$$

It follows that $\#(\Psi_0^-) \leq [1, 2, 2]$ and $\#(\Psi_0^+) \leq [2, 2, 0]$. Therefore, we have $\min(\nu_0^+, \nu_0^-) \leq [2, 5, 0]$.

(1) Suppose that $\mu_\alpha \neq 0$. By a similar argument used in **(3.B)**, Theorem 1 can be shown for this case.

(2) Suppose that $\mu_\alpha = 0$. Let $\Lambda_L = \Lambda(\alpha_2, \alpha_3, \alpha_4)$. Then L is a fundamental subgroup of G, and $L \cong \mathrm{Sp}(2,1)$. Clearly, $\Psi_0^+ \subset \Lambda_L$, so, we have $\nu_{0,L}^- = \nu_0^-$. We shall consider the condition

(6.2.q): $\Lambda_{L,S}^0 = \Lambda_L$.

(i) Suppose that (2.6.q) does not hold. Then $\alpha_2 \notin \Lambda_{K,\perp}$. Thus, it is easy to see that $\langle \wedge, \alpha_2 \rangle > 0$. Then it is easily verified that $r_1, r_2 \notin \Lambda_{K,\perp}$, hence, $r_1, r_2 \notin \Psi_0^+$. Here $r_1 = e_3 + e_4$ and $r_2 = -\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$. Thus, it follows that $\#(\Psi_0^+) = 0$. Therefore, $\min(\nu_0^+, \nu_0^-) = \nu_0^- = c_0 = 1$. By 6.1 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary for $c > c_0 = 1$. By 8.3 of [**BK1**], $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = c_0 = 1$. Thus, for this case, Theorem 1 holds by continuity argument (cf. [**KS**]).

(ii) Suppose that (2.6.q) holds. Then, by 6.1 in [**BK1**] $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is not infinitesimally unitary when $c > \nu_0 = c_0 = 5$ or $\min(\nu_0^+, \nu_0^-) - 2 = c_0' < c < c_0$. By 11.1 in [**BK1**], it is easy to see that $U^L(M_L A N_L, \sigma_L, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = \nu_{0,L}^- - 2 = \nu_0^- - 2 = 3$. By Table 1.2 in [**BK1**], we have

$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_1 \right\rangle \geq 0, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_2 \right\rangle \geq \left[\frac{1}{2}, 0, \frac{1}{2}\right],$$
$$\left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_3 \right\rangle \geq \frac{1}{2}, \quad \left\langle \lambda_0 + \frac{1}{2}\alpha, \alpha_4 \right\rangle \geq \left[\frac{-1}{4}, 0, \frac{1}{4}\right]. \tag{2.6.1}$$

By (2.6.1), $\langle \lambda_0 + \frac{1}{2}\alpha, \beta \rangle \geq 0$ for all $\beta \in \Lambda_L(u)$. It follows from 8.2 and 8.3 of [**BK1**] that $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible for $0 < c < c_0' = 3$. Therefore, by continuity argument (cf. [**KS**]), it is easy to see that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for $0 < c \leq c_0' = 3$. Moreover in Section 4, we shall show that $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary for $c = c_0 = 5$ (cf. Lemma 4.3). Hence, (B.1) holds in Theorem 1 for this case. Clearly, for this case there is a gap $(c_0', c_0) = (3, 5)$ that is called the gap (B.1).

Summarizing the results of (1) and (2), Theorem 1 follows for $\alpha = \alpha_3$.

It follows from **(6.A)** and **(6.B)** that Theorem 1 is proved for case **(6)** since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_3\}$.

(1′) $\theta_C = (1, 1, 0, 0)$.

(1′.A) Let $\alpha = \alpha_1$. By similar methods used in case **(1)**, it is easily verified that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [2, 1, 0]$. Therefore, for this case, Theorem 1 can be shown by a similar argument used in case **(1)**.

(1′.B) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easily verified that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [2, 1, 0]$. Therefore, for this case, Theorem 1 can be shown by a similar argument used in **(3.B)**.

It follows from (**1′.A**) and (**1′.B**) that Theorem 1 is proved for case (**1′**) since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_4\}$.

(**2′**) $\theta_C = (0,1,1,0)$. Set $z = \pm 1$. It is easy to see that

$$\Lambda_K^+ = \left\{-e_1, e_4, -e_1 \pm e_4, e_2 \pm e_3, -\frac{1}{2}(e_1 + ze_2 - ze_3 \pm e_4)\right\},$$

$$\Lambda_n^+ = \Big\{e_2, e_3, -e_1 \pm e_2, -e_1 \pm e_3, e_2 \pm e_4, e_3 \pm e_4, \\ -\frac{1}{2}(e_1 + ze_2 + ze_3 \pm e_4)\Big\},$$

$$\Pi_K = \left\{\alpha_3, \alpha_4, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4), \alpha_1, e_2 + e_3\right\}, \quad \delta_K = -5e_1 + 2e_2 + e_4.$$

(**2′.A**) Let $\alpha = \alpha_2$. Clearly, we have

$$\Phi^- = \left\{e_3, -e_1 + e_3, e_2 - e_4, -\frac{1}{2}(e_1 - e_2 - e_3 + e_4)\right\},$$

$$\Phi_\alpha^- = \left\{e_4, -e_1 + e_4, e_2 - e_3, -\frac{1}{2}(e_1 - e_2 + e_3 - e_4)\right\},$$

$$\Phi^+ = \left\{-e_1 - e_3, e_2 + e_4, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4)\right\},$$

$$\Phi_\alpha^+ = \left\{-e_1 - e_4, e_2 + e_3, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4)\right\}.$$

By Table 2.1 of [**BK1**], the following formulas are easily verified

$$\begin{aligned} &\langle \Lambda, \alpha_1\rangle \geq 0, && \langle \Lambda, \alpha_2\rangle \geq [0,1,2], \\ &\langle \Lambda, \alpha_3\rangle \geq \left[\frac{1}{2}, 0, 0\right], && \langle \Lambda, \alpha_4\rangle \geq \frac{1}{2}[-3,-2,-1]. \end{aligned} \tag{2.2′.1}$$

It follows from (2.2′.1) that

$$\begin{aligned} \#(\Psi_0^-) &\leq [1,2,2], & \#(\Psi_1^-) &\leq [2,1,0], \\ \#(\Psi_o^+) &\leq [2,0,0], & \#(\Psi_1^+) &\leq [0,1,0]. \end{aligned}$$

Let $\beta = -\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$ and $\beta' = -\frac{1}{2}(e_1 - e_2 + e_3 + e_4)$. It is easy to see that β and β' are compact and strongly orthogonal to α. Thus, the fact that $\langle \lambda_0, s\rangle = 0, s = \beta, \beta'$ is in contradiction to nondegeneracy. So, it follows that $\langle \lambda_0, -e_1\rangle > 0$. Then we shall consider the case where $\lambda_0 = (t,0,0,0), t \in \mathbf{Z}, t > 1$ for $\mu_\alpha = -1$. Let $r_3 = \beta = e_3^*$, $r_2 = e_2 - e_3 = e_2^* - e_3^*$ and $r_1 = \alpha_2 = e_1^* - e_2^*$. Let $\Lambda_L = \Lambda(r_1, r_2, r_3)$. Then L is a standard subgroup of G and $L \cong \mathrm{SO}(5,2)$. The restriction λ_0^* of λ_0 to L can be written as $\lambda_0^* = \frac{1}{2}te_3^*$. Clearly, if $\mu_\alpha = -1$, then we have $t \in 2\mathbf{Z}, t > 0$.

Under these conditions, it follows from (2.2′.1) that $-e_1 - e_4$, $r \notin \Psi_0^+ \cup \Psi_1^+$ if $\mu_\alpha = -1$, $r \notin \Psi_1^+$ if $\mu_\alpha = 0$. Here $r = -\frac{1}{2}(e_1 + e_2 - e_3 + e_4)$. Thus by (2.2′.1) we obtain $\min(\nu_0^+, \nu_0^-) \leq [2,1,0]$.

Hence, similar arguments as used in case **(2.A)** show that Theorem 1 holds for this case. Similarly, if (2.2.q) holds, $\mu_\alpha = -1$, and $e_4 \notin \Psi_1^-$, then there is a gap $(c_0', c_0) = (1,2)$.

(2′.B) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easily verified that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [2,1,0]$, therefore, by a similar argument used in **(3.B)**, Theorem 1 can be shown for $\alpha = \alpha_4$.

