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A LARGE DEVIATION PRINCIPLE FOR SYMMETRIC MARKOV PROCESSES NORMALIZED BY FEYNMAN–KAC FUNCTIONALS

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Abstract

We establish a large deviation principle for the occupation distribution of a symmetric Markov process normalized by Feynman–Kac functional. The obtained theorem means a large deviation from a ground state, not from an invariant measure.

1. Introduction

Let $\mathbf{M} = (\Omega, X_t, \mathbb{P}_x, \zeta)$ be an m-symmetric irreducible Markov process on a locally compact separable metric space X. Here ζ is the lifetime and m is a positive Radon measure with full support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(X; m)$ generated by $\mathbf{M}$ (for the definition, see (2.1)). We denote by $\mathcal{P}$ the set of probability measures with the weak topology, and for a positive Green-tight Kato measure μ (Definition 2.1) define the function I^μ on the set $\mathcal{P}$ by

$$I^\mu(\nu) = \begin{cases} \mathcal{E}^\mu(\sqrt{f}, \sqrt{f}) & \text{if} \quad \nu = f \cdot m, \ \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty & \text{otherwise}, \end{cases} \tag{1.1}$$

where $\mathcal{E}^\mu = \mathcal{E} - (\,\cdot\,,\ \cdot\,)_\mu$. Given $\omega \in \Omega$ with $0 < t < \zeta(\omega)$, let $L_t(\omega) \in \mathcal{P}$ be the normalized occupation distribution: for a Borel set A of X

$$L_t(\omega)(A) = \frac{1}{t} \int_0^t 1_A(X_s(\omega))\, ds,$$

where 1_A is the indicator function of the set A. We denote by A_t^μ the positive continuous additive functional with Revuz measure μ. One of authors proved Donsker–Varadhan type large deviation principle with rate function I^μ.

Theorem 1.1 ([24]). *Assume that the Markov process* $\mathbf{M}$ *possesses the strong Feller property and the tightness property* (*see* (III) *in* Section 2).

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(i) *For each open set* $G \subset \mathcal{P}$

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x(e^{A_t^\mu};\ L_t \in G,\ t < \zeta) \geq -\inf_{\nu\in G} I^\mu(\nu).$$

(ii) *For each closed set* $K \subset \mathcal{P}$

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x(e^{A_t^\mu};\ L_t \in K,\ t < \zeta) \leq -\inf_{\nu\in K} I^\mu(\nu).$$

Varadhan [29] gave an abstract formulation for the large deviation principle. The statement in Theorem 1.1 is slightly different from his formulation. In fact, the rate function I^μ is not always non-negative because it is defined by the Schrödinger form $\mathcal{E}^\mu$, not by the Dirichlet form $\mathcal{E}$. Furthermore, Theorem 1.1 does not represent a large deviation from a invariant measure because the Markov process is allowed to be explosive. By this reason, we consider the normalized probability measure $Q_{x,t}$ on $\mathcal{P}$ defined by, for a Borel set $B \subset \mathcal{P}$,

$$Q_{x,t}(B) = \frac{\mathbb{E}_x(e^{A_t^\mu};\ L_t \in B, t < \zeta)}{\mathbb{E}_x(e^{A_t^\mu};\ t < \zeta)},$$

and prove that the family of probability measures $\{Q_{x,t}\}_{t>0}$ obeys the large deviation principle as $t \to \infty$ in the sense of Varadhan's formulation. In other words, $\{Q_{x,t}\}_{t>0}$ satisfies the *full* large deviation principle with a *good* rate function in the sense of [11, Section 2.1]. This is the main theorem of this paper (Theorem 4.1). The rate function is given by

$$J(\nu) := I^\mu(\nu) - \lambda_2(\mu), \quad \nu \in \mathcal{P}. \tag{1.2}$$

Here $\lambda_2(\mu)$ is the bottom of the spectrum of the Schrödinger type operator $\mathcal{L} + \mu$, where $\mathcal{L}$ is the generator of the Markov process:

$$\lambda_2(\mu) = \inf\{\mathcal{E}^\mu(u, u) \colon u \in \mathcal{D}(\mathcal{E}),\ \|u\|_2 = 1\}.$$

To obtain the main theorem, we need to show that the rate function J is good, that is, enjoys the properties (i)–(iv) in Lemma 4.1. In particular, we must show that J has a unique zero point, that is, the existence of a ground state ϕ_0 of the operator $\mathcal{L}+\mu$. In order to show the existence of a ground state, we usually use the L^2-weak compactness of the set $\{u \in \mathcal{D}(\mathcal{E})\colon \mathcal{E}^\mu(u,u) \leq l\}$ ($l \in \mathbb{R}$) and the lower semi-continuity of the Schrödinger form $\mathcal{E}^\mu$ with respect to the L^2-weak topology (e.g. [17]); however we can not derive these properties from our general setting. Hence we here use the following properties instead, the tightness of the level set $\{\nu \in \mathcal{P}\colon I^\mu(\nu) \leq l\}$ and the lower semi-continuity of the function I^μ with respect to the weak topology. This is a key to the proof of the goodness of the rate function J. We would like to emphasis that the tightness follows from the condition (III) and the Green-tightness of μ, and the lower semi-continuity of I^{μ} follows from a variational formula for the Schrödinger form (Proposition 2.1), that is, the identification of the Schrödinger form with the modified I-function defined in (2.8). The latter is an extension of a well-known fact due to Donsker and Varadhan that for a symmetric Markov process, the I-function is identical with the Dirichlet form. On account of Lemma 4.1, we can regard the main theorem as a large deviation from the ground state of the Schrödinger operator.

In [10], [25], [28], L^p-independence of growth bounds of non-local Feynman–Kac semigroups have been considered. In this paper we also deal with non-local Feynman–Kac transforms and extended Theorem 1.1 to symmetric Markov processes with non-local Feynman–Kac functional (Theorem 2.1). The existence of ground states implies the existence of a quasi-stationary distribution, $\eta(B) := \int_B \phi_0(x)\,dm(x)/\int_X \phi_0(x)\,dm(x)$. In [16], they prove that if a Markov semigroup is intrinsically ultracontractive, then the measure η is the so-called Yaglom limit and a unique quasi-stationary distribution. In the last section, we will give an extension of this fact to generalized Feynman–Kac semigroups by employing Fukushima's ergodic theorem.

2. Symmetric Markov processes with non-local Feynman–Kac functionals

Let X be a locally compact separable metric space and $\mathcal{B}(X)$ the Borel σ-field. Adjoining an extra point ∞ to the measurable set $(X, \mathcal{B}(X))$, we set $X_\infty = X \cup \{\infty\}$ and $\mathcal{B}(X_\infty) = \mathcal{B}(X) \cup \{B \cup \{\infty\} \colon B \in \mathcal{B}(X)\}$. Let $\mathbf{M} = (\Omega, X_t, \mathbb{P}_x, \zeta)$ be a right Markov process on X with lifetime $\zeta := \inf\{t > 0 \colon X_t = \infty\}$. We define the semigroup and the resolvent by

$$p_t f(x) = \mathbb{E}_x(f(X_t);\, t < \zeta), \quad R_\beta f(x) = \int_0^\infty e^{-\beta t} p_t f(x)\,dt$$

for a bounded Borel function f on X. We assume that the Markov process $\mathbf{M}$ is m-symmetric, $(p_t f, g)_m = (f, p_t g)_m$, where m is a positive Radon measure with full support. Let $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ be the Dirichlet form on $L^2(X; m)$ generated by $\mathbf{M}$:

$$\begin{cases} \mathcal{D}(\mathcal{E}) = \left\{u \in L^2(X; m) \colon \lim_{t \to 0} \dfrac{1}{t}(u - p_t u, u)_m < \infty \right\}, \\ \mathcal{E}(u, v) = \lim_{t \to 0} \dfrac{1}{t}(u - p_t u, v)_m. \end{cases} \tag{2.1}$$

For basic materials on right processes and associated Dirichlet forms (quasi-regular Dirichlet forms), we refer to [7], [18].

