PACIFIC JOURNAL OF MATHEMATICS
Vol. 176, No. 2, 1996

CLASSIFICATION OF THE STABLE HOMOTOPY TYPES OF STUNTED LENS SPACES FOR AN ODD PRIME

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For an odd prime p we obtain the complete classification of the stable homotopy types of stunted lens spaces modulo p by adapting the ideas introduced by Feder, Gitler and Mahowald in the study of the 2 primary problem. Advantage is taken of the stable, p-local decomposition of stunted lens spaces. We check that the classification is realized by J-homology and cohomology groups as in the case of real projective spaces.

1. Introduction.

Throughout this paper p will denote an odd prime. The infinite dimensional mod p lens space L admits a CW structure having a cell in each dimension. Let L^b be the b^{th} dimensional skeleton of L, the stunted lens space L_a^b is defined as the quotient space of L^b by L^{a-1}. In this paper we obtain the complete classification of the stable homotopy types of stunted lens spaces mod p. The techniques use the well known fact that stunted lens spaces are either Thom complexes or reduced Thom complexes (i.e. collapsing to a point the bottom Thom sphere) of multiples of the canonical complex line bundle over finite dimensional lens spaces, in particular the J order of such bundles gives sufficient algebraic conditions to obtain stable equivalences among stunted lens spaces. More precisely two stunted lens spaces each with N cells, are stably equivalent provided there is a congruence modulo p^s on the dimension of the top cells, where s is roughly the integral part of the quotient $N/(2p-2)$. On the other hand by using the Adams operation ψ^{p+1} one can see that a congruence modulo p^{s-1} is also necessary. The problem is to determine the optimal congruence for classification in each case. It is well known that the congruence modulo p^s gives the classification when the spaces are S-reducible or S-coreducible, and it is easy to see that the same is true when the total number of complete Moore cells is not divisible by $p-1$. In this paper we show that the congruence modulo p^{s-1} gives the optimal condition in all remaining cases.

The starting point in the classification of the homotopy types of stunted lens spectra is the following theorem which is a classical consequence of the calculation of the Adams's J groups for finite dimensional lens spaces [**7**].

Theorem 1.1 ([**7**, **8**]). *Let m, n, k be positive integers with* $k = s(p-1) + r$ *and* $0 \leq r < p-1$.

a) *If $m \equiv n \pmod{p^s}$, then $L_{2m+\delta}^{2m+2k+\epsilon} \simeq L_{2n+\delta}^{2n+2k+\epsilon}$ for any $\epsilon, \delta \in \{0,1\}$*

b) *If $L_{2m+\delta}^{2m+2k+\epsilon} \simeq L_{2n+\delta}^{2n+2k+\epsilon}$ for some $\epsilon, \delta \in \{0,1\}$, then $m \equiv n \pmod{p^{s-1}}$, and if in addition $r > 0$ or if one of the spaces is either S-reducible or S-coreducible,then in fact $m \equiv n \pmod{p^s}$*

In view of Theorem 1.1 we can restrict attention to non S-reducible nor S-coreducible stunted lens spaces where the number of complete Moore cell is divisible by $p-1$. Theorems 1.1, 2.2, 3.6, 3.7 and 4.7 of this paper imply that in such remaining cases the classification is given by a congruence module p^{s-1} in the notation of Theorem 1.1. The final answer can be sumarized in the following result

Theorem 1.2. *For δ, $\epsilon \in \{0, 1\}$ two spaces $L_{2n+\delta}^{2n+2l+\epsilon}$ and $L_{2m+\delta}^{2m+2l+\epsilon}$ which are neither S-reducible nor S-coreducible are of the same stable homotopy type if and only if $n \equiv m \pmod{p^e}$ where e is the integral part of the quotient $(l-1)/(p-1)$.*

The paper is organized as follows, in section two we settle the classification in some easy cases and identify the obstructions for the existence of (optimal) stable equivalences. In section three, these obstructions are shown to be independent of the spaces involved, obtaining a partial classification which is completed in section four by using the stable, p-local decomposition of stunted lens spaces into a wedge of $p-1$ summands. In section five we compute the relevant J-groups to show that the classification is determined by these graded abelian groups just as in [**3**]. In the final section we identify the obstructions arising in the classification of the summands in the p-local decomposition of stunted lens spectra. An analysis of these obstructions will appear elsewhere.

The paper represents a portion of the author's Ph. D. thesis written at the University of Rochester with the guidance of Samuel Gitler. It is a pleasure to thank Professor Gitler for his useful advise and continuous motivation.

2. Some sufficient conditions.

Let ξ be the realification of the canonical complex line bundle over the infinite dimensional lens space and let σ denote its stable class, restrictions of σ to skeleta L^b will also be denoted by σ. The methods used in [**5**] can be carried over to odd primes to classify the homotopy types of the Thom spectra and reduced Thom spectra for multiples of the bundle $\sigma^{\frac{p-1}{2}}$ (defined over finite skeleta of L). It follows from [**7**] that the classification of the homotopy types of such spectra is equivalent to the corresponding one for stunted lens spectra and therefore we will work towards the classification of the former ones.

Notation 2.1. Let α denote a vector bundle over L^a (for some large enough number a) whose stable class is $\sigma^{\frac{p-1}{2}}$. Restrictions of α to lower skeleta will be denoted by α too.

Remark. The essential property we need on α is that it has order p^s over $L^{2s(p-1)}$ but order p^{s-1} over $L^{2s(p-1)-1}$. There is another natural choice for such a budle: Following [**2**] the usual action of Σ_p on $\mathbf{C}^p$ restricts to an action on the hyperplane $z_1 + \cdots + z_p = 0$ and defines a representation $\Sigma_p \to U(p-1)$ giving rise to a $p-1$ dimensional complex vector bundle β over $B\Sigma_p$, then the realification of the restriction of β under the map $B\mathbf{Z}/p \to B\Sigma_p$ produces a vector bundle α' with the required properties. The referee of the paper has observed that the classification of homotopy types of stunted lens spectra could equally be obtained by working directly with the bundle β to obtain stable identifications among stunted $(B\Sigma_p)_{(p)}$ spaces. Section 6 of the paper uses this approach to obtain sufficient conditions for the existence of optimal stable equivalences among stunted $(B\Sigma_p)_{(p)}$ spaces with both top and bottom integral cells.

We will say that the top cell of L_a^b is integral (torsion) if b is odd (even), likewise L_a^b is said to have an integral (torsion) bottom cell when a is even (odd). The next theorem together with 1.1 b) gives the classification for stunted lens spectra with no integral cells, it follows the ideas of [**5**]. The notation " $\overline{T}$ " means pinching the bottom integral cell of the Thom spectrum T.