It follows from **(2′.A)** and **(2′.B)** that Theorem 1 is proved for case **(2′)** since $\Pi \cap \Lambda_n^+ = \{\alpha_2, \alpha_4\}$.

(3′) $\theta_C = (1,0,0,1)$. It is easy to see that

$$\Lambda_K^+ = \left\{e_2, e_3, -e_1 \pm e_4, e_2 \pm e_3, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 - e_4)\right\},$$
$$\Lambda_n^+ = \Big\{-e_1, e_4, -e_1 \pm e_2, -e_1 \pm e_3, e_2 \pm e_4, e_3 \pm e_4,$$
$$-\frac{1}{2}(e_1 \pm e_2 \pm e_3 + e_4)\Big\},$$
$$\Pi_K = \left\{\alpha_1, e_3, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4), -e_1 - e_4\right\},$$
$$2\delta_K = -4e_1 + 3e_2 + e_3 + 2e_4.$$

(3′.A) Let $\alpha = \alpha_2$. It is clear that

$$\Phi^- = \left\{-e_1 + e_3, e_2 - e_4, -\frac{1}{2}(e_1 \pm e_2 - e_3 + e_4)\right\},$$
$$\Phi_\alpha^- = \left\{-e_1 + e_4, e_2 - e_3, -\frac{1}{2}(e_1 \pm e_2 + e_3 - e_4)\right\},$$
$$\Phi^+ = \{e_4, -e_1 - e_3, e_2 + e_4\},$$
$$\Phi_\alpha^+ = \{e_3, -e_1 - e_4, e_2 + e_3\}.$$

By Table 1.2 in [**BK1**], the following formulas are easily verified

$$\langle \Lambda, \alpha_1 \rangle \geq 0, \qquad \langle \Lambda, \alpha_2 \rangle \geq 1 + \mu_\alpha,$$
$$\langle \Lambda, \alpha_3 \rangle \geq \left[0, -1, -\frac{3}{2}\right], \quad \langle \Lambda, \alpha_4 \rangle \geq \frac{3}{2}. \tag{2.3′.1}$$

It follows from (2.3$'$.1) that

$$\#(\Psi_0^-) \leq [1,1,2], \quad \#(\Psi_1^-) \leq [0,2,1],$$
$$\#(\Psi_0^+) \leq [2,2,0], \quad \#(\Psi_1^+) \leq [0,0,1].$$

Let $\beta = -\frac{1}{2}(e_1 + e_2 - e_3 - e_4)$. It is clear that β is compact and strongly orthogonal to α. Therefore, the fact that $\langle \lambda_0, \beta \rangle = 0$ is in contradiction to nondegeneracy. Thus, it follows that $\langle \lambda_0, \beta \rangle > 0$. Therefore, we have $\langle \lambda_0, \alpha_3 \rangle > 0$ or $\langle \lambda_0, \alpha_4 \rangle > 0$. Under these conditions, it follows from (2.3$'$.1) that $-\frac{1}{2}(e_1 \pm e_2 + e_3 - e_4) \notin \Psi_1^-$ if $\mu_\alpha = 0$. Hence, by (2.3$'$.1), we obtain $\min(\nu_0^+, \nu_0^-) \leq [2,3,1]$.

Thus, similar arguments as used in case **(3.A)** show that Theorem 1 holds for this case. Similarly, if (2.3.q) holds and $\mu_\alpha = 0$, then there is a "gap" $(c_0', c_0) = (2,3)$ that will be considered in Section 4.

(3$'$.B.a) Let $\alpha = \alpha_3$. By similar methods used in **(3.B)**, it is easily verified that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [0,1,2]$. Therefore, by a simlar argument used in **(3.B)**, Theorem 1 can be proved for this case.

(3$'$.B.b) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easily verified that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [0,1,2]$. Therefore, by a similar argument used in **(3.B)**, Theorem 1 can be proved for this case.

It follows from **(3$'$.A)**, **(3$'$.B.a)** and **(3$'$.B.b)** that Theorem 1 is proved for case **(3$'$)** since $\Pi \cap \Lambda_n^+ = \{\alpha_2, \alpha_3, \alpha_4\}$.

(4$'$) $\theta_C = (0,1,0,1)$. It is easy to see that

$$\Lambda_K^+ = \left\{-e_1, e_3, -e_1 \pm e_3, e_2 \pm e_4, -\frac{1}{2}(e_1 + ze_2 \pm e_3 - ze_4), z = \pm 1\right\},$$
$$\Lambda_n^+ = \left\{e_2, e_4, -e_1 \pm e_2, -e_1 \pm e_4, e_2 \pm e_3, e_3 \pm e_4,\right.$$
$$\left. -\frac{1}{2}(e_1 + ze_2 \pm e_3 + ze_4), z = \pm 1\right\}.$$
$$\Pi_K = \left\{e_3, -\frac{1}{2}(e_1 + e_2 + e_3 - e_4), e_2 \pm e_4\right\},$$
$$2\delta_K = -5e_1 + 2e_2 + e_3.$$

(4$'$.A.a) Let $\alpha = \alpha_1$. It is clear that

$$\Phi^- = \left\{e_2, , -e_1 + e_2, -\frac{1}{2}(e_1 - e_2 + e_3 - e_4)\right\},$$
$$\Phi_\alpha^- = \left\{e_3, -e_1 + e_3, -\frac{1}{2}(e_1 + e_2 - e_3 - e_4)\right\};$$

$$\Phi^+ = \left\{e_3 \pm e_4, -e_1 - e_2, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4)\right\},$$
$$\Phi^+_\alpha = \left\{e_2 \pm e_4, -e_1 - e_3, -\frac{1}{2}(e_1 - e_2 + e_3 + e_4)\right\}.$$

By Table 1.2 in [**BK1**], the following formulas are easily verified

$$\langle \Lambda, \alpha_1 \rangle \geq -1 + \mu_\alpha, \quad \langle \Lambda, \alpha_2 \rangle \geq 1 - \mu_\alpha,$$
$$\langle \Lambda, \alpha_3 \rangle \geq \frac{1}{2}, \qquad \langle \Lambda, \alpha_4 \rangle \geq -\frac{1}{2}. \tag{2.4$'$.1}$$

It follows from $(2.4', 1)$ that

$$\#(\Psi_0^-) = 0, \quad \#(\Psi_1^-) \leq [0,0,2],$$
$$\#(\Psi_0^+) \leq 3, \quad \#(\Psi_1^+) = 0.$$

Let $\beta = -\frac{1}{2}(e_1 + e_2 + e_3 - e_4)$ and $\beta' = \frac{1}{2}(e_1 - e_2 - e_3 + e_4)$. It is clear that β and β' are compact and strongly orthogonal to α. Then the fact that $\langle \lambda_0, s \rangle = 0$, $s = \beta, \beta'$ is in contradiction to nondegeneracy. Hence, it follows that $\langle \lambda_0, -e_1 \rangle > 0$. Under these conditions,it follows from (2.4$'$.1) that $-\frac{1}{2}(e_1 + e_2 - e_3 - e_4) \notin \Psi_1^-$ and $-e_1 - e_3, -\frac{1}{2}(e_1 - e_2 + e_3 + e_4) \notin \Psi_0^+$ if $\mu_\alpha = 1$. Thus $\min(\nu_0^+, \nu_0^-) \leq [0,1,2]$.

Thus, similar arguments as used in cases **(4.A.a)** show that Theorem 1 holds. Similarly, if (2.2.q) holds and $\mu_\alpha = 1$, then there is a gap $(c_0', c_0) = (1,2)$ that will be considered in Section 4.

(**4$'$.A.b**) Let $\alpha = \alpha_2$. Under the reflection in α, then the data of the case (**4$'$.A.b**) with $\alpha = \alpha_2$ are replaced by the data of the case (**2**,.**A**) with $\alpha = \alpha_2$. Hence, similar arguments as used in (**2$'$.A**) show that for this case Theorem 1 holds. Similarly, if (2.2.q) holds, $\mu_\alpha = 1$ and $e_3 \notin \Psi_1^+$, then there is a gap $(c_0', c_0) = (1,2)$.

(**4$'$.B.a**) Let $\alpha = \alpha_3$. By similar methods used in **(4.B)**, it is easy to see that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [0,1,2]$. By a similar argument used in **(4.B)**, we can show that Theorem 1 holds for this case.