We impose three assumptions on $\mathbf{M}$.

(I) (*Irreducibility*) If a Borel set A is p_t-invariant, i.e., $p_t(1_A f)(x) = 1_A p_t f(x)$ m-a.e. for any $f \in L^2(X; m) \cap \mathcal{B}_b(X)$ and $t > 0$, then A satisfies either $m(A) = 0$ or $m(X \setminus A) = 0$. Here $\mathcal{B}_b(X)$ is the space of bounded Borel functions on X.

(II) (*Strong Feller property*) For each t, $p_t(\mathcal{B}_b(X)) \subset C_b(X)$, where $C_b(X)$ is the space of bounded continuous functions on X.
(III) (*Tightness*) For any $\epsilon > 0$, there exists a compact set K such that

$$\sup_{x \in X} R_1 1_{K^c}(x) \leq \epsilon.$$

Here 1_{K^c} is the indicator function of the complement of the compact set K.

The assumption (II) implies that $\mathbf{M}$ satisfies the *absolute continuity condition*, that is, its transition probability $p_t(x, \cdot)$ is absolutely continuous with respect to m for each $t > 0$ and $x \in X$. As a result, the resolvent kernel is also absolutely continuous with respect to m, $R_\beta(x, dy) = R_\beta(x, y)m(dy)$. By [14, Lemma 4.2.4] the density $R_\beta(x, y)$ is assumed to be a non-negative Borel function such that $R_\beta(x, y)$ is symmetric and β-excessive in x and in y. Under the absolute continuity condition, "quasi everywhere" statements are strengthened to "everywhere" ones. Moreover, we can defined notions without exceptional set, for example, *smooth measures in the strict sense* or *positive continuous additive functional in the strict sense* (cf. [14, Section 5.1]). Here we only treat the notions in the strict sense and omit the phrase "in the strict sense".

We denote S_{00} the set of positive Borel measures μ such that $\mu(X) < \infty$ and $R_1\mu(x)$ $(= \int_X R_1(x,y)\mu(dy))$ is uniformly bounded in $x \in X$. A positive Borel measure μ on X is said to be *smooth* if there exists a sequence $\{E_n\}_{n=1}^\infty$ of Borel sets increasing to X such that $1_{E_n} \cdot \mu \in S_{00}$ for each n and

$$\mathbb{P}_x\left(\lim_{n\to\infty} \sigma_{X\setminus E_n} \geq \zeta\right) = 1, \quad \forall x \in X,$$

where $\sigma_{X\setminus E_n}$ is the first hitting time of $X \setminus E_n$. The totality of smooth measures is denoted by S_1.

If an additive functional $\{A_t\}_{t\geq 0}$ is positive and continuous with respect to t for each $\omega \in \Lambda$, it is said to be a *positive continuous additive functional* (PCAF in abbreviation). By [14, Theorem 5.1.7], there exists a one-to-one correspondence between positive smooth measures and PCAF's (*Revuz correspondence*): for each smooth measure μ, there exists a unique PCAF $\{A_t\}_{t\geq 0}$ such that for any positive Borel function f on X and γ-excessive function h $(\gamma \geq 0)$, that is, $e^{-\gamma t} p_t h \leq h$,

$$\lim_{t\to 0} \frac{1}{t}\mathbb{E}_{h\cdot m}\left(\int_0^t f(X_s)\, dA_s\right) = \int_X f(x)h(x)\mu(dx). \tag{2.2}$$

Here $\mathbb{E}_{h\cdot m}(\cdot) = \int_X \mathbb{E}_x(\cdot)h(x)m(dx)$. We denote by A_t^μ the PCAF corresponding to the smooth measure μ. For a signed Borel measure $\mu = \mu^+ - \mu^-$, let $|\mu| = \mu^+ + \mu^-$. When $|\mu|$ is a smooth measure, we define $A_t^\mu = A_t^{\mu^+} - A_t^{\mu^-}$ and $A_t^{|\mu|} = A_t^{\mu^+} + A_t^{\mu^-}$.

Following Chen [4], we introduce classes of potentials.

DEFINITION 2.1. (i) A signed Borel measure μ is said to be the *Kato measure* (in notation, $\mu \in \mathcal{K}$), if $|\mu| \in S_1$ and

$$\lim_{t\to 0} \sup_{x\in X} \mathbb{E}_x(A_t^{|\mu|}) = 0.$$

(ii) A measure $\mu \in \mathcal{K}$ is said to be in the class $\mathcal{K}_\infty$, if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that for all measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{x\in X} \int_{K^c \cup B} R_1(x, y)|\mu|(dy) \leq \epsilon.$$

(iii) A signed Borel measure μ is said to be in the class $\mathcal{S}_\infty$, if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that for all measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{(x,z)\in X\times X\setminus d} \int_{K^c \cup B} \frac{R_1(x, y)R_1(y, z)}{R_1(x, z)}|\mu|(dy) \leq \epsilon.$$

It is known in [2] that μ belongs to $\mathcal{K}$ if and only if

$$\lim_{\beta\to\infty} \sup_{x\in X} \int_X R_\beta(x, y)|\mu|(dy) = 0, \tag{2.3}$$

and in [4] that

$$\mathcal{S}_\infty \subset \mathcal{K}_\infty \subset \mathcal{K}. \tag{2.4}$$

We denote that $(N, H) = (N(x, dy), H_t)$ is the Lévy system of $\mathbf{M}$, that is, N is a kernel on $(X_\infty, \mathcal{B}(X_\infty))$ with $N(x, \{x\}) = 0$ and H is a positive continuous additive functional of $\mathbf{M}$ such that for any non-negative measurable function F on $X \times X$ vanishing on the diagonal set and any $x \in X$,

$$\mathbb{E}_x\left(\sum_{0<s\leq t} F(X_{s-}, X_s); t < \zeta\right) = \mathbb{E}_x\left(\int_0^t \int_X F(X_s, y)N(X_s, dy)\, dH_s\right).$$

We denote by μ^H be the smooth measure corresponding to H_t.

DEFINITION 2.2. Let F be a bounded measurable function on $X \times X$ vanishing on the diagonal set.
(i) F is said to be in the class $\mathcal{A}_\infty$, if for any $\epsilon > 0$ there exist a compact subset K and a positive constant $\delta > 0$ such that for all measurable set $B \subset K$ with $|\mu|(B) < \delta$,

$$\sup_{(x,z)\in X\times X\setminus d} \int_{((K\setminus B)\times(K\setminus B))^c} \frac{R_1(x, y)|F(y, z)|R_1(z, w)}{R_1(x, w)} N(y, dz)\mu^H(dy) \leq \epsilon.$$

(ii) F is said to be in the class $\mathcal{A}_2$, if $F \in \mathcal{A}_\infty$ and

$$\mu_{|F|}(dx) = \left(\int_X |F(x,y)|N(x,dy)\right)\mu^H(dx) \in \mathcal{S}_\infty.$$

For properties and examples of $\mathcal{A}_\infty$ and $\mathcal{A}_2$, see [4], [5]. In the remainder of this paper, we assume that F is symmetric, $F(x,y) = F(y,x)$. We write $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ if $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{A}_2$.