Theorem 2.2. *Let $k = s(p-1)$, $t = p^{s-1}$ and α is as in* 2.1. *For any integer n there is a stable equivalence*

$$\overline{(L^{2k})^{n\alpha}} \simeq \overline{(L^{2k})^{(n+t)\alpha}}.$$

Proof. $(n+t)\alpha$ is classified by the composite

$$L^{2k} \xrightarrow{\Delta} L^{2k} \times L^{2k} \xrightarrow{1\times c} L^{2k} \times S^{2k} \xrightarrow{n\alpha\times\theta} BO \times BO \to BO$$

where θ fits in the following diagram since $t\alpha$ is trivial over L^{2k-1} in view of the calculations in [**7**]

$$\begin{tikzcd} L^{2k-1} \arrow[r] & L^{2k} \arrow[r, "t\alpha"] \arrow[d, "c"'] & BO(t\,|\alpha|) \\ & S^{2k} \arrow[ur, "\theta"'] & \end{tikzcd}$$

It is well known that $(S^{2k})^{\theta} = S^{t|\alpha|} \cup_{J\theta} e^{t|\alpha|+2k}$. Let $g = (1 \times c) \circ \Delta$, $T(g) =$ Thomification of g, and g' the induced map by $T(g)$. Then one has a diagram

$$\begin{tikzcd}
(L^{2k})^{(n+t)\alpha} \arrow[r, "T(g)"] \arrow[d, "pinch"] & (L^{2k})^{n\alpha} \wedge (S^{2k})^{\theta} & (L^{2k})^{n\alpha} \wedge S^{t|\alpha|} \arrow[l] \arrow[d, "(pinch) \wedge 1"] & (L^{2k})^{n\alpha} \wedge S^{t|\alpha|+2k-1} \arrow[l, "1 \wedge J\theta"'] \\
\overline{(L^{2k})^{(n+t)\alpha}} \arrow[r, "g'"'] & \left[\overline{(L^{2k})^{n\alpha} \wedge (S^{2k})^{\theta}}\right]^{(t+n)|\alpha|+2k}_{(t+n)|\alpha|+1} \arrow[r, bend right, "r"'] & \overline{(L^{2k})^{n\alpha}} \wedge S^{t|\alpha|} \arrow[l] & S^{n|\alpha|} \wedge S^{t|\alpha|+2k-1} \arrow[l, "\gamma"'] \arrow[u, "i \wedge 1"']
\end{tikzcd}$$

where the second space of the bottom row stands for the $(t+n)|\alpha| + 2k$ dimensional skeleton of $(L^{2k})^{n\alpha} \wedge (S^{2k})^{\theta}$ with the bottom cell collapsed to a point. Here rows pointing backwards are cofibrations. Clearly the right square commutes and γ is null homotopic, so that we get the retraction r as shown. Using the definition of g we can see that $r \circ g'$ induces isomorphisms in mod p cohomology, except possibly in the top dimension (which is the first dimension where there are two generators mapping non trivially under g'). Therefore it suffices to check that $r^*(U_{n\alpha} y^k \wedge U_\theta) = U_{n\alpha} y^k \wedge U_\theta$ where U_λ denotes the Thom class of λ and y is the generator in $H^2(L, \mathbf{Z}/p)$. This follows easily using naturality of the Bockstein operator β together with the fact that $\beta(U) = 0$ for both Thom classes and the formula $\beta(xy^{k-1}) = y^k$ where x is the generator in $H^1(L, \mathbf{Z}/p)$. $\square$

Same type of techniques can be applied to classify stunted lens spaces with integral top cell. The following result identifies the "obstruction" for the existence of a stable equivalence among two such spaces.

Proposition 2.3. *Let $k = s(p-1)$ and let α be as in 2.1. Let $\beta_s \in \pi^S_{2k-1}$ be a generator of the p-local component of ImJ. If the composite*

$$S^{n|\alpha|+2k} \xrightarrow{\beta_s} S^{n|\alpha|+1} \xrightarrow{i} \overline{(L^{2k})^{n\alpha}}$$

is stably null homotopic then for any integer t with $\nu_p(t) \geq s-1$, there is a stable equivalence

$$\overline{(L^{2k+1})^{n\alpha}} \simeq \overline{(L^{2k+1})^{(n+t)\alpha}}.$$

Proof. The proof is similar to that of 2.2 with minor differences: We consider the diagram

$$\begin{tikzcd}
L^{2k-1} \arrow[r] & L^{2k+1} \arrow[r, "t\alpha"] \arrow[d, "c"] & BO \\
& S^{2k} \vee S^{2k+1} \arrow[r] & S^{2k} \arrow[u, "\theta"']
\end{tikzcd}$$

Here θ exists because $\pi_{odd}(BO)$ is 2-torsion and $t\alpha$ is an element of p-torsion, moreover, killing in advance any other torsion, we can choose θ so that $J\theta$ is a multiple of β_s. Now we continue as in the proof of 2.2 to get

$$\begin{tikzcd}
\left[\left(L^{2k+1}\right)^{n\alpha} \wedge \left(S^{2k}\right)^{\theta}\right]_{(n+t)|\alpha|+1}^{(n+t)|\alpha|+2k+1} \arrow[d, bend right=60, "r"'] & \overline{\left(L^{2k+1}\right)^{(n+t)\alpha}} \arrow[l, "g'"'] \\
\overline{\left(L^{2k+1}\right)^{n\alpha}} \wedge S^{t|\alpha|} \arrow[u] & \\
\left(S^{n|\alpha|} \vee S^{n|\alpha|+1}\right) \wedge S^{t|\alpha|+2k-1} \arrow[u, "\gamma"] \arrow[r, "i \wedge 1"] & \left(L^{2k+1}\right)^{n\alpha} \wedge S^{t|\alpha|+2k-1} \arrow[ul, "pinch \wedge J\theta"']
\end{tikzcd}$$

where the column is a cofibration. Similar considerations as in 2.2 together with the present hypothesis show that γ is null homotopic and we have a retraction r as shown. In order to see that $r \circ g'$ is isomorphism in mod p cohomology, we need to check that the behavior of r^* on the last two top cells is the correct one, namely, we want λ to be zero in

i) $r^*(U_{n\alpha} \cdot y^k \wedge U_\theta) = U_{n\alpha} \cdot y^k \wedge U_\theta + \lambda\,(U_{n\alpha} \wedge U_\theta \cdot i)$ and

ii) $r^*(U_{n\alpha} \cdot xy^k \wedge U_\theta) = U_{n\alpha} \cdot xy^k \wedge U_\theta + \lambda\,(U_{n\alpha} \cdot x \wedge U_\theta \cdot i)$

where x and y are the usual generators for the mod p cohomology of the infinite lens space and i for the sphere.