(**4$'$.B.b**) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easy to see that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [0,1,2]$. By a similar argument used in **(3.B)**, Theorem 1 can be proved for this case.

Therefore, it follows from (**4$'$.A**), (**4$'$.B.a**) and (**4$'$.B.b**) that Theorem 1 is proved for case (**4$'$**) since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_3, \alpha_4\}$.

(**5$'$**) $\theta_C = (1,0,1,0)$.

(**5$'$.A.a**) Let $\alpha = \alpha_1$. Under the reflection in α, then the data of the case **(5.A.a)** with $\alpha = \alpha_1$ are replaced by the data of the case **(1.A)** with

$\alpha = \alpha_1$. Therefore, by a similar argument used in case **(1.A)**, Theorem 1 can be shown for this case.

(5′.A.b) Let $\alpha = \alpha_2$. Under the reflection in α, then the data of the case (**5′.A.b**) with $\alpha = \alpha_2$ are replaced by the data of the case (**3′.A**) with $\alpha = \alpha_2$. Therefore, by a similar argument used in case (**3′.A**), Theorem 1 can be shown for this case.

(5′.B.b) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easy to see that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [2, 1, 0]$. By a similar argument used in **(3.B)**, Theorem 1 can be shown for this case.

It follows from (**5′.A.a**), (**5′.A.b**) and (**5′.B.b**) that Theorem 1 is proved for case (**5′**) since $\Pi \cap \Lambda_n^+ = \{\alpha_2, \alpha_3, \alpha_4\}$.

(6′) $\theta_C = (0, 0, 1, 1)$.

(6′.A) Let $\alpha = \alpha_1$. Under the reflection in α, then the data of the case (**6′.A**) with $\alpha = \alpha_1$ are replaced by the data of the case (**4′.A.a**) with $\alpha = \alpha_1$. Therefore, by similar argument as used in case (**4′.A.a**), Theorem 1 can be shown for this case. Similarly, if (2.4.q) holds and $\mu_\alpha = -1$, then there is a gap $(c_0', c_0) = (1, 2)$ that will be considered in Section 4.

(6′.B.a) Let $\alpha = \alpha_3$. For this case, as in the case **(6.B)**, it is easy to see that $\min(\nu_0^+, \nu_0^-) = c_0 \leq [0, 5, 2]$ and $c_0' \leq [0, 3, 2]$. By a similar argument used in case **(6.B)**, we can show that for $\mu_\alpha \neq 0$, Theorem 1 holds and, for $\mu_\alpha = 0$, (B.1) holds in Theorem.

(6′.B.b) Let $\alpha = \alpha_4$. By similar methods used in **(3.B)**, it is easy to see that $\min(\nu_0^+, \nu_0^-) = c_0 = c_0' \leq [0, 1, 2]$. By a similar argument used in **(3.B)**, Theorem 1 can be shown for this case.

It follows from (**6′.A**), (**6′.B.a**) and (**6′.B.b**) that Theorem 1 is proved for case (**6′**) since $\Pi \cap \Lambda_n^+ = \{\alpha_1, \alpha_3, \alpha_4\}$.

The proof of Theorem 1 is complete. □

3. The Reducibility for the Gaps.

The reducibility of the standard induced representations of G is important in the study of unitary representations of G. B. Speh and D.A. Vogan [**SV**], and Barbasch and D.A. Vogan [**BV**] gave an algorithm for computing composition series of the standard induced representations of G. Baldoni-Silva and A.W. Knapp [**BK2**] use Vogan's algorithm mentioned above to determine some irreducibility questions that arise in [**BK1**]. In this section, we shall use Vogan's algorithm to determine some reduciblility questions that arise in the discussions for the gaps mentioned in Section 2.

By the results of Section 2, it is clear that in the cases of **(4.A.a)**,(1),(ii) and of **(3.A)**,(2),(ii), there are the gaps (A.1) and (A.2) respectively. The case **(4.A.a)**,(1),(ii) is called the case of gap (A.1), and for this case we have
(3.1)

$$\lambda_0 = \lambda_{0,b} = (-1,0,0,0), \mu_\alpha = 1, \wedge = \frac{1}{2}(-3,1,1,1), \nu = \frac{1}{2}\alpha, c_0' = 1, c_0 = 2.$$

The case **(3.A)**,(2),(ii) is called the case of gap (A.2) and for this case, we have
(3.2)

$$\lambda_0 = \lambda_{0,b} = \frac{1}{2}(-3,1,0,0), \mu_\alpha = 0, \wedge = (-3,0,0,3), \nu = \frac{1}{2}2\alpha, c_0' = 2, c_0 = 3.$$

The data given by (3.1) and (3.2) are called the data of gap (A.1) and of gap (A.2) respectively.

First we shall use some notations given by D. Barbasch and D.V. Vogan [**BV**].

Let $R(\lambda_0 \otimes \nu) = \{r \in \Lambda \mid 2\langle\gamma, r\rangle/\langle r, r\rangle = (\gamma, r) \in \mathbf{Z}\}$. Here $\gamma = (\lambda_0 \otimes \nu) = \lambda_0 + \nu$. It is clear that $R(\lambda_0 \otimes \nu)$ has a decomposition

$$R(\lambda_0 \otimes \nu) = R^{++} \cup R_0 \cup R^{--}$$

of the roots according to whether their inner products with γ are positive, zero, or negative. Let $\phi = \alpha$. Then $R(\lambda_0 \otimes \nu)^\alpha = R(\lambda_0 \otimes \nu)$. Choose a positive root system R_0^+ so that

(a) $R_0^+ \supset \Lambda_-^+ \cap R_0$.

(b) If $r \in R_0$ and $(-\theta)r \in R^{++}$, then $r \in R_0^+$.

(c) If $r, (-\theta)r \in R_0^+, r \neq (-\theta)r$, then both belong to R_0^+, or neither does.

Define Π^R be the simple root system of the positive root system $R^+(\lambda_0 \otimes \nu) = R^{++} \cup R_0^+$. If $-\theta$ does not preserve $R^+(\lambda_0 \otimes \nu)$, then

$$\alpha = \sum_{r \in \Pi^R} n_r r$$

with n_r a nonegative rational number. We define

$$\Pi_{\text{crit}} = \{r \in \Pi^R \mid n_r \neq 0\}.$$

Let $C(\lambda_0 \otimes \nu)^\alpha$ be the span of Π_{crit}.

Lemma 3.1. *For the case of gap* (A.1), (*resp. of gap* (A.2)), Π_{crit} *is given by* (3.1.1) (*resp. by* (3.1.2)) *below. It is isomorphic to a subset of*

Λ^+ *containing* α *and the isomorphism* φ *preserveing the additional structure* (1) – (5) *described in* [**BV**], (*cf. p.* 384 *in* [**BV**]).

Proof. It follows from the data of gap (A.1) given by (3.1) that

$$\gamma = (\lambda_0 \otimes \nu) = \lambda_0 + \frac{1}{2}\alpha = \left(-1, \frac{1}{2}, -\frac{1}{2}, 0\right).$$

It is easily verified that

$$\begin{aligned} &R^+\left(\left(-1, 0, \frac{1}{2}, -\frac{1}{2}\right)\right) \\ &= \left\{-e_1, e_2, -e_3, e_4, e_2 \pm e_3, -e_1 \pm e_4, -\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\right\}. \end{aligned}$$

It follows that

$$\Pi^R = \left\{e_2 + e_3, -e_3, -\frac{1}{2}(e_1 + e_2 - e_3 + e_4), e_4\right\}.$$

It is easily shown that

$$\Pi_{\text{crit}} = \{e_2 + e_3 = r_1^*, -e_3 = r_2^*\}. \tag{3.1.1}$$

Here r_1^*, r_2^* can be written as $r_1^* = e_1^* - e_2^*, r_2^* = e_2^*$.

It is easy to see that $\varphi\Pi_{\text{crit}} = \{e_2 - e_3, e_3\}$ where the isomorphism φ is the reflection in the hyperplane orthogonal to the root e_3. Clearly $e_3 \in \Lambda_K$, hence, φ preserves the additional structure (1)–(5) given by [**BV**], (cf. p. 384 in [**BV**]).