For $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ define the AF $A_t^{\mu+F}$ by

$$A_t^{\mu+F} = A_t^\mu + \sum_{0<s\le t} F(X_{s-}, X_s),$$

and the generalized Feynman–Kac semigroup $\{p_t^{\mu+F}\}_{t\ge 0}$ by

$$p_t^{\mu+F} f(x) = \mathbb{E}_x\left(e^{A_t^{\mu+F}} f(X_t);\ t < \zeta\right), \quad f \in \mathcal{B}_b(X).$$

For $F \in \mathcal{A}_2$, we define the symmetric Dirichlet form $(\mathcal{E}_F, \mathcal{D}(\mathcal{E}))$ as follows: for $u, v \in \mathcal{D}(\mathcal{E})$

$$\begin{aligned} \mathcal{E}_F(u,v) &= \mathcal{E}^{(c)}(u,v) + \mathcal{E}^{(k)}(u,v) \\ &\quad + \frac{1}{2}\int_{X\times X} (u(x)-u(y))(v(x)-v(y))e^{F(x,y)}N(x,dy)\mu^H(dx), \end{aligned} \tag{2.5}$$

where $\mathcal{E}^{(c)}$ and $\mathcal{E}^{(k)}$ are the local part and the killing part of the Dirichlet form $(\mathcal{E},\mathcal{D}(\mathcal{E}))$ in Beurling–Deny formula ([14, Theorem 3.2.1]). Fundamental properties of non-local Feynman–Kac transforms were earlier studied by J. Ying [31], [32]. It is known in [8] that $\{p_t^{\mu+F}\}_{t\ge 0}$ is the semigroup generated by the Schrödinger form $(\mathcal{E}^{\mu+F}, \mathcal{D}(\mathcal{E}))$:

$$\mathcal{E}^{\mu+F}(u,v) = \mathcal{E}_F(u,v) - \int_X u(x)v(x)\,d\mu_{F_1}(x) - \int_X u(x)v(y)\,d\mu(x), \tag{2.6}$$

where $F_1 = \exp(F) - 1$. The form $\mathcal{E}^{\mu+F}$ is also written as

$$\begin{aligned} \mathcal{E}^{\mu+F}(u,v) = \mathcal{E}(u,v) &- \int_{X\times X} u(x)v(y)F_1(x,y)N(x,dy)\,d\mu^H(y) \\ &- \int_X u(x)v(x)d\mu(x), \quad u, v \in \mathcal{D}(\mathcal{E}). \end{aligned}$$

Let $\mathcal{P}$ be the set of probability measures on X equipped with the weak topology. We define the function $I^{\mu+F}$ on $\mathcal{P}$ by

$$I^{\mu+F}(\nu) = \begin{cases} \mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) & \text{if} \quad \nu = f\cdot m,\ \sqrt{f} \in \mathcal{D}(\mathcal{E}), \\ \infty & \text{otherwise.} \end{cases}$$

Let $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ and define $\kappa(\mu + F)$ by

$$\kappa(\mu + F) = \lim_{t\to\infty} \frac{1}{t} \log\| p_t^{\mu+F} \|_{\infty,\infty}.$$

We see from [1] that $\kappa(\mu + F)$ is finite. If $\alpha > \kappa(\mu + F)$ and $f \in \mathcal{B}_b(X)$, we define the resolvent $R_\alpha^{\mu+F}$ by

$$R_\alpha^{\mu+F} f(x) = \mathbb{E}_x\left(\int_0^\infty e^{-\alpha t + A_t^{\mu+F}} f(X_t)\, dt\right).$$

We set

$$\mathcal{D}_+(\mathcal{H}^{\mu+F}) = \left\{R_\alpha^{\mu+F} f : \alpha > \kappa(\mu + F),\ f \in L^2(X; m) \cap C_b(X),\ f \geq 0 \text{ and } f \not\equiv 0\right\}.$$

Each function $\phi = R_\alpha^{\mu+F} f \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ is strictly positive because $\mathbb{P}_x(\sigma_O < \zeta) > 0$ for any $x \in X$ by the assumption (I). Here O is a non-empty open set $\{x \in X : f(x) > 0\}$ and $\sigma_O = \inf\{t > 0 : X_t \in O\}$. We define the generator $\mathcal{H}^{\mu+F}$ by

$$\mathcal{H}^{\mu+F} u = \alpha u - f, \quad u = R_\alpha^{\mu+F} f \in \mathcal{D}_+(\mathcal{H}^{\mu+F}).$$

Let h be the function defined by $h(x) = \mathbb{E}_x(\exp(A_\zeta^{\mu+F}))$. We may assume that $\mu + F$ is gaugeable, that is, $\sup_{x\in X} h(x) < \infty$. In fact, it is enough to prove Theorem 2.1 and Theorem 4.1 below for the β-subprocess, $\mathbb{P}_x^{(\beta)} = e^{-\beta t}\mathbb{P}_x$. Moreover, we see that every $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ becomes gaugeable with respect to the β-subprocess of $\mathbf{M}$ for a large enough β. In fact, we see from [5, Theorem 3.4] that $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$ is gaugeable with respect to the β-subprocess if and only if

$$\begin{aligned} \inf\Big\{&\mathcal{E}_F(u, u) + \int_X u(x)^2(\mu^- + \mu_{F_1^-})(dx) + \beta \int_X u(x)^2 m(dx) : \\ &\int_X u(x)^2(\mu^+ + \mu_{F_1^+})(dx) = 1\Big\} > 1, \end{aligned} \tag{2.7}$$

where F_1^+ and F_1^- is the positive and negative part of F_1. Since by (3.1)

$$\begin{aligned} &\mathcal{E}_F(u, u) + \int_X u(x)^2(\mu^- + \mu_{F_1^-})(dx) + \beta \int_X u(x)^2 m(dx) \\ &\geq e^{-\|F^-\|_\infty}\left(\mathcal{E}(u, u) + \beta \int_X u(x)^2 m(dx)\right) \geq \frac{e^{-\|F^-\|_\infty}}{\|R_\beta(\mu^+ + \mu_{F_1^+})\|_\infty}, \end{aligned}$$

and the right hand side tends to ∞ as $\beta \to \infty$ because of $\mu^+ + \mu_{F_1^+} \in \mathcal{K}$, (2.7) holds for a large β.