The first relation follows by using the Bockstein operator as in 2.2, the second one is the main trick of [**5**] and [**3**]: Under the present hypothesis the top cell to be attached in the above diagram (i.e. the one corresponding to $U_{n\alpha} \cdot x \wedge U_\theta \cdot i$) actually splits off over $\overline{(L^{2k})^{n\alpha}} \wedge S^{t|\alpha|}$ so that r induces a retraction r' from the cofiber of $\gamma' : \left(S^{n|\alpha|} \vee S^{n|\alpha|+1}\right) \wedge S^{t|\alpha|+2k-1} \to \overline{\left(\mathrm{L}^{2k}\right)^{n\alpha}} \wedge S^{t|\alpha|}$ into $\overline{\left(\mathrm{L}^{2k}\right)^{n\alpha}} \wedge S^{t|\alpha|}$ and this forces $\lambda = 0$ in ii) above, moreover r' restricts to a retraction like the one used in the proof of 2.2, thus the final conclusion follows from the next lemma. $\square$

Lemma 2.4. *Suppose X and Y are stable finite p-local CW complexes with respective attaching maps $S^n \xrightarrow{\pi_X} X$ and $S^n \xrightarrow{\pi_Y} Y$ for integral top cells in the respective cofibers X' and Y'. Suppose given maps f and h fitting in the diagram*

$$\begin{tikzcd}
S^n \arrow[r, "\pi_X"] \arrow[d, "q"] & X \arrow[r] \arrow[d, "f"] & X' \arrow[d, "h"] \\
S^n \arrow[r, "\pi_Y"'] & Y \arrow[r] & Y'
\end{tikzcd}$$

If f and h induce isomorphisms in mod p cohomology, then $X' \simeq Y'$.

Proof. Let g be the homotopy inverse for f; multiplying by g both sides of $f \circ \pi_X \simeq \pi_Y \circ q$ yields $\pi_X \simeq (q \circ g) \circ \pi_Y$. But since q is prime to p, $q \circ g$ is still a homotopy equivalence so that the map it produces from the cofiber of π_Y to that of π_X induces isomorphisms in integral homology and is therefore a stable equivalence. □

3. Symmetric maps.

In this section we use the ideas of [**4**] about projective classes, as suggested by [**5**] to obtain information on the stable classification of stunted lens spaces. We use the following conventions: whenever we refer to a space we mean the corresponding suspension spectrum, in particular the degree p map on a given space makes sense and will be denoted simply by p. The Moore spectrum $S^{a-1} \cup_p e^a$ will also be denoted by $P^a(p)$.

We start with the following well known result (cf. [**2, 10**]).

Proposition 3.1. *For each positive integer l there is a map $L^{2l} \xrightarrow{g_l} L^{2l-2(p-1)}$ such that $L^{2l} \xrightarrow{g_l} L^{2l-2(p-1)} \hookrightarrow L^{2l}$ agrees with the degree p map on L^{2l}, moreover the restriction of g_{l+1} to L^{2l} factors as $L^{2l} \xrightarrow{g_l} L^{2l-2(p-1)} \hookrightarrow L^{2l+2}$.*

If no confusion arises we will simply denote the map g_l by g. Consider the following diagram where columns as well as the diagonal are cofibrations

$$\begin{tikzcd}
 & & S^{2l} \arrow[dr, "f"] & \\
 & S^{2l-1} \cup_p e^{2l} \arrow[ur] \arrow[rr] & & S^{2l-2p+2} \\
S^{2l-1} \arrow[ur] & L^{2l} \arrow[u] \arrow[rr, "g"] & & L^{2l-2p+2} \arrow[u] \\
 & L^{2l-2} \arrow[u] \arrow[rr] & & L^{2l-2p+1} \arrow[u]
\end{tikzcd}$$

If the composite $S^{2l-1} \to S^{2l-1} \cup_p e^{2l} \to S^{2l-2p+2}$ is null homotopic then there is a map f as shown. Say $l = a(p-1) + b \quad 0 \leq b < p-1$. Since $p^{a-1}\sigma^b$, as an element of $KU(L^{2l-2p+2})$, is the image of some non zero element $\mu \in KU(S^{2l-2p+2})$ and $0 \neq p^a\sigma^b \in KU(L^{2l})$, it follows that $f^*(\mu) \neq 0$, thus f^* is monic, but this is a contradiction to the fact that any positive dimensional element in the stable homotopy groups of the sphere is nilpotent (in the case $b = 0$ take $p^{a-2}\sigma^{p-1}$ rather than $p^{a-1}\sigma^b$). Thus after pinching the proper skeleta the map $g : L^{2l} \to L^{2l-2p+2}$ induces an extension $\overline{A} : P^{2l}(p) \to S^{2l-2p+2}$ of the Adams map A.

We use the previous remarks to choose explicit generators of ImJ as done in [**4**] for the mod 2 case. Let θ_1 be a vector bundle over $S^{2(p-1)}$ that pulls back to $\sigma^{\frac{p-1}{2}} \in KO(L^{2p-2})$ and such that $J\theta_1$ is a generator of the p-local component in the image of the J-homomorphism. Let $\beta_1 = J\theta_1$ $(= A)$ and suppose chosen $\beta_1, \dots, \beta_s$ such that

(1) β_i lies in the stem $2i(p-1)-1$

(2) $\beta_i \in \langle \beta_{i-1}, A, p \rangle \qquad i > 1, \qquad$ and

(3) $\beta_i = J\theta_i$ is a generator of the p-local component in the image of the J-homomorphism where θ_i is a vector bundle over $S^{2i(p-1)}$ that pulls back to $p^{i-1}\sigma^{\frac{p-1}{2}} \in KO(L^{2i(p-1)})$.

$$\begin{tikzcd}
 & & S^{2s(p-1)+2p-2} \arrow[rrd, "f"] & & \\
S^{2s(p-1)+2p-3} \arrow[r, "p"] & S^{2s(p-1)+2p-3} \arrow[r, "i"] & P^{2(s+1)(p-1)}(p) \arrow[u, "c"] \arrow[r, "\overline{A}"] & S^{2s(p-1)} \arrow[r, "\theta_s"] & BO \\
 & L^{2(s+1)(p-1)-1} \arrow[u, "c"] \arrow[r, "i"] & L^{2(s+1)(p-1)} \arrow[u, "c"] \arrow[r, "g"] & L^{2s(p-1)} \arrow[u, "c"] &
\end{tikzcd}$$

We construct the next β_{s+1}. Since $KO\left(S^{2s(p-1)+2p-3}\right) = 0$ the composite $\theta_s \circ A$ is null so that the Toda bracket of (2) makes sense and there is a map f as shown. The commutativity of the right square says that f is a multiple prime to p of a generator of $KO(S^{2(s+1)(p-1)})$. Let $m(k)$ be the order of $J(S^{2k})$ and let $n(k) = m(k)/p^{\nu_p(m(k))}$. If f represents the integer l, pick an integer x such that $l + xp \equiv 0 \bmod n\left((s+1)(p-1)\right)$. Let $\theta_{s+1} = f + xp$. Then θ_{s+1} still pulls back to $p^s\sigma^{\frac{p-1}{2}}$, $J\theta_{s+1}$ is a generator of the p-local component of $J(S^{2(s+1)(p-1)})$ and by construction $J\theta_{s+1} \in \langle \beta_s, \beta_1, p \rangle$, so we can take $\beta_{s+1} = J\theta_{s+1}$.