It follows from the data of gap (A.2) given by (3.2) that

$$\gamma = (\lambda_0 \otimes \nu) = \lambda_0 + \alpha = \frac{1}{2}(-3, 1, 2, -2).$$

Set $x = (x_1, x_2, x_3, x_4), x_i = \pm 1, i = 1, 2, 3, 4$. By computing, we have

$$R^+\left(\frac{1}{2}(-3, 1, 2, -2)\right) = \left\{-e_1, e_2, e_3, -e_4, -e_1 \pm e_2, e_3 \pm e_4, \frac{-\eta_x}{2}x\right\}.$$

Here $\eta_x = -1$ if $x = (1, -1, 1, -1), (1, 1, 1, -1)$, $\eta_x = 1$ otherwise.

Hence, it is easy to see that

$$\Pi^R = \left\{-e_3 - e_4, -\frac{1}{2}(1, 1, -1, -1), \frac{1}{2}(1, 1, 1, -1), -\frac{1}{2}(1, -1, 1, 1)\right\}.$$

Therefore, we obtain

$$\Pi_{\text{crit}} = \left\{-e_3 - e_4 = r_3^*, -\frac{1}{2}(1, 1, -1, -1) = r_2^*, \frac{1}{2}(1, 1, 1, -1) = r_1^*\right\}. \tag{3.1.2}$$

Here r_1^*, r_2^*, r_3^* can be written as $r_1^* = e_1^* - e_2^*, r_2^* = e_2^* - e_3^*, r_3^* = 2e_3^*$.

Clearly, $\varphi\Pi_{\text{crit}} = \{\alpha_2, \alpha_3, \alpha_4\}$ where $\varphi = \varphi_1\varphi_2$. Here φ_1 (resp. φ_2) is the reflection in the hyperplane orthogonal to α_4 (resp. e_3). Clearly, the isomorphism φ preserves the additional structure (1)-(5) since α_4 and e_3 are roots in Λ_K.

The proof is complete. □

If $x = x_1e_1^* + x_2e_2^*$ in case (A.1) (resp. $x = x_1e_1^* + x_2e_2^* + x_3e_3^*$ in case (A.2)), then $(x_1, x_2)^*$ (resp. $(x_1, x_2, x_3)^*$) is called the coordinate of x for Π_{crit}.

We shall directly use some results given by B. Speh and D.A. Vogan [**SV**] and D.A. Vogan [**V2**] and we shall introduce some notations given by Baldoni-Silva and A.W. Knapp [**BK2**].

Let L be the standard subalgebra of g with $\Lambda_L = \Lambda(S), S \subset \Lambda$, and let L denote the standard subgroup of G with Lie algebra L also.

Let us fix a compact Cartan subgroup B_L of L with Lie algebra $b_L = b \cap L$. We shall be working with some Cartan subalgebras $b_{-,L} + \mathbf{a}$ where $b_{-,L} = b_- \cap b_L$ and $\mathbf{a} \subset g_-$ formed by Cayley transform relative to a succession of noncompact roots in an ordered set $\{...\}$ and we can write $\mathbf{a} \Rightarrow \{...\}$ for $\mathbf{a}$. Let A be the subgroup of G with Lie algebra $\mathbf{a}$. For subgroup A, there is a standard cuspidal parabolic subgroup $P_L = M_LAN_L$ of L. Let $\lambda_{0,L}, \mu_{\alpha,L}$ and ν_L be the restriction of λ_0, μ_α and ν to L which are defined by (3.1b) of [**BK1**] respectively. Here λ_0, μ_α and ν are the data given by the cases of gap (A.1) or of gap (A.2). Let σ_L be the representation determined by $\lambda_{0,L}, \mu_{\alpha,L}$ and ν_L. If $\gamma_L = \lambda_{0,L} + \nu_L$ is singular, then there is regular $\gamma_{0,L}$ obtained by adding to γ_L a suitable parameter that is dominant integral for $\Lambda_L^+ = \Lambda_L \cap \Lambda^+$ and adjusting $\mu_{\alpha,L}$ compatible. Let $\sigma_{0,L}$ be the representation determined by $\gamma_{0,L}$ and $\mu_{\alpha,L}$. We denote by $U^L(M_LAN_L, \sigma_{0,L}, \nu_L)$ and $J^L(M_LAN_L, \sigma_{0,L}, \nu_L)$ the induced representation for group L and its Langlands quotient respectively. Let $\pi(\gamma_L; \{...\})$ and $\bar{\pi}(\gamma_L; \{...\})$ be the global characters of $U^L(M_LAN_L, \sigma_{0,L}, \nu_L)$ and of $J^L(M_LAN_L, \sigma_{0,L}, \nu_L)$ respectively. (Baldoni-Silva and Knapp's notations in [**BK2**] differs slightly from this: they use $\pi(\gamma_L, \mathbf{a} \leftrightarrow \{...\})$ and $\bar{\pi}(\gamma_L, \mathbf{a} \leftrightarrow \{...\})$ for $\pi(\gamma_L; \{...\})$ and $\bar{\pi}(\gamma_L; \{...\})$ respectively.)

In the following, we shall directly use the notations given by [**BK2**]. For each $\beta \in \Lambda_L$, let s_β denote the wall-crossing functor which acts on the local expression for a global character by the reflection in the hyperplane orthogonal to β (or the reflection in the hyperplane orthogonal to β on E_R). We say that β is in the τ- invariant of $\bar{\pi}(\gamma_L; \{...\})$ (denoted by $\beta \in \tau(\bar{\pi}(\gamma_L; \{...\}))$) if $s_\beta\bar{\pi}(\gamma_L; \{...\}) = 0$. Let ϕ denote the empty set.

Lemma 3.2. *Let $\Lambda_L = \Lambda(\Pi_{\text{crit}})$. Then $U^L(M_L A N_L, \sigma_L, \nu_L)$ is reducible.*

Proof. First, we show the lemma for the case of the gap (A.1). By (3.1), we have $\gamma = \lambda_0 + \nu = \frac{1}{2}(-2, 1, -1, 0)$ (in system given by (1.1)). It is easy to see that

$$(\gamma, r_1^*) = 0, (\gamma, r_2^*) = 1.$$

Thus $\gamma_L = \frac{1}{2}(1,1)^*$ is dominant for Π_{crit}. Clearly, $\alpha = (1,1)^*$, so, $(\gamma_L, \alpha) = 1$. Since $\mu_{\alpha,L} = 1$,α (or σ_L) is a cotangent case. Hence α does not satisfy the parity condition.

We number the simple roots of simple Lie algebra B_2 (from left to right) as 1 and 2 (2 is shorter).

Let $\Pi^\vee = \{r_1^* + 2r_2^*, -r_2^*\}$ and $\beta = r_2^* = (0.1)^*$. For convenience, let s_2 denote s_β. Clearly, $s_2\gamma_L$ is dominant for $\Pi^\vee$ and the set of singular roots in $\Pi^\vee$ for $s_2\gamma_L = \frac{1}{2}(1,-1)^*$ is the set $\{(1,1)^*\} = \{1\}$. It is easy to see that $\alpha \notin s_2\Pi^\vee = \Pi_{\text{crit}}$.

Since α does not satisfy the parity condition and α is a simple root in $\Pi^\vee$, moreover, $s_2\gamma_L$ is $\Pi^\vee$ dominant and is integral, by Theorem 1.2 of [**BK2**], we have

$$\pi(s_2\gamma_L; \alpha) = \bar{\pi}(s_2\gamma_L; \alpha). \tag{3.2.1}$$

Since β is complex, it follows from Theorem 1.5 of [**BK2**] that

$$s_2\pi(s_2\gamma_L; \alpha) = \pi(s_2 s_2\gamma_L; \alpha) = \pi(\gamma_L; \alpha). \tag{3.2.2}$$

By Theorem 1.6 of [**BK2**], we have

$$s_2\pi(\gamma_L; \alpha) = \bar{\pi}(s_2\gamma_L; \alpha) + \bar{\pi}(\gamma_L; \alpha) + \Theta_0. \tag{3.2.3}$$

Here Θ_0 must occur on the right side of (3.2.1) and must have the simple root 2 in their τ-invariants. Clearly, $s_2\gamma_L$ is dominant for $\Pi^\vee$ so it is easy to see that $\theta((0,-1)^*) = (1,0)^*$ is a positive root, hence, by Theorem 1.4 of [**BK2**], we have $\tau(\bar{\pi}(\gamma_L; \alpha)) = \phi$. Thus $\Theta_0 = 0$.

Clearly, γ_L is dominant for $s_2\Pi^\vee$. By Theorem 1.4 of [**BK2**], $\tau(\bar{\pi}(\gamma_L; \alpha)) = \{(0,1)^*\} = \{2\}$ since $\theta((0,1)^*) = (-1,0)^*$ is negative root (the number of $(0,1)^*$ is 2 in $s_2\Pi^\vee$).