We define the function I_h on $\mathcal{P}$ by

$$I_h(\nu) = -\inf_{\substack{\phi\in\mathcal{D}_+(\mathcal{H}^{\mu+F})\\ \epsilon>0}} \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h}\,d\nu. \tag{2.8}$$

The gauge function $h(x)$ satisfies $0 < c \le h(x) \le C < \infty$. Indeed, it follows from Proposition 2.2 in [4] and (2.4) that for $\mu \in \mathcal{K}_\infty$ and $F \in \mathcal{A}_2$, $\sup_{x\in X}\mathbb{E}_x(A_\zeta^{|\mu|+|F|}) < \infty$. Hence, by Jensen's inequality,

$$\inf_{x\in X}\mathbb{E}_x(\exp(A_\zeta^{\mu+F})) \ge \exp\left(-\sup_{x\in X}\mathbb{E}_x(A_\zeta^{|\mu|+|F|})\right) > 0.$$

Let us define the function I_α on $\mathcal{P}$ by

$$I_\alpha(\nu) = -\inf_{\substack{u\in\mathcal{B}_b^+(X)\\ \epsilon>0}} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}u+\epsilon h}{u+\epsilon h}\right)d\nu$$

Lemma 2.1. *It holds that*

$$I_\alpha(\nu) \le \frac{I_h(\nu)}{\alpha}, \quad \nu\in\mathcal{P}.$$

Proof. For $u = R_\alpha^{\mu+F}f \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ and $\epsilon > 0$, set

$$\phi(\alpha) = -\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}u+\epsilon h}{u+\epsilon h}\right)d\nu.$$

Then, noting that $(d/d\alpha)(R_\alpha^{\mu+F}u) = -R_\alpha^{\mu+F}(R_\alpha^{\mu+F}u) = -(R_\alpha^{\mu+F})^2u$, we have

$$\frac{d\phi}{d\alpha}(\alpha) = -\int_X \frac{R_\alpha^{\mu+F}u - \alpha(R_\alpha^{\mu+F})^2u}{\alpha R_\alpha^{\mu+F}u+\epsilon h}\,d\nu = \int_X \frac{\mathcal{H}^{\mu+F}(R_\alpha^{\mu+F})^2u}{\alpha R_\alpha^{\mu+F}u+\epsilon h}\,d\nu.$$

Since

$$\begin{aligned}&(\alpha(R_\alpha^{\mu+F})^2u - R_\alpha^{\mu+F}u)(\alpha^2(R_\alpha^{\mu+F})^2u+\epsilon h)\\ &-(\alpha(R_\alpha^{\mu+F})^2u - R_\alpha^{\mu+F}u)(\alpha R_\alpha^{\mu+F}u+\epsilon h)\end{aligned}$$

equals $\alpha(\alpha(R_\alpha^{\mu+F})^2u - R_\alpha^{\mu+F}u)^2 \ge 0$, we have

$$\frac{\alpha(R_\alpha^{\mu+F})^2u - R_\alpha^{\mu+F}u}{\alpha R_\alpha^{\mu+F}u+\epsilon h} \ge \frac{\alpha(R_\alpha^{\mu+F})^2u - R_\alpha^{\mu+F}u}{\alpha^2(R_\alpha^{\mu+F})^2u+\epsilon h},$$

and thus

$$\begin{aligned}\int_X \frac{\mathcal{H}^{\mu+F}(R_\alpha^{\mu+F})^2 u}{\alpha R_\alpha^{\mu+F} u + \epsilon h}\, d\nu &\geq \int_X \frac{\mathcal{H}^{\mu+F}(R_\alpha^{\mu+F})^2 u}{\alpha^2 (R_\alpha^{\mu+F})^2 u + \epsilon h}\, d\nu \\ &= -\frac{1}{\alpha^2}\left(-\int_X \frac{\mathcal{H}^{\mu+F}(R_\alpha^{\mu+F})^2 u}{(R_\alpha^{\mu+F})^2 u + \epsilon h/\alpha^2}\, d\nu \right) \\ &\geq -\frac{1}{\alpha^2} I_h(\nu).\end{aligned}$$

Therefore

$$\phi(\infty) - \phi(\alpha) = \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h} \right) d\nu \geq -\frac{I_h(\nu)}{\alpha},$$

which implies

$$-\inf_{\substack{u \in \mathcal{D}_+(\mathcal{H}^{\mu+F}) \\ \epsilon > 0}} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} u + \epsilon h}{u + \epsilon h} \right) d\nu \leq \frac{I_h(\nu)}{\alpha}.$$

Since $\|\beta R_\beta^{\mu+F} f\|_\infty \leq C\|f\|_\infty$, $\beta > 0$, and $\beta R_\beta^{\mu+F} f(x) \to f(x)$ as $\beta \to \infty$,

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}(\beta R_\beta^{\mu+F} f) + \epsilon h}{\beta R_\beta^{\mu+F} f + \epsilon h} \right) d\nu \xrightarrow{\beta\to\infty} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} f + \epsilon h}{f + \epsilon h} \right) d\nu. \tag{2.9}$$

Define the measure ν_α by

$$\nu_\alpha(A) = \int_X \alpha R_\alpha^{\mu+F}(x, A)\, d\nu(x), \quad A \in \mathcal{B}(X).$$

Given $v \in \mathcal{B}_b^+(X)$, take a sequence $\{g_n\}_{n=1}^\infty \subset C_b^+(X) \cap L^2(X; m)$ such that

$$\int_X |v - g_n|\, d(\nu_\alpha + \nu) \to 0 \quad \text{as} \quad n \to \infty.$$

We then have

$$\int_X |\alpha R_\alpha^{\mu+F} v - \alpha R_\alpha^{\mu+F} g_n|\, d\nu \leq \int_X \alpha R_\alpha^{\mu+F}(|v - g_n|)\, d\nu = \int_X |v - g_n|\, d\nu_\alpha \to 0$$

as $n \to \infty$, and so

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} g_n + \epsilon h}{g_n + \epsilon h} \right) d\nu \xrightarrow{n\to\infty} \int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} v + \epsilon h}{v + \epsilon h} \right) d\nu. \tag{2.10}$$

Hence, combining (2.9) and (2.10)

$$\inf_{u\in\mathcal{D}_+(\mathcal{H}^{\mu+F})}\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}u+\epsilon h}{u+\epsilon h}\right)d\nu=\inf_{u\in\mathcal{B}_b^+(X)}\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}u+\epsilon h}{u+\epsilon h}\right)d\nu,$$

which implies the lemma. □

Lemma 2.2. *If $I_h(\nu)<\infty$, then ν is absolutely continuous with respect to m.*

Proof. By a similar argument in the proof of [12, Lemma 4.1], we obtain this lemma. Indeed, for $a>0$ and $A\in\mathcal{B}(X)$, set $u(x)=a1_A(x)+1\in\mathcal{B}_b^+(X)$. Then

$$\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F}u+\epsilon h}{u+\epsilon h}\right)d\nu=\int_X \log\left(\frac{a\alpha R_\alpha^{\mu+F}(x,A)+\alpha R_\alpha^{\mu+F}(x,X)+\epsilon h}{a1_A(x)+1+\epsilon h}\right)d\nu.$$

Define the measure ν_α as in the proof of Lemma 2.1. Put

$$c_\alpha=\int_X \alpha R_\alpha^{\mu+F}(x,X)\,d\nu(x)\quad(=\nu_\alpha(X)).$$

We see from Lemma 2.1 and Jensen's inequality that

$$\log(a\nu_\alpha(A)+c_\alpha+\epsilon h)\geq\nu(A)\log(a+1+\epsilon h)+\nu(A^c)(1+\epsilon h)-\frac{I_h(\nu)}{\alpha},$$

and by letting $\epsilon\to 0$

$$\log(a\nu_\alpha(A)+c_\alpha)\geq\nu(A)\log(a+1)-\frac{I_h(\nu)}{\alpha}.$$

Since $\log x\leq x-1$ for $x>0$, we have

$$a\nu_\alpha(A)+c_\alpha-1\geq\nu(A)\log(a+1)-\frac{I_h(\nu)}{\alpha},$$

and so

$$\nu_\alpha(A)-\nu(A)\geq\frac{-I_h(\nu)/\alpha+\nu(A)(\log(a+1)-a)+1-c_\alpha}{a}.$$