We study the symmetric properties of the generators of ImJ by using the previous construction. In [**9**] the definition for a map to be symmetric is given and it is shown that any stable class in the homotopy of spheres is realized in some dimension by a symmetric map. We will need a slightly different notion of stable symmetricity. Recall that a map $f : S^{2n+1} \to S^{2m}$ is symmetric if it factors through the map $\pi_{2n+1} : S^{2n+1} \to L^{2n+1}$ that attaches the top cell of L^{2n+2}. Then to say that f is stably symmetric would mean in [**9**] that for some $t > 0$, $\Sigma^{2t} f$ can be factored through the attaching map $\pi_{2n+2t+1} : S^{2n+2t+1} \to L^{2n+2t+1}$; instead we will require the factorization through $\Sigma^{2t}\pi_{2n+1}$. The second difference is that we will use Thom complexes of α to represent stunted lens spaces (cf. 2.1 and its previous remarks). The following definition of stable symmetricity is more restrictive than the one used in [**9**].

Definition 3.2. Let α be as in 2.1 and $k = s(p-1)$. A map $S^{m|\alpha|+2k-1} \longrightarrow S^{m|\alpha|}$ is called stably symmetric if it can be stably factored through a map homotopic to the attaching map $S^{m|\alpha|+2k-1} \longrightarrow (L^{2k-1})^{m\alpha}$ whose cofiber is $(L^{2k})^{m\alpha}$.

Proposition 3.3. *Let $k = s(p-1)$ and let β_s and β_{s+1} be as in* (2). *If $S^{m|\alpha|+2k-1} \xrightarrow{\beta_s} S^{m|\alpha|}$ is stably symmetric, then so is $S^{m|\alpha|+2k+2p-3} \xrightarrow{\beta_{s+1}} S^{m|\alpha|}$.*

Proof. By [**7**] we have $J(\alpha) = J(u\xi)$ where u is prime to p (see 2.1 and its previous remarks), let $t = (s+1)(p-1)$, then for some $e \gg 0$

$$\Sigma^{2mu+2e}\left(L^{2t}\right)^{m\alpha} = \Sigma^{m|\alpha|+2e}\left(L^{2t}\right)^{mu\xi} = \Sigma^{m|\alpha|+2e} L_{2mu}^{2mu+2t}$$

and $S^{m|\alpha|+2t-1} \xrightarrow{\beta_{s+1}} S^{m|\alpha|}$ is stably symmetric if and only if its $2mu+2e$ *th* suspension stably factors through the attaching map

$$\Sigma^{m|\alpha|+2e}\pi_{2mu+2t-1} : S^{m|\alpha|+2mu+2t+2e-1} \longrightarrow \Sigma^{m|\alpha|+2e} L^{2mu+2t-1}.$$

Similar considerations hold with β_s and k replacing β_{s+1} and t respectively. Thus we need to show that the existence of a stable commutative diagram

$$\begin{tikzcd} S^{2mu+2k-1} \arrow[rr, "\beta_s"] \arrow[dr, "\pi_{2mu+2k-1}"'] & & S^{2mu} \\ & L^{2mu+2k-1} \arrow[ur, "\phi"'] & \end{tikzcd}$$

implies the existence of the corresponding one for β_{s+1}:

$$\begin{tikzcd} S^{2mu+2t-1} \arrow[rr, "\beta_{s+1}"] \arrow[dr, "\pi_{2mu+2t-1}"'] & & S^{2mu} \\ & L^{2mu+2t-1} \arrow[ur, dotted] & \end{tikzcd}$$

Consider the following stable commutative diagram (where the map g is as in 3.1 and $l = mu$)

$$\begin{tikzcd}
 & S^{2l+2t-1} \arrow[r] \arrow[rr, bend left, "A"] & P^{2l+2t} \arrow[r] & S^{2l+2k} & \\
 & & L^{2l+2t} \arrow[u, "c"] \arrow[r, "g"] & L^{2l+2k} \arrow[u, "c"] & \\
\Sigma^{-1}L^{2l+2t-1} \arrow[r, "c"] & S^{2l+2t-2} \arrow[r, "\pi"] & L^{2l+2t-2} \arrow[u, "i"] \arrow[r] & L^{2l+2k-1} \arrow[u, "i"] \arrow[r, "\phi"] & S^{2l} \\
S^{2l+2t-2} \arrow[u, "\Sigma^{-1}\pi_{2l+2t-1}"] \arrow[r, "p"] & S^{2l+2t-2} \arrow[u, equal] \arrow[r, "i"] \arrow[rr, bend right, "A"'] & P^{2l+2t-1} \arrow[u, "\pi"] \arrow[r] & S^{2l+2k-1} \arrow[u, "\pi_{2l+2k-1}"] \arrow[r, "\beta_s"] & S^{2l} \arrow[u, equal]
\end{tikzcd}$$

Since both bottom rows in the diagram start with cofibrations, it follows that a representative of $\langle \beta_s, A, p\rangle$ can be factored through $\pi_{2l+2t-1}$. The indeterminacy of this Toda bracket is given by (suspensions of) maps of either of the following form

i) $S^{2l+2t-1} \xrightarrow{p} S^{2l+2t-1} \to S^{2l}$ or

ii) $S^{2l+2t-1} \to S^{2l+2k-1} \xrightarrow{\beta_s} S^{2l}$

since in case (i), p factors through $\pi_{2l+2t-1}$, all such composites factor in the same way. On the other hand, all composites in (ii) are null homotopic since β_s is p-torsion but the p-component of the $2p-2$ stem is trivial. The result follows from (2). □

The next proposition starts the induction suggested by 3.3 to show that certain maps are stably symmetric.

Proposition 3.4. *Let $k = s(p-1)$ and let q and m be positive integers with $q < k < 2q+1$. If $m\alpha$ is trivial over L^{2q} but not over L^{2k}, then $\beta_s : S^{m|\alpha|+2k-1} \to S^{m|\alpha|}$ is stably symmetric*

Proof. Let $a = \nu_p(m)$ with $m = p^a l$ (so that $a < s$). By hypothesis there is a vector bundle μ over L^{2k}_{2q+1} that pulls back to $m\alpha$ under the collapsing map. Then $\mu' = p^{s-a-1}\mu$ pulls back to $t\alpha$ where $t = lp^{s-1}$.

$$\begin{tikzcd}
 & & BO \\
L^{2k-1} \arrow[r] \arrow[dd] & L^{2k} \arrow[ur, "t\alpha"] \arrow[dr] \arrow[dd] & \\
 & & S^{2k} \arrow[dd, equal] \\
L^{2k-1}_{2q+1} \arrow[r, "j"] & L^{2k}_{2q+1} \arrow[uuur, "\mu'"'] \arrow[dr] & \\
 & & S^{2k} \arrow[uuuu, bend right, "\theta"']
\end{tikzcd}$$