Clearly, the set $\{2\}$ is disjoint the singular root set $\{1\}$. Therefore, by Theorem 1.3 of [**BK2**], it follows from (3.2.2) and (3.2.3) that $U^L(M_L A N_L, \sigma_L, \frac{1}{2}\alpha)$ is reducible into two pieces for the case of gap (A.1).

Now we shall prove the lemma for the case of gap (A.2). By (3.2), we have $\gamma = \lambda_0 + \nu = \frac{1}{2}(-3, -1, 2, -2)$ (in the system given by (1.1)). It is easy to see that

$$(\gamma, r_1^*) = (\gamma, r_2^*) = 1, (\gamma, r_3^*) = 0.$$

Thus $\gamma_L = (2,1,0)^*$ that is dominant for Π_{crit}. Clearly, $\alpha = (2,0,0)^*$, so, $(\gamma_L, \alpha) = 2$. Since $\mu_{\alpha,L} = 0$,α (or σ_L) is a tangent case. Hence α does not satisfy the parity condition.

We number the simple roots of simple Lie algebra C_3 from left to right as 1,2 and 3 (3 is longer).

Let $\Pi^\vee = \{r_3^* + 2r_2^* + 2r_1^* = 2e_1^*, -r_1^* = -e_1^* + e_2^*, -r_2^* = -e_2^* + e_3^*\}$.

Let $\beta_1 = (-1,0,1)^*$, $\beta_2 = (-1,1,0)^*$ and $\beta_2' = (0,-1,1)^*$. Let $s_i = s_{\beta_i}, i = 1,2$ and $s_2' = s_{\beta_2'}$. It is easily verified that $\alpha = (2,0,0)^*$ is a simple root in $\Pi^\vee$. Clearly, $s_2 s_1 s_2' \gamma_L = (0,1,2)^*$ is dominant for $\Pi^\vee$ and is integral. The set of singular roots in $\Pi^\vee$ for $s_2 s_1 s_2' \gamma_L$ is $\{3\}$. It is clear that $\alpha \notin s_2' s_1 s_2 \Pi^\vee = \Pi_{\mathrm{crit}}$.

Since $\alpha \in \Pi^\vee$ and α does not satisfy the parity condition, by Theorem 1.2 of [**BK2**], we have

$$\pi(s_2 s_1 s_2' \gamma_L; \alpha) = \bar{\pi}(s_2 s_1 s_2' \gamma_L; \alpha). \tag{3.2.4}$$

Clearly, β_2 is a complex root, thus by Theorem 1.5 of [**BK2**], we have

$$s_2 \pi(s_2 s_1 s_2' \gamma_L; \alpha) = \pi(s_1 s_2' \gamma_L; \alpha). \tag{3.2.5}$$

By Theorem 1.6 of [**BK2**], it follows from (3.2.4) and (3.2.5) that

$$\pi(s_1 s_2' \gamma; \alpha) = \bar{\pi}(s_2 s_1 s_2' \gamma_L; \alpha) + \bar{\pi}(s_1 s_2' \gamma_L; \alpha) + \Theta_1. \tag{3.2.6}$$

By Theorem 1.4 of [**BK2**], we obtain

$$\tau(\bar{\pi}(s_2 s_1 s_2' \gamma_L; \alpha)) = \{1\}, \tau(\bar{\pi}(s_1 s_2' \gamma_L; \alpha)) = \{2\}. \tag{3.2.7}$$

Thus, by Theorem 1.6 of [**BK2**] it follows from (3.2.7) and (3.2.4) that $\Theta_1 = 0$. Clearly, β_1 is complex, so, by Theorem 1.5 of [**BK2**], we have

$$s_1 \pi(s_1 s_2' \gamma_L; \alpha) = \pi(s_2' \gamma_L; \alpha). \tag{3.2.8}$$

By Theorem 1.6 of [**BK2**], it follows from (3.2.7) and (3.2.8) that

$$\pi(s_2' \gamma_L; \alpha) = -\bar{\pi}(s_2 s_1 s_2' \gamma_L; \alpha) + \bar{\pi}(s_1 s_2' \gamma_L; \alpha) + \bar{\pi}(s_2' \gamma_L; \alpha) + \Theta_2. \tag{3.2.9}$$

By Theorem 1.4 of [**BK2**], we obtain

$$\tau(\bar{\pi}(s_2' \gamma_L; \alpha)) = \{1,2\}. \tag{3.2.10}$$

Clearly, we have $\Theta_2 = c_2 \pi(s_2 s_1 s_2' \gamma_L; \alpha)$ by (3.2.6) and (3.2.7), where c_2 is a constant. Using similar methods used in [**BK2**], it is easily verified that $c_2 = 1$ by Theorem 1.7 of [**BK2**]. Therefore, by (3.2.9), we have

$$\pi(s_2' \gamma_L; \alpha) = \bar{\pi}(s_1 s_2' \gamma_L; \alpha) + \bar{\pi}(s_2' \gamma_L; \alpha). \tag{3.2.11}$$

It is easy to see that β_2' is a m-compact root. Clearly, $2 \in \tau(\bar{\pi}(s_1 s_2' \gamma_L; \alpha))$ by (3.2.7) and $2 \in \tau(\bar{\pi}(s_2'\gamma_L; \alpha))$ by (3.2.10). Thus, by Theorem 1.6 of [**BK2**], it follows from (3.2.11) that

$$-\pi(\gamma_L; \alpha) = -\bar{\pi}(s_1 s_2' \gamma_L; \alpha) - \bar{\pi}(\gamma_L; \alpha). \tag{3.2.12}$$

It is easy to see that the sets $\{2\}$ and $\{1, 2\}$ are disjoint from the set $\{3\}$ of the set of singular roots, hence, by Theorem 1.3 of [**BK2**], it follows from (3.2.7), (3.2.10) and (3.2.12) that $U^L(M_L A N_L, \sigma_L, \alpha)$ is reducible into two pieces for the case of gap (A.2). The proof is complete. □

Combining Lemma 3.1, Lemma 3.2 and Theorem of [**BV**], we obtain the following lemma immediately.

Lemma 3.3. *For gap* (A.1) *or gap* (A.2), $U(MAN, \sigma, \frac{1}{2}c_0'\alpha)$ *is reducible.*

D.A. Vogan [**V1**], [**V3**] and Speh and Vogan [**SV**] used the θ-stable parabolic subgroups of g^C to study of unitary representations of semisimple Lie groups. We shall not need their detailed construction. It is enough to have the following result.

For each subset $S^\dagger$ of Λ, let $-S^\dagger = \{-r \mid r \in S^\dagger\}$. Let S be a given subset of Λ. A subset $S^\dagger$ of Λ is said to be a supplement of S in Λ if $S^\dagger$ satisfies the following condition:

(s.1) $S^\dagger \cap \Lambda(S) = \phi$,

(s.2) $S^\dagger \cup \Lambda(S) \cup -S^\dagger = \Lambda$,

(s.3) $S^\dagger \cap -S^\dagger = \phi$,

(s.4) there exists a ζ in ib' such that $\langle \zeta, r\rangle \geq 0$ for all $r \in S^\dagger$,

(s.5) there is a positive root system Λ_0^+ in the root system $\Lambda_0 = \{r \in \Lambda \mid \langle \zeta, r\rangle = 0\}$ such that $\Lambda_0^+ \cap S^\dagger = \Lambda_0 \cap S^\dagger$.

For fixed $S \subset \Lambda$ and a fixed supplement $S^\dagger$ of S in Λ, define

$$l^C = b^C + \sum_{r \in \Lambda(S)} g_r^C, u = u^C = \sum_{r \in S^*} g_r^C.$$

For convenience, let Λ_l and $\Lambda(u)$ denote $\Lambda(S)$ and $S^\dagger$ respectively. Let

$$\mathbf{q} = l^C + u. \tag{3.e}$$

It is easily verified that the following conditions are satisfied:

(a) $\theta(\mathbf{q}) = \mathbf{q}$, (since for any $r \in \Lambda, \theta(e_r) = \pm e_r$).

(b) $l^C = \bar{\mathbf{q}} \cap \mathbf{q}$, (by Lemma 1.1, (s.1) and (s.3)).

(c) $g^C = \bar{u} + l^C + u$, (by Lemma 1.1, (s.2) and (s.3)).