Noting that $\log(a+1)-a<0$, we have

$$\nu_\alpha(A)-\nu(A)\geq\frac{-I_h(\nu)/\alpha+(\log(a+1)-a)+1-c_\alpha}{a}$$

for all $A\in\mathcal{B}(X)$ and

$$\begin{aligned}\nu(A)-\nu_\alpha(A)&=1-c_\alpha+(\nu_\alpha(A^c)-\nu(A^c))\\&\geq\frac{-I_h(\nu)/\alpha+(\log(a+1)-a)+(1-c_\alpha)(a+1)}{a}\end{aligned}$$

for all $A \in \mathcal{B}(X)$. Therefore we can conclude that

$$\sup_{A \in \mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a+1) + I_h(\nu)/\alpha + (1 - c_\alpha)(a+1)}{a}.$$

Note that $c_\alpha \to 1$ as $\alpha \to \infty$. Then since

$$\limsup_{\alpha \to \infty} \sup_{A \in \mathcal{B}(X)} |\nu(A) - \nu_\alpha(A)| \leq \frac{a - \log(a+1)}{a}$$

and the right-hand side converges to 0 as $a \to 0$, the lemma follows. □

Proposition 2.1. *It holds that for* $\nu \in \mathcal{P}$

$$I_h(\nu) = I^{\mu+F}(\nu).$$

Proof. We follow the argument of the proof of [12, Theorem 5]. Suppose that $I_h(\nu) = l < \infty$. By Lemma 2.2, ν is absolutely continuous with respect to m. Let us denote by f its density and let $f^n = \sqrt{f} \wedge n$. Since $\log(1-x) \leq -x$ for $-\infty < x < 1$ and

$$-\infty < \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} < 1,$$

we have

$$\begin{aligned}\int_X \log\left(\frac{\alpha R_\alpha^{\mu+F} f^n + \epsilon h}{f^n + \epsilon h}\right) f\, dm &= \int_X \log\left(1 - \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h}\right) f\, dm \\ &\leq -\int_X \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} f\, dm,\end{aligned}$$

and then

$$\int_X \frac{f^n - \alpha R_\alpha^{\mu+F} f^n}{f^n + \epsilon h} f\, dm \leq I_\alpha(f \cdot m).$$

By letting $n \to \infty$ and $\epsilon \to 0$, we have

$$\int_X \sqrt{f}(\sqrt{f} - \alpha R_\alpha^{\mu+F} \sqrt{f})\, dm \leq I_\alpha(f \cdot m) \leq \frac{I_h(f \cdot m)}{\alpha},$$

which implies that $\sqrt{f} \in \mathcal{D}(\mathcal{E})$ and $\mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) \leq I_h(f \cdot m)$.

Let $\phi \in \mathcal{D}_+(\mathcal{H}^{\mu+F})$ and define the semigroup P_t^ϕ by

$$P_t^\phi f(x) = \mathbb{E}_x\left(e^{A_t^{\mu+F}} \frac{(\phi + \epsilon h)(X_t)}{(\phi + \epsilon h)(X_0)} \exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi + \epsilon h}(X_s) ds\right) f(X_t)\right).$$

Then, P_t^ϕ is $(\phi+\epsilon h)^2 m$-symmetric and satisfies $P_t^\phi 1 \le 1$. Given $\nu = f\cdot m \in \mathcal{P}$ with $\sqrt{f} \in \mathcal{D}(\mathcal{E})$, set

$$S_t^\phi \sqrt{f}(x) = \mathbb{E}_x\left(e^{A_t^{\mu+F}} \exp\left(-\int_0^t \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h}(X_s)\,ds\right)\sqrt{f}(X_t)\right).$$

Then

$$\begin{aligned}\int_X (S_t^\phi \sqrt{f})^2\,dm &= \int_X (\phi+\epsilon h)^2 \left(P_t^\phi\left(\frac{\sqrt{f}}{\phi+\epsilon h}\right)\right)^2 dm \\ &\le \int_X (\phi+\epsilon h)^2 P_t^\phi\left(\left(\frac{\sqrt{f}}{\phi+\epsilon h}\right)^2\right) dm \\ &\le \int_X (\phi+\epsilon h)^2 \left(\frac{\sqrt{f}}{\phi+\epsilon h}\right)^2 dm \\ &= \int_X f\,dm.\end{aligned}$$

Hence

$$0 \le \lim_{t\to 0} \frac{1}{t}(\sqrt{f} - S_t^\phi\sqrt{f}, \sqrt{f})_m = \mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) + \int_X \frac{\mathcal{H}^{\mu+F}\phi}{\phi+\epsilon h} f\,dm,$$

and thus $\mathcal{E}^{\mu+F}(\sqrt{f}, \sqrt{f}) \ge I_h(f\cdot m)$. □

We now obtain a generalization of Theorem 1.1 in exactly the same way as the proof of it (cf. [10], [28]):

Theorem 2.1 ([24]). *Assume* (I), (II) *and* (III). *Suppose that* $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$.
(i) *For each open set* $G \subset \mathcal{P}$

$$\liminf_{t\to\infty} \frac{1}{t} \log \mathbb{E}_x(e^{A_t^{\mu+F}};\ L_t \in G, t<\zeta) \ge -\inf_{\nu\in G} I^{\mu+F}(\nu).$$

(ii) *For each closed set* $K \subset \mathcal{P}$

$$\limsup_{t\to\infty} \frac{1}{t} \log \sup_{x\in X} \mathbb{E}_x(e^{A_t^{\mu+F}};\ L_t \in K,\ t<\zeta) \le -\inf_{\nu\in K} I^{\mu+F}(\nu).$$

3. The existence of ground states

We first recall an inequality ([19]): for $\mu \in \mathcal{K}$,

$$\int_X \tilde{u}^2\,d\mu \le \|R_\alpha \mu\|_\infty(\mathcal{E}(u,u) + \alpha(u,u)_m), \quad u \in \mathcal{D}(\mathcal{E}). \tag{3.1}$$

Let $\lambda_2(\mu + F)$ be the bottom of the spectrum of $\mathcal{H}^{\mu+F}$:

$$\lambda_2(\mu + F) = \inf\{\mathcal{E}^{\mu+F}(u, u) \colon u \in \mathcal{D}(\mathcal{E}),\ \|u\|_2 = 1\}. \tag{3.2}$$

Proposition 3.1. *Assume* (I), (II) *and* (III). *There exists a unique ground state* $\phi_0 \in \mathcal{D}(\mathcal{E})$: $\lambda_2(\mu + F) = \mathcal{E}^{\mu+F}(\phi_0, \phi_0)$.