Since $L^{2k-1} \to L^{2k-1}_{2q+1}$ is monic in KO-theory, μ' restricts trivially under the inclusion map j and we can choose a vector bundle θ over S^{2k} that induces μ' under the collapsing map $L^{2k}_{2q+1} \to S^{2k}$ and such that $J\theta$ is a generator of the p-local component of ImJ.

$$\begin{tikzcd}
(L^{2q})^{m\alpha} \arrow[r] & (L^{2k})^{m\alpha} \arrow[r] \arrow[dd, "c"'] & \overline{(L^{2k}_{2q+1})^{\mu}} \arrow[r, "\theta_\mu"] \arrow[d, "\varphi", "\simeq"'] & S^1 \arrow[d, "p^{s-a-1}"] \\
 & & \overline{(L^{2k}_{2q+1})^{\mu'}} \arrow[r, "\theta_{\mu'}"] \arrow[d] & S^1 \\
 & S^{2k} \arrow[r, equal] & \overline{(S^{2k})^{\theta}} \arrow[ur, "J\theta"'] &
\end{tikzcd}$$

Here the first three spaces on the top row form a cofibration and the composite of the last two maps is null homotopic. The maps θ_μ and $\theta_{\mu'}$ fit in the Puppe sequence defining the corresponding reduced Thom spectra (which are taken so that all Thom classes are zero dimensional). Since $k < 2q+1$, the space L^{2k}_{2q+1} is a double suspension and from [**4**, 3.3 and 3.4] both spectra $\overline{(L^{2k}_{2q+1})^{\mu}}$ and $\overline{(L^{2k}_{2q+1})^{\mu'}}$ are equivalent to the suspension spectrum of L^{2k}_{2q+1} (with zero dimensional bottom cell), furthermore there is a stable equivalence $\varphi : \overline{(L^{2k}_{2q+1})^{\mu}} \to \overline{(L^{2k}_{2q+1})^{\mu'}}$ that yields the above diagram commutative. It is then clear that $J\theta \circ c$ is null homotopic, from which the results follows.

As in the proof of 3.3 let $J(\alpha) = J(u\xi)$ with p not dividing u. The following result is the key to get the classification of stunted lens spaces with at most one integral cell. $\square$

Corollary 3.5. *Let $k = s(p-1)$ and u as above. If $s > \nu_p(un+k+1)$ then the composite $S^{n|\alpha|+2k} \xrightarrow{\beta_s} S^{n|\alpha|+1} \xrightarrow{i} \overline{(L^{2k+1})^{n\alpha}}$ is stably null homotopic.*

Proof. Let m be a positive integer such that $um + un + k + 1 \equiv 0 \bmod p^s$, then a stable dual of $\overline{(L^{2k+1})^{n\alpha}}$ is $(L^{2k})^{m\alpha}$. Thus, it is enough to prove that the composite $(L^{2k})^{m\alpha} \xrightarrow{c} S^{m|\alpha|+2k} \xrightarrow{\beta_s} S^{m|\alpha|+1}$ is stably null homotopic provided $s > \nu_p(m)$. Note that the previous composite is stably null homotopic if and only if $\beta_s : S^{m|\alpha|+2k-1} \to S^{m|\alpha|}$ is stably symmetric, thus by 3.3 we can restrict to the case $s = \nu_p(m) + 1$. If $\nu_p(m) = 0$, the result follows easily from the action of the Steenrod operation $\mathcal{P}^1$ on the cohomology of the lens space. If $\nu_p(m) \geq 1$, take $q = \nu_p(m)(p-1)$ in 3.4 to get the result. $\square$

The next result together with 1.1, completes the classification for stunted lens spaces with integral top cell.

Theorem 3.6. *Let m and n be positive integers and $k = s(p-1)$, let u be prime to p and such that $J(\alpha) = J(u\xi)$ as in 3.5.*

1. *The space $\overline{(L^{2k+1})^{n\alpha}}$ is S-reducible if and only if $\nu_p(un+k+1) \geq s$.*
2. *Two lens spaces $\overline{(L^{2k+1})^{m\alpha}}$ and $\overline{(L^{2k+1})^{n\alpha}}$ which are not S-reducible, are stably equivalent if and only if $\nu_p(m-n) \geq s-1$.*

Proof. The first claim is well known, for the second let $t = p^{s-1}$, if there is a $q \in \{0, 1, \dots, p-1\}$ such that the composite

$$S^{(n+qt)|\alpha|+2k} \xrightarrow{\beta_s} S^{(n+qt)|\alpha|+1} \xrightarrow{i} \overline{(L^{2k})^{(n+qt)\alpha}}$$

is stably null homotopic, then we are done by 2.3 (note that in such situation the spaces involved are not S-reducible), otherwise whenever $\overline{(L^{2k+1})^{(n+qt)\alpha}}$ is not S-reducible, we have by 3.5 and 1 of this theorem, a factorization of $i \circ \beta_s$ in the form

$$\begin{tikzcd}
 & S^{(n+qt)|\alpha|+2k} \arrow[dr, "\pi_q"] & \\
S^{(n+qt)|\alpha|+2k} \arrow[ur, "f_q"] \arrow[r, "\beta_s"] & S^{(n+qt)|\alpha|+1} \arrow[r, "i"] & \overline{(L^{2k})^{(n+qt)\alpha}}
\end{tikzcd}$$

where π_q is the attaching map for the top cell of $\overline{(L^{2k+1})^{(n+qt)\alpha}}$. Our assumption implies $\nu_p(f_q) = 0$ so that π_q and the composite $i \circ \beta_s$ are equal up to a unit in the $(n+qt)|\alpha| + 2k$ homotopy group of $\overline{(L^{2k})^{(n+qt)\alpha}}$, thus π_q is independent of q and so is the cofiber $\overline{(L^{2k+1})^{(n+qt)\alpha}}$. In detail, a stable commutative diagram

$$\begin{tikzcd}
S^{(nq+t)|\alpha|+1} \arrow[r, "i"] \arrow[d, "u_1"] & \Sigma^{qt|\alpha|}\overline{(L^{2k})^{n\alpha}} \overset{\mathrm{def}}{=} L_1 \arrow[d, "h"] \\
S^{(nq+t)|\alpha|+1} \arrow[r, "i"] & \overline{(L^{2k})^{(n+qt)\alpha}} \overset{\mathrm{def}}{=} L_2
\end{tikzcd}$$

where h is a stable equivalence and u_1 is prime to p, can be extended to the following diagram provided neither $\overline{\left(\mathrm{L}^{2k+1}\right)^{n\alpha}}$ nor $\overline{\left(\mathrm{L}^{2k+1}\right)^{(n+qt)\alpha}}$ are S-reducible (the dimension of the spheres has been suppressed)

$$\begin{tikzcd}
 & & S \arrow[dr, "\pi_1"] \arrow[dd, "u_2"] & \\
S \arrow[urr, "f_1"] \arrow[r, "\beta_s"'] \arrow[dd, "u_1"'] & S \arrow[rr, "i"'] \arrow[dd, "u_1"'] & & L_1 \arrow[dd, "h"] \\
 & & S \arrow[dr, "\pi_2"] & \\
S \arrow[urr, "f_2"] \arrow[r, "\beta_s"'] & S \arrow[rr, "i"'] & & L_2
\end{tikzcd}$$

where π_i are the corresponding attaching maps and where u_2 is taken (prime to p) so that

$$f_2 u_1 \equiv u_2 f_1 \pmod{|\pi_{top\ dim}(L_2)|},$$

it follows that $h \circ \pi_1 = \pi_2 \circ u_2$ and an application of 2.4 to the square formed by π_1, π_2, u_2 and h finishes the proof. □

Dually we have the classification of stunted lens spaces with bottom integral cell.