(d) (3.e) is Levi decompostion of $\mathbf{q}$ with Levi factor l, (by (s.4), (s.5) and (s.3)).

By (a), (b), (c), (d) and (3.e), we obtain the following Lemma immediately.

Lemma 3.4. *With above notations, $\mathbf{q}$ is a θ-stable parabolic subalgebra of g^C and $\mathbf{q}$ is determined by the subset S and $S^\dagger$ of Λ.*

The subalgebra $\mathbf{q}$ defined by (3.e) is called to be the θ- stable parabolic subalgebra determined by $(S, S^\dagger)$. Let $l = l^C \cap g$ and L be the normalizer of $\mathbf{q}$ in G. Clearly, $\Lambda_L = \Lambda(S)$.

Lemma 3.5. *With above notations, for gap* (A.2), $U(MAN, \sigma, \frac{1}{2}3\alpha)$ *is irreducible.*

Proof. We number the simple roots of simple Lie algebra B_3 from left to right as 1, 2 and 3 (3 is shorter).

Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$ and let $\Lambda_l = \Lambda_L = \Lambda(\alpha_1, \alpha_2, \alpha_3)$. Let

$$S^\dagger = \left\{-e_1 \pm e_2, -e_1 \pm e_3, -e_1 \pm e_4, -e_1, -\frac{\eta_x}{2}x, x = (e_1 \pm e_2 \pm e_3 \pm e_4)\right\}.$$

Here $\eta_x = -1$ if $x = (1,-1,1,-1)$ or $x = (1,1,1,-1)$, $\eta_x = 1$ otherwise, (cf. the coordinates given by (1.1)). It is easily verified that the subset S and $S^\dagger$ of Λ safisfy the condition (s.1)–(s.5) (letting $\zeta = \gamma = \lambda_0 + \nu = \frac{1}{2}(-3,1,3,-3)$). Let $\mathbf{q}$ be the θ-stable parabolic subalgebra determined by $(S, S^\dagger)$. Then l^C is its Levi factor. Let $\Lambda_L^+ = \Lambda_L \cap \Lambda^+$ and $\Lambda'^+ = \Lambda_L^+ \cup \Lambda(u)$. Define

$$\lambda_{0,L} = \lambda_0 - \delta(u), \mu_{\alpha,L} = \mu_\alpha, \wedge_L = \wedge - 2\delta(u \cap p).$$

Here $\delta(u)$ (resp. $\delta(u \cap p)$) is the half sum of the roots (resp. noncompact roots) in $\Lambda(u) = S^\dagger$ (cf. (3.1b) in [**BK1**]). Let χ_L be such that $\chi_L(\gamma_\alpha)$ is consistently with $\mu_{\alpha,L}$. Then $(\lambda_{0,L}, \Lambda_L^+, \chi_L)$ leads to a well-defined standard induced series of representations $U^L(M_L A N_L . \sigma_L, \frac{1}{2}3\alpha)$. Here $M_L = M \cap L$ and $N_L = N \cap L$ and N is defined in G for Λ'^+. Let $\gamma_L = \lambda_{0,L} + \frac{1}{2}3\alpha$.

It is easily verified that $\delta(u) = \frac{1}{2}(-9,0,2,-2)$. By the data of gap (A.2) given by (3.2), we have

$$\lambda_{0,L} = \lambda_0 - \delta(u) = \frac{1}{2}[(-3,1,0,0,) - (-9,0,2,-2)] = \frac{1}{2}(6,1,-2,2).$$

It follows that

$$\gamma_L = \lambda_{0,L} + \frac{3}{2}\alpha = \frac{1}{2}[(6,1,-2,2) + (0,0,3,-3)] = \frac{1}{2}(6,1,1,-1).$$

It is easy to see that $(\gamma_L, \alpha) = 1$ is odd. Since $\mu_{\alpha,L} = 0, \alpha$ (or σ_L) is a tangent case. Therefore, α satisfies the parity condition. Let $\Pi_L = \{\alpha_1 \alpha_2, \alpha_3\} = \Pi_L^\vee$.

Let $\beta = \alpha_3$ and $s_3 = s_\beta$. It is clear that $s_3\gamma_L = \frac{1}{2}(6,1,1,1)$ is $\Pi_L = \Pi_L^\vee$ dominant and the set of the singular roots in $\Pi_L = \Pi_L^\vee$ for $s_3\gamma_L$ is $\{1,2\}$. Therefore, since α satisfies the parity condition, by Theorem 1.1 of [**BK2**], we have

$$(3.5.1) \qquad \pi(s_3\gamma_L;\alpha) = \bar{\pi}(s_3\gamma_L;\alpha) + \bar{\pi}(s_3\gamma_L;\phi) + \bar{\pi}(s_\alpha s_3\gamma_L;\phi).$$

Clearly, β is a complex root, so, by Theorem 1.5 of [**BK2**], we have

$$(3.5.2) \qquad s_3\pi(s_3\gamma;\alpha) = \pi(\gamma;\alpha).$$

By Theorem 1.4 of [**BK2**], we have

$$(3.5.3) \qquad \tau(\bar{\pi}(s_3\gamma_L;\alpha)) = \{2\}, \tau(\bar{\pi}(s_3\gamma_L;\phi)) = \{1\}, \tau(\bar{\pi}(s_\alpha s_3\gamma_L;\phi)) = \{3\}.$$

By Theorem 1.6 of [**BK2**], it follows from (3.5.1) and (3.5.2) that

$$(3.5.4) \qquad \pi(\gamma_L;\alpha) = \bar{\pi}(s_3\gamma_L;\alpha) + \bar{\pi}(\gamma_L;\alpha) + \Theta_1 + \varpi.$$

Here

$$\varpi = \bar{\pi}(s_3\gamma_L;\phi) + \bar{\pi}(s_3\gamma_L;\beta) + \Theta_2 - \bar{\pi}(s_\alpha s_3\gamma_L;\phi).$$

By Theorem 1.4 of [**BK2**], we obtain

$$(3.5.5) \qquad \tau(\bar{\pi}(\gamma_L;\alpha)) = \{3\}, \quad \tau(\bar{\pi}(s_3\gamma_L;\beta)) = \{1,3\}.$$

It follows from (3.5.1) and (3.5.3) that $\Theta_1 = c_1(\bar{\pi}(s_\alpha s_3\gamma_L;\phi))$ by Theorem 1.6 of [**BK2**]. Here c_1 is a constant. Using similar methods used in [**BK2**], by Theorem 1.7 of [**BK2**], we obtain $c_1 = 1$.

Thus, by the results given above, we have

$$(3.5.6) \qquad \pi(\gamma_L;\alpha) = \bar{\pi}(s_3\gamma_L;\alpha) + \bar{\pi}(\gamma_L;\alpha) + \bar{\pi}(s_3\gamma_L;\phi) + \bar{\pi}(s_3\gamma_L;\beta) + \Theta_2.$$

It is easily shown that if $C(\Theta_2)$ is a irreducible character which occurs in Θ_2, then the τ-invariant of $C(\Theta_2)$ must contain 1 where 1 is the compact root $e_2 - e_3 = \alpha_1$ in $\Pi_L = \Pi_L^\vee$. Thus, by (3.5.3), (3.5.5) and (3.5.6), it is easily verified that only the τ-invariant of the second term in the right side of (3.5.6) that is $\pi(\gamma_L;\alpha)$ is disjoint from the set $\{1,2\}$ of the singular roots, so, by Theorem 1.3 of [**BK2**], $U^L(M_LAN_L,\sigma_L,\frac{1}{2}3\alpha)$ is irreducibe.

It is easily verified that $\langle\beta,(\lambda_0+\nu)\rangle \geq 0$ for all $\beta \in \Lambda(u) = S^\dagger$. Here $\nu = \frac{1}{2}3\alpha$. Hence, by 4.17 of [**SV**], $U(MAN,\sigma,\frac{1}{2}3\alpha)$ is irreducible. The proof is complete. $\square$

4. The Gaps and the Isolated Representations.

In this section, the chief idea to prove the unitarity is to use the arguments given by D.A. Vogan [**V1**] and D.A. Vogan and G.J. Zuckerman [**VZ**] as in Baldoni-silva and A.W. Knapp [**BK1**].