Proof. Let $\{u_n\}$ be a minimizing sequence of the right-hand side of (3.2), i.e., $\|u_n\|_2 = 1$ and $\lambda_2(\mu + F) = \lim_{n\to\infty} \mathcal{E}^{\mu+F}(u_n, u_n)$. Put $\mu' = |\mu| + |\mu_{F_1}|$. Since $\mathcal{E}(u_n, u_n) \le c \cdot \mathcal{E}_F(u_n, u_n)$ $(c = \exp(-\|F\|_\infty))$ and $\int_X u_n^2 d\mu' \le \|R_\alpha \mu'\|_\infty \cdot (\mathcal{E}(u_n, u_n) + \alpha)$,

$$\begin{aligned}\mathcal{E}^{\mu+F}(u_n, u_n) &= \mathcal{E}_F(u, u) - \int_X u_n^2\, d\mu' \\ &\ge \frac{1}{c}\mathcal{E}(u_n, u_n) - \|R_\alpha \mu'\|_\infty (\mathcal{E}(u_n, u_n) + \alpha \\ &= \left(\frac{1}{c} - \|R_\alpha \mu'\|_\infty\right)\mathcal{E}(u_n, u_n) - \alpha \|R_\alpha \mu'\|_\infty.\end{aligned}$$

Taking α large enough so that $c\|R_\alpha \mu'\|_\infty < 1$ on account of (2.3), we have

$$\sup_n \mathcal{E}(u_n, u_n) \le \frac{c(\sup_n \mathcal{E}^{\mu+F}(u_n, u_n) + \alpha \|R_\alpha \mu'\|_\infty)}{1 - c\|R_\alpha \mu'\|_\infty} < \infty.$$

We see from the assumption (III) that for any $\epsilon > 0$ there exists a compact set K such that

$$\sup_n \int_{K^c} u_n^2\, dm \le \|R_1 1_{K^c}\|_\infty \cdot \left(\sup_n \mathcal{E}(u_n, u_n) + 1\right) < \epsilon.$$

As a result, the subset $\{u_n^2 \cdot m\}$ of $\mathcal{P}$ is tight. Hence there exists a subsequence $u_{n_k}^2 \cdot m$ which converges to a probability measure ν weakly. Since the function $I^{\mu+F}$ is lower semi-continuous by Proposition 2.1,

$$I^{\mu+F}(\nu) \le \liminf_{k\to\infty} I^{\mu+F}(u_{n_k}^2 \cdot m) = \liminf_{k\to\infty} \mathcal{E}^{\mu+F}(u_{n_k}, u_{n_k}) < \infty.$$

Therefore ν can be written as $\nu = \phi_0^2 m$, $\phi_0 \in \mathcal{D}(\mathcal{E})$ by Proposition 2.1 and $\lambda_2(\mu + F) = \mathcal{E}^{\mu+F}(\phi_0, \phi_0)$, that is, ϕ_0 is the ground state. The uniqueness of the ground state follows from the irreducibility (I) (e.g. [9, Proposition 1.4.3]). □

We also know from the proof above that the level set $\{\nu \in \mathcal{P} \colon I^{\nu+F}(\nu) \le l\}$ is compact.

4. Large deviations from ground states

Given $\omega \in \Omega$ with $0 < t < \zeta(\omega)$, we define the occupation distribution $L_t(\omega) \in \mathcal{P}$ by

$$L_t(\omega)(A) = \frac{1}{t}\int_0^t 1_A(X_s(\omega))\,ds$$

for a Borel set A of X, where 1_A is the indicator function of the set A.

Define the probability measure $Q_{x,t}$ on $\mathcal{P}$ by

$$Q_{x,t}(B) = \frac{\mathbb{E}_x(e^{A_t^{\mu+F}};\ L_t \in B,\ t < \zeta)}{\mathbb{E}_x(e^{A_t^{\mu+F}};\ t < \zeta)}, \quad B \in \mathcal{B}(\mathcal{P}). \tag{4.1}$$

We define the function J on $\mathcal{P}$ by

$$J(\nu) = I^{\mu+F}(\nu) - \lambda_2(\mu + F). \tag{4.2}$$

We then have the next lemma by Proposition 2.1 and Proposition 3.1.

Lemma 4.1. *The function J satisfies*:
(i) $0 \le J(\nu) \le \infty$.
(ii) *J is lower semicontinuous.*
(iii) *For each $l < \infty$, the set $\{\nu \in \mathcal{P} \colon J(\nu) \le l\}$ is compact.*
(iv) *$J(\phi_0^2 \cdot m) = 0$ and $J(\nu) > 0$ for $\nu \ne \phi_0^2 \cdot m$.*

REMARK 4.1. Let $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ the bilinear form on $L^2(X;\ \phi_0^2 m)$ defined by

$$\begin{cases} \mathcal{E}^{\phi_0}(u, v) = \mathcal{E}^{\mu+F}(u\phi_0, u\phi_0) - \lambda_2(\mu + F)(u\phi_0, u\phi_0)_m, \\ \mathcal{D}(\mathcal{E}^{\phi_0}) = \{u \in L^2(X;\ \phi_0^2 m) \colon u\phi_0 \in \mathcal{D}(\mathcal{E})\}. \end{cases}$$

We then see that $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$ is a Dirichlet form and $\mathcal{E}^{\phi_0}$ is expressed by

$$\mathcal{E}^{\phi_0}(u, v) = \int_X \phi_0^2 d\mu^c_{\langle u,v\rangle} + \int_{X\times X\setminus\triangle} (u(x) - u(y))(v(x) - v(y))\phi_0(x)\phi_0(y)J(dx, dy).$$

Here $\mu^c_{\langle u,v\rangle}$ is the local part of energy measure ([6]). We then see that

$$J(\nu) = I_{\mathcal{E}^{\phi_0}}(\nu),$$

where $I_{\mathcal{E}^{\phi_0}}$ is defined by

$$I_{\mathcal{E}^{\phi_0}}(\nu) = \begin{cases} \mathcal{E}^{\phi_0}(\sqrt{f}, \sqrt{f}) & \text{if } \quad \nu = f \cdot \phi_0^2 m,\ \sqrt{f} \in \mathcal{D}(\mathcal{E}^{\phi_0}), \\ \infty & \text{otherwise.} \end{cases} \tag{4.3}$$

We then have the main theorem:

Theorem 4.1. *Assume* (I), (II) *and* (III). *Let* $\mu + F \in \mathcal{K}_\infty + \mathcal{A}_2$. *Let* $\{Q_{x,t}\}_{t>0}$ *be a family of probability measures defined in* (4.1). *Then* $\{Q_{x,t}\}_{t>0}$ *obeys a large deviation principle with rate function* J:

(1) *For each open set* $G \subset \mathcal{P}$

$$\liminf_{t\to\infty} \frac{1}{t} \log Q_{x,t}(G) \geq -\inf_{\nu\in G} J(\nu).$$

(2) *For each closed set* $K \subset \mathcal{P}$

$$\limsup_{t\to\infty} \frac{1}{t} \log Q_{x,t}(K) \leq -\inf_{\nu\in K} J(\nu).$$

Corollary 4.1. *The measure* $Q_{x,t}$ *converges to* $\delta_{\phi_0^2 \cdot m}$ *weakly.*