Theorem 3.7. *Let m and n be positive integers and $k = s(p-1)$*

1. *The lens space $(L^{2k})^{n\alpha}$ is S-coreducible if and only if $\nu_p(n) \geq s$.*
2. *Two lens spaces $(L^{2k})^{n\alpha}$ and $(L^{2k})^{m\alpha}$ which are not S-coreducible, are stably equivalent if and only if $\nu_p(m-n) \geq s-1$.*

4. Stunted lens spaces with two integral cells.

The only stunted lens spectra we are missing to classify are those of the form $L_{2n}^{2n+2k+1}$. Their p-localization decomposes as a wedge of $p-1$ summands. By 1.1 (b) we only have to consider the case $k \equiv 0 \pmod{p-1}$ and under such condition the integral cells belong to different stable summands, therefore the desired classification is an easy consequence of the last two theorems. The section begins with an explicit calculation of these remarks. In the sequel q denotes $2(p-1)$ and all spaces will be localized at p.

It is well known that L stably decomposes as a wedge of $p-1$ spaces $B(i)$ $1 \leq i \leq p-1$ where the i^{th} wedge summand has a CW structure with a cell in every positive dimension congruent with $2i-1$ and $2i$ modulo q (the spaces $B(i)$ as well as the decomposition are realized after a single suspension, moreover $B(p-1)$ is the p localization of the classifying space for the symmetric group in p letters). We denote the stunted spaces obtained from $B(i)$ as follows

Notation 4.1. Let $B(i)^n$ denote the skeleton of $B(i)$ formed by the first n positive dimensional cells, thus $B(i)^{2n}$ is a complex of dimension $2i+(n-1)q$ whereas $B(i)^{2n+1}$ is $2i-1+nq$ dimensional. Let $B(i)^n_m$ denote the quotient space of $B(i)^n$ by $B(i)^{m-1}$. We will say that the top cell in $B(i)^n_m$ is integral if n is odd and that its bottom cell is integral if m is even. $B(p-1)^n_m$ will be denoted by B^n_m.

To simplify the statement of the next result we adopt the following convention: For positive integers i, n and m with $m \leq n$ and $i = a(p-1)+r$ where $1 \leq r \leq p-1$, $B(i)^n_m$ will stand for the stunted space $B(r)^{n+2a}_{m+2a}$. Thus the bottom cell of $B(i)^{2n+1}_{2m}$ lies in dimension $2i+(m-1)q$ and the top cell in dimension $2i-1+nq$.

Proposition 4.2. *Let ϵ, $\delta \in \{0, 1\}$ and let a, l and i be positive integers. Define $\rho(l) = 1+p+\cdots+p^{l-1}$. Then there is a stable equivalence*

$$B(i)^{2a+2l+\epsilon}_{2a+\delta} \simeq B(i+1)^{2a+2\rho(l)+2l+\epsilon}_{2a+2\rho(l)+\delta}.$$

Proof. By 1.1 (a) we have a stable equivalence

$$L^{2i-1+(a+l)q}_{2i+(a-1)q} \simeq L^{2i-1+(a+l)q+2p^l}_{2i+(a-1)q+2p^l}.$$

Each one of these spectra decomposes as a wedge of $p-1$ summands in such a way that two given stable factors have no cells in a common dimension. Therefore the above identification yields stable equivalences among the respective summands. In particular, since $2p^l = q\rho(l)+2$, the required equivalence for $\delta = 0$ and $\epsilon = 1$ follows by looking at the summands carrying the lowest (and highest) cell. The general situation follows by pinching the bottom cells or removing the top cells. □

Iteration of the equivalence in 4.2 yields

Corollary 4.3. *With the notation as in 4.2, there are stable equivalences*

$$B(i)^{2a+2l+\epsilon}_{2a+\delta} \simeq B(i)^{2a+2p^l+2l+\epsilon}_{2a+2p^l+\delta}.$$

Using 2.2 instead of 1.1 in the proof of 4.2 we obtain the classification of stunted B spectra with no integral cells.

Theorem 4.4. *Let a, b, l and i be positive integers.*

1. *There are stable equivalences $B(i)^{2a+2l}_{2a+1} \simeq B(i+1)^{2a+2\rho(l-1)}_{2a+2\rho(l-1)+1}$.*
2. *The spaces $B(i)^{2a+2l}_{2a+1}$ and $B(i)^{2b+2l}_{2b+1}$ are stably equivalent if and only if $\nu(a-b) \geq l-1$.*

Proof. The existence of the equivalences follows as in the proof or 4.2 applying 2.2; that they are optimal follows from standard arguments using Adams operations (eg. [**5**]). □

Note. In [**2**] the stunted spectra B_{2a+1}^{2a+2l} are identified as reduced Thom spectra, from which equivalences $B_{2a+1}^{2a+2l} \simeq B_{2a+2p^l+1}^{2a+2p^l+2l}$ are obtained.

Similarly, 3.7 is the key for the classification of stunted B spectra with an integral bottom cell.

Theorem 4.5. *Let a, b, i and l be positive integers.*

1. *$B(i)_{2a}^{2a+2l}$ is S-coreducible if and only if $\nu\left((a-1)(p-1)+i\right) \geq l$.*
2. *If neither $B(i)_{2a}^{2a+2l}$ nor $B(i+1)_{2a+2\rho(l-1)}^{2a+2\rho(l-1)+2l}$ are S-coreducible then they are stably equivalent.*
3. *Two spaces $B(i)_{2a}^{2a+2l}$ and $B(i)_{2b}^{2b+2l}$, which are not S-coreducible, are stably equivalent if and only if $\nu(a-b) \geq l-1$.*

Proof. Part 1 is easily proven by identifying $B(i)_{2a}^{2a+2l}$ as the stable summand carrying the bottom cell in a stunted lens spaces. Part 2 is proven as in 4.2 using 3.7. For part 3 we need to identify the non coreducible spectra among

$$B(i)_{2a}^{2a+2l},\ B(i)_{2a+2p^{l-1}}^{2a+2p^{l-1}+2l}, \dots, B(i)_{2a+qp^{l-1}}^{2a+qp^{l-1}+2l}.$$

If neither of these is coreducible, the result follows by iteration of the equivalences in 2; otherwise only one of them is coreducible, say $B(i)_{2a}^{2a+2l}$, then by 2 we get equivalences

$$B(i+1)_{2a+2\rho(l-1)}^{2a+2\rho(l-1)+2l} \simeq B(i+2)_{2a+4\rho(l-1)}^{2a+4\rho(l-1)+2l} \simeq \cdots \simeq B(i+p-1)_{2a+q\rho(l-1)}^{2a+q\rho(l-1)+2l}$$

but by 4.2 we have for $j = 1, \dots, p-1$:

$$B(i+j)_{2a+2j\rho(l-1)}^{2a+2j\rho(l-1)+2l} \simeq B(i)_{2a+2j\rho(l-1)-2j\rho(l)}^{2a+2j\rho(l-1)-2j\rho(l)+2l} = B(i)_{2a-2jp^{l-1}}^{2a-2jp^{l-1}+2l}$$

giving the desired identifications. 4.4 implies that these are best possible. □

By S-duality (or using 3.6) we get the classification of stunted B spectra with top integral cell.