Now we bring in intertwining operator. We shall use the notations of [**KS**] and of [**BK1**] without redefining. According to [**KZ1**], the intertwining operator that defines the Hermitian form at ν is

$$\sigma(w)A_P(w,\sigma,\nu) \tag{4.1}$$

apart from normalization. Here w is a representative in K of the nontrivial element of $W(A:G)$. We may assume that this operator is positive definite (on each K-type) relative to $L^2(K,V^\sigma)$ for ν small and positive.

Let E be a finite-dimensional subspace of the domain of (4.1) equal to the sum of number of K-types, and let $T(z): E \to E$ be the restriction to E of $\sigma(w)A_P(w,\sigma,\frac{1}{2}(c_0'-z)\alpha)$, for complex z with $\mid z \mid< 1$. Let E_k be the subspace of E defined by (13.2) in [**BK1**]. We say that $T(z)$ has only a simple zero at $z=0$ if $E_2=0$.

Lemma 4.1. *With the above notations, for the case of gap* (A.1) *or gap* (A.2)

$$E_0 = E, E_1 = E \cap \ker T(0) \quad \text{and} \quad E_2 = 0.$$

Proof. First we shall consider the case of gap (A.2). Let A_* be the subgroup of G built from $\alpha = e_3 - e_4$, $\alpha' = e_3 + e_4$ and $\alpha'' = -e_1$. (By [**C**] (or by [**Su**]), it is easy to see that the Lie algebra of A_* is contained in a Cartan subalgebra of g.) For A_*, let $P_* = M_*A_*N_*$ be the real rank three standard parabolic subgroup of G. Clearly $\Lambda_{m_*} = \{e_2\}$ where m_* is the Lie algebra of M_*. Let λ_0^* and σ_* be the restriction of λ_0 and σ to P_* respectively. For restricted roots relative to this parabolic subgroup, we can use a system of type $A_1 \oplus B_2$ with $f_1+f_2 = \text{cayley}(\alpha)$,$f_1-f_2 = \text{cayley}(\alpha')$ and $f = \text{cayley}(\alpha'')$.

We can choose w in (4.1) to be a representative in K of the reflection $s_{f_1+f_2}$ in $W(A_*:G)$, and the techniques of [**KS**] show that

$$A_P\left(w,\sigma,\frac{1}{2}c\alpha\right) \subseteq A_{P_*}\left(w,\sigma_*,\frac{1}{2}c(f_1+f_2)\right). \tag{4.2}$$

Actually since we can discard invertible operators in our analysis by Lemma 13.2 of [**BK1**], we can simply write $s_{f_1+f_2}$ directly in place of w and Proposition 7.8 of [**KS**] allows us to factor the right side of (4.2) according to a cocycle relation as

$$A_{P_*}\left(w,\sigma_*,\frac{1}{2}c(f_1+f_2)\right) = A_{P_*,3}A_{P_*,2}A_{P_*,1}, \tag{4.3}$$

$$A_{P_*,3} = A_{P_*}\left(s_{f_2}, s_{f_1-f_2}s_{f_2}\sigma_*, -\frac{1}{2}c(f_1-f_2)\right),$$
$$A_{P_*,2} = A_{P_*}\left(s_{f_1-f_2}, s_{f_2}\sigma_*, \frac{1}{2}c(f_1-f_2)\right),$$
$$A_{P_*,1} = A_{P_*}\left(s_{f_2}, \sigma_*, \frac{1}{2}c(f_1+f_2)\right)$$

(cf. (13.5) of [**BK1**]). Let $\mathbf{a}_*$ be the subspace generated by $\{f_1, f_2, f\}$ over $\mathbf{R}$. It is easy to see that $\mathbf{a}_*$ is a commutative subalgebra in g_-

Let us examine $A_{P_*,1}$ here more closely. This operator depends only on data in the subgroup of G given as the centralizer $Z_1 = Z_G(\ker(f_2))$, and by means of kind of identification in Proposition 7.5 of [**KS**], it can be identified with a standard intertwining operator of Z_1.

Clearly, $\ker f_2$ in $\mathbf{a}_*$ is a subapace generated by $\{f_1, f\}$ over $\mathbf{R}$. Thus $Z_1 \cong \mathrm{SO}(3,2)$ and we can write the Dynkin diagram of Lie algebra $\mathbf{z}_1$ of Z_1 as

$$\bullet \Rightarrow \bullet.$$

Here the left (resp. right) $\bullet$ denotes $e_2 - e_4$ (resp. $f_2 = \mathrm{Cayley}(e_4)$). Let $\mathbf{a}_{*,1}$ be the subspace generated by f_2 over $\mathbf{R}$. Clearly $\mathbf{a}_{*,1} \subset \mathbf{z}_1$. Let $A_{*,1}$ be the subgroup of Z_1 with Lie algebra $\mathbf{a}_{*,1}$. For $A_{*,1}$, there is a standard parabolic subgroup $P_{*,1} = M_{*,1}A_{*,1}N_{*,1}$ of Z_1.

Let $m_{*,1}$ denote the Lie algebra of $M_{*,1}$. Then $\Lambda_{m_{*,1}} = \Lambda(e_2)$ and $m_{*,1} = m_*$. Clearly $M_{*,1} \subset M$ and $N_{*,1} \subset N$. Relative to this system, the restriction of λ_0 to $M_{*,1}$ can be write as $(\frac{1}{2}, 0)$ in coordinates (x_2, x_4) (cf. (1.1)). It is easy to see that the restriction of σ to $M_{*,1}$ (denoted by $\sigma_{*,1}$) is a tangent case. Hence, since f_2 is short in $\mathbf{z}_1$ and $c = 2$ is even, by 8.3 of [**BK1**], the induced representation $U^{Z_1}(M_{*,1}A_{*,1}N_{*,1}, \sigma_{*,1}, \frac{1}{2}2f_2)$ is irreducible. Therefore at $z = 0$,$T_1(z)$ is invertible. Thus 13.2 of [**BK1**] allows to discard the opertor T_1 on the right side of (4.3) from our analysis, and in similar fashion we can discard T_3 in the right side of (4.3).

Let us examine more closely the operator $A_{P_*,2}$. This operator depends only on data in the subgroup $Z_2 = Z_G(\ker(f_1 - f_2))$ and again can be identifitied with a standard intertwining operator for Z_2. Here the relevant fact about the identification is that if the operator for Z_2 is diagonal with diagonal entries having at most a simple zero at $z = 0$, then the same thing is ture of the operator in G.

Clearly, $\ker(f_1 - f_2)$ in $\mathbf{a}_*$ is the subspace generated by $\{f_1 + f_2, f\}$ over $\mathbf{R}$. Thus $Z_2 \cong \mathrm{SL}(2,\mathbf{R}) \otimes \mathrm{SO}(3)$ and we can write the Dynkin diagram of Lie algebra $\mathbf{z}_2$ of Z_2 as

$$\circ \quad \bullet.$$

Here $\bullet$ denotes $f_1 - f_2$ corresponding to the subgroup Z_2', $Z_2' \cong \mathrm{SL}(2,\mathbf{R})$

and $\circ$ denotes e_2 corresponding to the subgroup $Z_2'', Z_2'' \cong \mathrm{SO}(3)$. Clearly, $Z_2 = Z_2' \otimes Z_2''$. Let $\mathbf{a}_{*,2}$ be the subspace generated by $f_1 - f_2$ over $\mathbf{R}$. Clearly $\mathbf{a}_{*,2} \subset \mathbf{z}_2$ and $\mathbf{a}_{*,2}$ is the Lie algebra of A'. Let $m_{*,2}$ denote the Lie algebra of subgroup Z_2''. Then $\Lambda_{m_{*,2}} = \Lambda(e_2)$ and $m_{*,2} = m_*$. Let $M_{*,2} = M_* \cap Z_2$ and $N_{*,2} = N_* \cap Z_2$. Then $P_{*,2} = M_{*,2}A'N_{*,2}$ is a standard parabolic subgroup of Z_2 for A'. It is easy to see that $M_{*,2} = Z_2'' = M_* \cong \mathrm{SO}(3)$. By a similar argument as in Section 13 of [**BK1**], (cf. p. 113 in [**BK1**]), it is easy to see that only the subgroup Z_2' of Z_2 is important to the operator $A_{P_*,2}$. As in Section 13 of [**BK1**], thus we can regard the operator $A_{P_*,2}$ (on a $(K \cap Z_2)$-type) as the tensor product of an identity operator by the restriction of this operator to a K-type of Z_2', ($Z_2' \cong \mathrm{SL}(2,\mathbf{R})$). The K-types for $\mathrm{SL}(2,\mathbf{R})$ have multiplicity one, and, thus any standard intertwining operator for $\mathrm{SL}(2,\mathbf{R})$ is scale for a given K-type and given ν. Let $T_2(z)$ be the restriction of $A_{P_*,2}$ on E. Using 13.3 and 13.4 of [**BK1**], by the analysis mentioned above, it is easy to see that at $z = 0, T(z)$ has only simple zero. It follows that $E_2 = 0$. By Lemma 3.3 we have $\ker T(0) \neq 0$, hence, $E_1 \neq 0$.