Proof. If a closed set K does not contain $\phi_0^2 \cdot m$, then $\inf_{x\in K} J(x) > 0$ by Lemma 4.1 (iv). Hence Theorem 4.1 (ii) says that $\lim_{t\to\infty} Q_{x,t}(K) = 0$ and $\lim_{t\to\infty} Q_{x,t}(K^c) = 1$. For a positive constant δ and a bounded continuous function f on the set of $\mathcal{P}$, define the closed set $K \subset \mathcal{P}$ by $K = \{\nu \in \mathcal{P} \colon |f(\nu) - f(\phi_0^2 \cdot m)| \geq \delta\}$. Then we have

$$\begin{aligned} &\left| \int_{\mathcal{P}} f(\nu) Q_{x,t}(d\nu) - f(\phi_0^2 \cdot m) \right| \leq \int_{\mathcal{P}} |f(\nu) - f(\phi_0^2 \cdot m)| Q_{x,t}(d\nu) \\ &= \int_K |f(\nu) - f(\phi_0^2 \cdot m)| Q_{x,t}(d\nu) + \int_{K^c} |f(\nu) - f(\phi_0^2 \cdot m)| Q_{x,t}(d\nu) \\ &\leq \delta Q_{x,t}(K^c) + 2\|f\|_\infty Q_{x,t}(K) \to \delta \end{aligned}$$

as $t \to \infty$. Since δ is arbitrary, the weak convergence follows. □

On account of Corollary 4.1, we can regard Theorem 4.1 as a genuine large deviation principle from the ground state.

5. Quasistationary distribution

In this section, we consider the existence of quasi-stationary distributions as an application of the existence of ground states. We continue with the setting of the preceding section.

Define the semigroup $\{p_t^{\phi_0}\}_{t\geq 0}$ on $L^2(X; \phi_0^2 m)$ generated by $(\mathcal{E}^{\phi_0}, \mathcal{D}(\mathcal{E}^{\phi_0}))$, that is

$$p_t^{\phi_0} f(x) = e^{\lambda_2(\mu+F)t} \frac{1}{\phi_0(x)} \mathbb{E}_x\left(e^{A_t^{\mu+F}} \phi_0(X_t) f(X_t)\right). \tag{5.1}$$

Let $\mathbf{M}^{\phi_0}=(\Omega, X_t, \mathbb{P}_x^{\phi_0})$ be the $\phi_0^2 m$-symmetric Markov process generated by the Markov semigroup $p_t^{\phi_0}$ in (5.1).

Set

$$\mathcal{P}_0=\left\{\nu\in\mathcal{P}\colon \int_X\sqrt{p_1^{\mu+F}(x,x)}\,d\nu(x)<\infty,\ \int_X\phi_0(x)\,d\nu(x)<\infty\right\}.$$

We then have

Theorem 5.1. *Assume that $m(X)<\infty$. Then for $\nu\in\mathcal{P}_0$ and $B\in\mathcal{B}(X)$*

$$\lim_{t\to\infty}e^{\lambda_2(\mu+F)t}\mathbb{E}_\nu\left(e^{A_t^{\mu+F}};\ X_t\in B\right)=\int_X\phi_0\,d\nu\int_B\phi_0\,dm.$$

Proof. Note that

$$e^{\lambda_2(\mu+F)t}\mathbb{E}_\nu\left(e^{A_t^{\mu+F}};\ X_t\in B\right)=\int_X\phi_0(x)\mathbb{E}_x^{\phi_0}\left(\frac{1_B}{\phi_0}(X_t)\right)d\nu(x).$$

Let $\{E_\lambda, 0\le\lambda<\infty\}$ be the spectral family of $(\mathcal{E}^{\phi_0},\mathcal{F}^{\phi_0})$. Then $\lim_{t\to\infty}p_t^{\phi_0}f=E_0 f$ in $L^2(X;\phi_0^2 m)$. Since $\mathcal{E}^{\phi_0}(E_0 f, E_0 f)=0$, $E_0 f$ equals $\int_X f\phi_0^2 dm$, m-a.e. by the irreducibility of $(\mathcal{E}^{\phi_0},\mathcal{F}^{\phi_0})$ (cf. [7, Theorem 5.2.13]). Note that $p_t^{\phi_0}(x,\ \cdot\ )\in L^2(X;\phi_0^2 m)$ because $\int_X p_t^{\phi_0}(x,y)^2\phi_0^2(y)\,dm(y)=p_{2t}^{\phi_0}(x,x)<\infty$. Put $c=\int_B\phi_0\,dm$. We then have

$$\begin{aligned}&\left|\int_X\phi_0(x)\mathbb{E}_x^{\phi_0}\left(\frac{1_B}{\phi_0}(X_t)\right)d\nu(x)-\int_X\phi_0\,d\nu\int_B\phi_0\,dm\right|\\ &=\left|\int_X\phi_0(x)\left(\int_X p_{1/2}^{\phi_0}(x,y)\left(\mathbb{E}_y^{\phi_0}\left(\frac{1_B}{\phi_0}(X_{t-1/2})\right)-c\right)\phi_0(y)^2\,dm(y)\right)d\nu(x)\right|.\end{aligned}\tag{5.2}$$

The right-hand side is dominated by

$$\int_X\phi_0(x)\sqrt{\int_X p_{1/2}^{\phi_0}(x,y)^2\phi_0^2(y)\,dm(y)}\,d\nu(x)\cdot\sqrt{\int_X\left(\mathbb{E}_y^{\phi_0}\left(\frac{1_B}{\phi_0}(X_{t-1/2})\right)-c\right)^2\phi_0^2(y)\,dm(y)}.$$

Since

$$p_t^{\phi_0}(x,y)=e^{\lambda_2(\mu+F)t}\frac{p_t^{\mu+F}(x,y)}{\phi_0(x)\phi_0(y)},$$

the first factor is equal to

$$\int_X\phi_0(x)\sqrt{p_1^{\phi_0}(x,x)}\,d\nu(x)=e^{(1/2)\lambda_2(\mu+F)}\int_X\sqrt{p_1^{\mu+F}(x,x)}\,d\nu(x)$$

and is finite by the assumption that $\nu\in\mathcal{P}_0$. Hence the right-hand side of (5.2) converges to zero as $t\to\infty$ because $1_B/\phi_0\in L^2(X;\phi_0^2 m)$. $\square$

Let η and $R_{\nu,t}$ be probability measures on X defined by

$$\eta(B) = \frac{\int_B \phi_0(x)\,dm(x)}{\int_X \phi_0(x)\,dm(x)}, \quad R_{\nu,t}(B) = \frac{\mathbb{E}_\nu\big(e^{A_t^{\mu+F}};\ X_t \in B\big)}{\mathbb{E}_\nu\big(e^{A_t^{\mu+F}};\ t < \zeta\big)} \quad \text{for} \quad B \in \mathcal{B}(X). \tag{5.3}$$

Corollary 5.1. *For $\nu \in \mathcal{P}_0$ and $B \in \mathcal{B}(X)$*

$$\lim_{t\to\infty} R_{\nu,t}(B) = \eta(B). \tag{5.4}$$

Note that the Dirac measure δ_x belongs to $\mathcal{P}_0$ and so the distribution $R_{\delta_x,t}$ converges to η for all $x \in X$. Hence Corollary 5.1 says that the semigroup $\{p_t^{\mu+F}\}_{t\geq 0}$ is *conditionally ergodic* and η is a *quasi-stationary distribution* of the semigroup $\{p_t^{\mu+F}\}_{t\geq 0}$: for any $t > 0$

$$R_{\eta,t} = \eta \tag{5.5}$$

(e.g. [16]). If the semigroup $\{p_t^{\mu+F}\}_{t\geq 0}$ is ultracontractive, $p_t^{\mu+F}(x, y) \leq c_t$, then $p_t^{\mu+F}(x, x)$ and $\phi_0(x)$ are bounded and $\mathcal{P}_0$ equals $\mathcal{P}$. Consequently, for any $\nu \in \mathcal{P}$, the distribution $R_{\nu,t}$ converges to η.