Theorem 4.6. *Let a, b, i and l be positive integers.*

1. *$B(i)_{2a+1}^{2a+2l+1}$ is S-reducible if and only if $\nu\left((a+l)(p-1)+i\right) \geq l$.*
2. *If neither $B(i)_{2a+1}^{2a+2l+1}$ nor $B(i+1)_{2a+2\rho(l-1)+1}^{2a+2\rho(l-1)+2l+1}$ are S-reducible then they are stably equivalent.*

3. *Two spaces* $B(i)_{2a+1}^{2a+2l+1}$ *and* $B(i)_{2b+1}^{2b+2l+1}$, *which are not S-reducible, are stably equivalent if and only if* $\nu\,(a-b) \geq l-1$.

The classification of stunted lens spectra with integral top and bottom cell is now a consequence of 4.5 and 4.6.

Theorem 4.7. *Let* m, n *and* s *be positive integers and let* $k = s(p-1)$.

1. *The stunted lens space* $L_{2n}^{2n+2k+1}$ *is*
   - i) *S-reducible if and only if* $\nu\,(n+k+1) \geq s$.
   - ii) *S-coreducible if and only if* $\nu\,(n) \geq s$.
2. *Two spaces* $L_{2n}^{2n+2k+1}$ *and* $L_{2m}^{2m+2k+1}$ *which are not S-reducible nor S-coreducible, are stably equivalent if and only if* $\nu\,(m-n) \geq s-1$.

5. Classification by J-groups.

In this section we show that the classification of the stable homotopy types of stunted lens spaces is determined by their J homology and cohomology groups, just as in the 2 primary case [**3**]. Heuristically, J gives the right answer since the stable classification of stunted lens spaces is given by Adams operations and S-reducibility and coreducibility. For the definition of J we follow the notation of [**2**] and [**10**]. When localized at p, bu decomposes as a wedge of suspensions of a spectrum ℓ whose homotopy is a polynomial algebra in a generator v_1 of degree q and Adams filtration 1. If $b \in \mathbf{Z}$ represents a generator of the units in $\mathbf{Z}/p^2$ then the Adams operation $\psi^b - 1 : \ell \to \ell$ lifts over $v_1 : \Sigma^q \ell \to \ell$ defining a map $\theta : \ell \to \Sigma^q \ell$. J is defined as the fibre of θ.

Lemma 5.1. *For integers* a, b *and* e, $a \equiv b \pmod{p^e}$ *if and only if*

$$\min\{e, \nu_p(i+a)\} = \min\{e, \nu_p(i+b)\} \qquad \forall i \geq 0.$$

Theorem 5.2. *Stunted* B *spaces with at most one integral cell as well as stunted lens spaces are stably classified by their* J*-homology and cohomology groups* (*with the proper shift in dimensions*). *In detail:*

- a) *Stunted spaces with no integral cells are stably classified by their J-homology groups; they are also classified by their J-cohomology groups.*
- b) *Stunted spaces with integral top cell are stably classified by their J-homology groups.*
- c) *Stunted spaces with integral bottom cell are stably classified by their J-cohomology groups.*

d) *Stunted lens spaces with two integral cells are stably classified by both their J-homology and J-cohomology groups.*

Proof. By S-duality and in view of the stable decomposition of stunted lens spaces, it suffices to consider parts a) and b) for stunted B spaces. Let m and n be integers with $0 < m \leq n$ and set $s = n - m + 1$ (=no. of Moore cells in B^{2n}_{2m-1}). In **[10]** it is shown that for $i \geq 0$,

$$J_{(i+n+1)q-1}(B^{2n}_{2m-1}) = \mathbf{Z}/p^a$$

where $a = \min\{s, \nu_p(i+n+1)+1\}$, thus the first part follows from 5.1. For spaces B^{2n+1}_{2m-1}, the cofibration $B^{2n}_{2m-1} \longrightarrow B^{2n+1}_{2m-1} \longrightarrow S^{(n+1)q-1}$ induces isomorphisms

$$J_{iq-1}\left(B^{2n}_{2m-1}\right) \xrightarrow{\simeq} J_{iq-1}\left(B^{2n+1}_{2m-1}\right)$$

thus according to 4.6 we need to verify that J-homology distinguishes S-reducibility on the spaces B^{2n+1}_{2m-1}. The fibration defining J induces exact sequences $(i > 0)$

$$\text{(1)} \qquad \ell_{(i+n+1)q-1}\left(B^{2n+1}_{2m-1}\right) \xrightarrow{\theta} \ell_{(i+n)q-1}\left(B^{2n+1}_{2m-1}\right) \to J_{(i+n+1)q-2}\left(B^{2n+1}_{2m-1}\right) \to 0$$

where the first two groups are isomorphic to $\mathbf{Z}_{(p)} \bigoplus \mathbf{Z}/p^s$. Let x denote the integral generator and y the torsion generator in the first group, similarly let x' and y' denote the corresponding generators in the second group. Using naturality of the action of θ in the cofibration $B^{2n}_{2m-1} \hookrightarrow B^{2n+1}_{2m-1} \to S^{(n+1)q-1}$ together with the well known action of θ on spheres and on stunted B-spaces with no integral cells (e.g. **[10]**), it is easy to check that

- $\theta(x) = u_2 p^{\nu(i)+1} x' + a y'$
- $\theta(y) = u_1 p^{\nu(i+n+1)+1} y'$

where a, u_1 and u_2 are integers and both u_1 and u_2 are prime to p. Using again naturality of θ in the inclusion $B^{2n+1}_{2m-1} \hookrightarrow B^{2n+2}_{2m-1}$ it follows that $a \equiv u_3 p^{\nu(i+n+1)} - u_2 p^{\nu(i)} \pmod{p^s}$.