By s aimilar argument used above, we can prove the lemma for case (A.1). (For case (A.1), let $\alpha = e_2 - e_3, \alpha' = e_2 + e_3$ and $\alpha'' = e_4$.) The proof is complete. □

The operator $T(z)$ is Hermitian for real z, and we can use it as in Section 3 of [**V1**] to define a nondegenerate Hermitian form on E_k/E_{k+1}, say with signature (p_k, q_k). Lemma 4.1 says that $p_k = q_k = 0$ for $k \geq 2$ and the positivity of $T(z)$ for $z > 0$ says that $q_0 = q_1 = 0$. According to Theorem 7.10 and Corollary 7.11 of [**V1**], the signature on E of $T(z)$ for small negative z is (p_0, p_1). Here $p_0 = \dim(E_0/E_1)$ and $p_1 = \dim(E_1/E_2) = \dim(E_1)$. Thus operator (4.1) is indefinite on any E large enough to contain the minimal K-type and a K-type that meets the (nontrivial) kernel of (4.1) at $\nu = \frac{1}{2}c_0'\alpha$. It follows from 8.3 of [**BK1**] and Lemma 3.5 that for gap (A.1) $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducble when $c_0' = 1 < c < c_0 = 2$, and for the gap (A.2), $U(MAN, \sigma, \frac{1}{2}c\alpha)$ is irreducible when $c_0' = 2 < c \leq c_0 = 3$. Therefore, by Lemma 4.1, we obtain the following lemma immediately.

Lemma 4.2.

(1) *For the case of gap* (A.1), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary when* $c_0' = 1 < c < c_0 = 2$.

(2) *For the case of gap* (A.2), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is not infinitesimally unitary when* $c_0' = 2 < c \leq c_0 = 3$.

Lemma 4.3.

(1) *For the case of gap* (A.1), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary when* $c = c_0 = 2$.

(2) *For the case of gap* (B.1), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary when* $c = c_0 = 5$.

Proof. Let $\rho_l = \rho_L$ denote the half sum of roots in the set

$$\{r \in \Lambda_l = \Lambda_L \mid \langle r, \alpha \rangle > 0\}.$$

First we shall show (1). From data of gap (A.1) given by (3.1) we have

$$\lambda_0 = \lambda_{0,b} = (-1,0,0,0),\ \wedge = \frac{1}{2}(-3,1,1,1),\ \mu_\alpha = 1.$$

Let $S = \{\alpha_1 = \alpha, \alpha_2\}$ and $\Lambda_L = \Lambda(S)$. Clearly, $\alpha = \alpha_1 \in \Lambda_L$ and

$$\Lambda_L = \{\pm(e_i - e_j) \mid 2 \le i < j \le 4\}.$$

Let $\Lambda^\wedge = \{-\frac{1}{2}(e_1 \pm e_2 \pm e_3 \pm e_4)\}$. Then

$$\Lambda_L(u) = \{-e_1 \pm e_i, e_i + e_j, -e_1, e_i, i = 2,3,4, j = 3,4, i < j\} \cup \Lambda^\wedge.$$

Let $\Lambda^\vee = \{-\frac{1}{2}(e_1 + ze_2 \pm e_3 - ze_4), z = \pm 1\}$. Then

$$\Lambda_L(u \cap p^C) = \{e_2, e_4, e_1 \pm e_2, -e_1 \pm e_4, e_2 + e_3, e_3 + e_4\} \cup \Lambda^\vee.$$

Thus we have $2\delta(u \cap p^C) = (-6,2,2,2)$. It is clear that

$$2\rho_L = (e_2 - e_3) + (e_2 - e_4) - (e_3 - e_4) = 2(e_2 - e_3) = 2\alpha.$$

Let λ be the parameter defined by (12.4) of [**BK1**]. Then we have

$$\lambda = \wedge - 2\delta(u \cap p^C) = \frac{1}{2}(-3,1,1,1) - (-6,2,2,2) = \frac{1}{2}(9,-3,-3,-3).$$

It is easily verified that $\langle \lambda, \beta \rangle = 0$ for all $\beta \in \Lambda_L$. Clearly, Λ_L has real rank one, so, by Proposition 12.4 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary when $c = c_0 = 2$.

Now, we shall show (2). Let $\Lambda_L = \Lambda(\alpha_2, \alpha_3, \alpha_4)$. Clearly, $\alpha = \alpha_3 \in \Lambda_L$. If (2.6.q) holds, then by the results of case **(6.B)**, we have

$$\lambda_0 = \frac{1}{2}(-3,1,1,0),\ \wedge = (2,-2,0,0).$$

By computing, we obtain

$$2\rho_L = 5\alpha,\ \lambda = \wedge - 2\delta(u \cap p^C) = (-2,2,0,0) - (-5,5,0,0) = (3,-3,0,0).$$

It is easily verified that $\langle \lambda, \beta \rangle = 0$ for all $\beta \in \Lambda_L$. Clearly Λ_L has real rank one, so, by Proposition 12.4 of [**BK1**], $J(MAN, \sigma, \frac{1}{2}c\alpha)$ is infinitesimally unitary when $c = c_0 = 5$.

The proof is complete. □

In Section 2 for the cases (4.$A.a$),(1),(ii) and (3.A),(1),(ii) we give certain statements for the gaps (A.1) and (A.2) respectively. By these statements, we can summarize Lemma 4.2 and 4.3 in the following proposition.

Proposition 4.1.
(1) *For the case of gap* (A.1), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary exactly when* $0 < c < c_0' = 1$ *or* $c = c_0 = 2$.
($J(MAN, \sigma, \alpha)$ *is an isolated unitary representation.*)
(2) *For the case of gap* (A.2), $J(MAN, \sigma, \frac{1}{2}c\alpha)$ *is infinitesimally unitary exactly when* $0 < c < c_0' = c_0 = 2$.

(From left to right, the circles in the Dynkin diagram of F_4 correspond the simple roots $\alpha_1, \alpha_2, \alpha_3$ and α_4.)

(1) $\theta_C = (1,-1,0,0) : \bullet - \circ \Rightarrow \circ - \circ.$	**(1′)** $\theta_C = (1,1,0,0) : \bullet - \circ \Rightarrow \circ - \bullet.$
(2) $\theta_C = (0,1,-1,0) : \circ - \bullet \Rightarrow \circ - \circ.$	**(2′)** $\theta_C = (0,1,1,0) : \circ - \bullet \Rightarrow \circ - \bullet.$
(3) $\theta_C = (1,0,0,-1) : \circ - \bullet \Rightarrow \bullet - \circ.$	**(3′)** $\theta_C = (1,0,0,1) : \circ - \bullet \Rightarrow \bullet - \bullet.$
(4) $\theta_C = (0,1,0,-1) : \bullet - \bullet \Rightarrow \bullet - \circ.$	**(4′)** $\theta_C = (0,1,0,1) : \bullet - \bullet \Rightarrow \bullet - \bullet.$
(5) $\theta_C = (1,0,-1,0) : \bullet - \bullet \Rightarrow \circ - \circ.$	**(5′)** $\theta_C = (1,0,1,0) : \bullet - \bullet \Rightarrow \circ - \bullet.$
(6) $\theta_C = (0,0,1,-1) : \bullet - \circ \Rightarrow \bullet - \circ.$	**(6′)** $\theta_C = (0,0,1,1) : \bullet - \circ \Rightarrow \bullet - \bullet.$

Table 1.1

(1) $\theta_C = (0,0,0,0) : \circ - \circ \Rightarrow \circ - \circ.$	(2) $\theta_C = (-1,-1,-1,1) : \circ - \circ \Rightarrow \bullet - \bullet.$
(3) $\theta_C = (1,1,1,1) : \circ - \circ \Rightarrow \bullet - \circ.$	(4) $\theta_C = (2,0,0,0) : \circ - \circ \Rightarrow \circ - \bullet.$

Table 1.2

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