From 4.6, B^{2n+1}_{2m-1} is S-reducible if and only if $\nu(n+1) \geq s$, thus we have to check that for integers n_1 and n_2 with $\nu(n_1+1) \geq s$ and $n_2 = n_1 + p^{s-1}$, the corresponding cokernels in the exact sequence (1) are not isomorphic; for instance taking $i = p^s$ we get $J_{(i+n_1+1)q-2}\left(B^{2n_1+1}_{2m_1-1}\right) = \mathbf{Z}/p^{s+1} \bigoplus \mathbf{Z}/p^s$, however the integral generator $x' \in \ell_{(i+n_2)q-1}\left(B^{2n_2+1}_{2m_2-1}\right)$ produces an element of order p^{s+2} in $J_{(i+n_2+1)q-2}\left(B^{2n_2+1}_{2m_2-1}\right)$. $\square$

6. Stunted B spaces with two integral cells.

The work done up to now can be considered as the odd primary version of [**5**], the cases left out in that paper correspond to the classification of stunted B spaces with both top and bottom integral cells that we have not considered. In this section we identify the obstructions arising in such classification. We will show elsewhere that a number of these obstructions are null homotopic, the methods however will be different from those used in [**3**] in the two primary case.

Recall $B = (B\Sigma_p)_{(p)}$ and let β the vector bundle defined in the remark below 2.1. If β_s represents the restriction of β to B^{2s+1}, then $B_{2n}^{2n+2s+1}$ can be identified with the Thom complex of $n\beta_s$ [**2**]. The stable dual for this spectrum can be obtained by comparing with the dual of a stunted lens space having it as a stable summand. With this in mind the following result is straightforward to check.

Lemma 6.1. *$B_{2a}^{2a+2s+1}$ is S-coreducible if and only if $\nu(a) \geq s$. It has stable dual $B^{-2a-1}_{-2a-2s-2}$.*

Remark 6.2. The spectrum $B^{-2a-1}_{-2a-2s-2}$ makes sense in view of 4.3, it has top cell in dimension $-aq-1$.

Let i' denote the inclusion into the next-to-the-bottom cell in $(B^{2s+1})^{a\beta}$, that is, i' is the composite $S^{(a+1)q-1} \hookrightarrow S^{aq} \vee S^{(a+1)q-1} \hookrightarrow (B^{2s+1})^{a\beta}$ (recall $q = 2(p-1)$). The next result is the odd primary version of the geometric idea of [**3**], it identifies obstructions for the classification of stunted B spaces with two integral cells.

Proposition 6.3. *Let n, s, t be positive integers with $\nu(t) \geq s-1$ and let β_s be as in 2.3. Assume that the following composites are stably null homotopic (the dimension of the spheres is implicit)*

$$S \xrightarrow{\beta_s} S \xrightarrow{i'} (B^{2s})^{n\beta} \qquad \textit{and} \qquad S \xrightarrow{\beta_s} S \xrightarrow{i'} (B^{2s})^{-(n+t+s+1)\beta}$$

Then $(B^{2s+1})^{n\beta} \simeq (B^{2s+1})^{(n+t)\beta}$.

Proof. $t\beta$ is trivial over B^{2s-1} so that there is a map θ that fits in the diagram

$$\begin{tikzcd} B^{2s+1} \arrow[r, "t\beta"] \arrow[d, "c'"'] & BO(tq) \\ S^{sq} \arrow[ur, "\theta"'] & \end{tikzcd}$$

and has $J(\theta) = \beta_s$. Since $(n+t)\beta$ is classified by

$$B^{2s+1} \xrightarrow{\Delta} B^{2s+1} \times B^{2s+1} \xrightarrow{1\times c'} B^{2s+1} \times S^{sq} \xrightarrow{n\beta \times \theta} BO \times BO \to BO\,,$$

the Thomification of $(1 \times c') \circ \Delta$ induces a map

$$\lambda :\ \left(B^{2s+1}\right)^{(n+t)\beta} \longrightarrow \left[\ \left(B^{2s+1}\right)^{n\beta} \wedge S^{tq}\ \right]\ \bigcup_{i\,\circ\, J\theta} \mathbf{e}^{(n+t+s)q}\ \bigcup_{i'\,\circ\, J\theta} \mathbf{e}^{(n+t+s+1)q-1}$$

where as above i' denotes the inclusion of the second cell in $\Sigma^{tq}\left(B^{2s+1}\right)^{n\beta}$, and i the inclusion of the bottom cell. As in [**3**] (or in 2.3), the hypothesis implies the existence of a retraction r

$$\left[\left(B^{2s+1}\right)^{n\beta}\wedge S^{tq}\right]\bigcup_{i\,\circ\, J\theta}\mathbf{e}^{(n+t+s)q}\bigcup_{i'\,\circ\, J\theta}\mathbf{e}^{(n+t+s+1)q-1}\longrightarrow\left[\left(B^{2s+1}\right)^{n\beta}\wedge S^{tq}\right]\bigcup_{i\,\circ\, J\theta}\mathbf{e}^{(n+t+s)q}$$

in such a way that the composite $r \circ \lambda$ induces isomorphisms in mod p cohomology except in dimension $(n+t+s)q$ where it looks like a folding map. The stable dual of this composite is a map $(B_1 \vee S^u) \bigcup_{\pi_1 \vee J\theta} \mathbf{e}^v \xrightarrow{g} \Sigma^{-w}\overline{B_2}$ of degree one on the top cell, where $u = -(n+s)q-1$, $v = -nq-1$, $w = -tq$, $B_1 = \left(B^{2s}\right)^{-(n+s+1)\beta}$, $\overline{B_2} = \left(B^{2s+1}\right)^{-(n+t+s+1)\beta}$, and π_1 is the attaching map for the top cell in $\left(B^{2s+1}\right)^{-(n+s+1)\beta}$. Define also $\overline{B_1}$ and B_2 similarly so to obtain the obvious cofibrations

$$S^{v-1} \xrightarrow{\pi_1} B_1 \xrightarrow{j_1} \overline{B_1} \quad \text{and} \quad S^{v-1} \xrightarrow{\pi_2} \Sigma^{-w}B_2 \xrightarrow{j_2} \Sigma^{-w}\overline{B_2}.$$

Let $f : B_1 \vee S^u \to \Sigma^{-w}B_2$ be the restriction of g to the $-nq-2$ dimensional skeleta. Thus the composite $f \circ (\pi_1 \vee J\theta)$ agrees with the attaching map π_2. Since $\pi_u\left(\Sigma^{-w}B_2\right)$ is generated by i' and $i \circ \beta_1$, and since $\beta_1 \circ \beta_s$ is null homotopic, the second half of our hypothesis implies that the restriction of f to the sphere S^u vanishes when precomposed with β_s, therefore the attaching map π_2 equals the composition of π_1 with the restriction of f to B_1. This produces a map $h : \overline{B_1} \to \Sigma^{-w}\overline{B}_2$ of degree one on the top cell and which induces isomorphisms in mod p cohomology in all other dimensions, thus h is the desired stable equivalence. □

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Received June 29, 1994 and revised November 29, 1994. The author held a fellowship from the CONACyT, México, during the time this research was done.

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