When the measure m is not finite, we assume the *intrinsic ultracontractivity* of $\{p_t^{\mu+F}\}_{t\geq 0}$, that is,

$$p_t^{\mu+F}(x, y) \leq C_t\phi_0(x)\phi_0(y). \tag{5.6}$$

In [16], they proved that for a (not necessary symmetric) Markov process, the intrinsic ultracontractivity is a sufficient condition for the measure η being a unique quasi-stationary distribution, and the equation (5.4) holds for any initial distribution. We would like to give another proof of this fact by using the next theorem due to Fukushima [13].

Theorem 5.2. *Assume that $m(X) < \infty$ and $\mathbf{M}$ is conservative, $p_t 1 = 1$, $t > 0$. Then for $f \in L^1(X; m)$,*

$$\lim_{t\to\infty} p_t f(x) = \frac{1}{m(X)} \int_X f(x)\,dm(x), \quad m\text{-a.e. and in} \quad L^1(X; m).$$

Note that $\mathbf{M}^{\phi_0}$ satisfies the assumptions in Theorem 5.2.

Theorem 5.3. *Assume that $\{p_t^{\mu+F}\}_{t\geq 0}$ is intrinsically ultracontractive. Then for any $\nu \in \mathcal{P}$ and any $B \in \mathcal{B}(X)$*

$$\lim_{t\to\infty} e^{\lambda_2(\mu+F)t}\mathbb{E}_\nu\big(e^{A_t^{\mu+F}};\ X_t \in B\big) = \int_X \phi_0\,d\nu \int_B \phi_0\,dm.$$

Consequently, the equation (5.4) *follows.*

Proof. First note that the upper bound (5.6) implies the lower bound ([9, Theorem 4.2.5]):

$$c_t\phi_0(x)\phi_0(y) \le p_t^{\mu+F}(x, y). \tag{5.7}$$

As a result,

$$\sup_{x\in X}\phi_0(x)\int_X \phi_0(y)\,dm(y) \le \frac{1}{c_t}\|p_t^{\mu+F}1\|_\infty < \infty.$$

Hence ϕ_0 belongs to $L^1(X; m)\cap L^\infty(X; m)$ and $1_B/\phi_0 \in L^1(X; \phi_0^2 m)$. Applying Theorem 5.2 to $\mathbf{M}^{\phi_0}$, we have

$$\mathbb{E}_y^{\phi_0}\left(\frac{1_B}{\phi_0}(X_t)\right) \to \int_B \phi_0\,dm, \quad m\text{-a.e. } y \quad \text{and} \quad L^1(X; \phi_0^2 m)$$

as $t\to\infty$. Since $p_{1/2}^{\phi_0}(x, \cdot)$ is bounded by the ultracontractivity, it follows from the equation (5.2) that

$$\lim_{t\to\infty}\int_X \phi_0(x)\mathbb{E}_x^{\phi_0}\left(\frac{1_B}{\phi_0}(X_t)\right)d\nu(x) = \int_X \phi_0\,d\nu\int_B \phi_0\,dm. \qquad \square$$

We finally consider the exponential integrability of hitting times of compact sets. Let $K\subset X$ be a compact set and D the complement of K, $D = X\setminus K$. We define the part (or absorbing) process X^D on D by

$$X_t^D = \begin{cases} X_t & t<\tau_D, \\ \Delta & t\ge\tau_D, \end{cases} \quad \tau_D = \inf\{t\ge 0\colon X_t\notin D\}.$$

Define the regular Dirichlet form $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$ on D by

$$\begin{cases} \mathcal{E}^D = \mathcal{E}, \\ \mathcal{D}(\mathcal{E}^D) = \{u\in\mathcal{D}(\mathcal{E})\colon u=0 \text{ q.e. on } K\}. \end{cases}$$

By [14, Theorem 4.4.3] the part process X^D is regarded as a Hunt process generated by $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$. We see from [4, Theorem 4.2] that m is in $\mathcal{K}_\infty$. We write $\mathcal{K}_\infty(R_1)$ for $\mathcal{K}_\infty$ to show the dependence. Let R_1^D be the 1-resolvent of X^D. The restriction m^D of m on D is in $\mathcal{K}_\infty(R_1^D)$. Indeed, let a compact set $\tilde{K}$ and a positive constant δ in the definition of $\mathcal{K}_\infty$ (Definition 2.1). We can suppose $K\subset\tilde{K}$. Let G be a relatively compact open set such that $K\subset G\subset\bar{G}\subset\tilde{K}$ and $m(G\setminus K)<\delta$. Then $\tilde{K}\cap G^c$ is a compact subset of D and

$$R_1^D 1_{(\tilde{K}\cap G^c)^c} = R_1^D 1_{\tilde{K}^c\cup(G\setminus K)} \le R_1 1_{\tilde{K}^c} + R_1 1_{G\setminus K} \le 2\epsilon.$$

Moreover, $R_1^D 1_B \le R_1 1_B$ for any Borel set $B \subset \tilde{K} \cap G^c$. Hence we have $m^D \in \mathcal{K}_\infty(R_1^D)$.

If X^D satisfies the irreducibility (I), it follows from [4, Theorem 4.1] that

$$\sup_{x\in D} \mathbb{E}_x(e^{\lambda\tau_D}) < \infty \Longleftrightarrow \lambda < \lambda^D,$$

where λ^D is the bottom of the spectrum of $(\mathcal{E}^D, \mathcal{D}(\mathcal{E}^D))$.

Noting that by (3.1)

$$1 \le \|R_1 1_D\|_\infty(\lambda^D + 1),$$

we see from (III) that

$$\lambda_D \uparrow \infty \quad \text{as} \quad K \uparrow X. \tag{5.8}$$

We can conclude that if for any compact set K, the part process X^D $(D = X \setminus K)$ is irreducible, then for any $\lambda > 0$ there exists a compact set K such that

$$\sup_{x\in X} \mathbb{E}_x\left(e^{\lambda\tau_D}\right) < \infty. \tag{5.9}$$

If $\mathbf{M}$ is conservative, τ_D equals the first hitting time σ_K of K, $\sigma_K = \inf\{t > 0 : X_t \in K\}$. Then the property (5.9) is called the *uniform hyper-exponential recurrence* in [30].

EXAMPLE 5.1 (One-dimensional diffusion processes). Let us consider a one-dimensional diffusion process $\mathbf{M} = (X_t, \mathbb{P}_x, \zeta)$ on an open interval $I = (r_1, r_2)$ such that $\mathbb{P}_x(X_{\zeta-} = r_1 \text{ or } r_2,\ \zeta < \infty) = \mathbb{P}_x(\zeta < \infty)$, $x \in I$, and $\mathbb{P}_a(\sigma_b < \infty) > 0$ for any $a, b \in I$. The diffusion $\mathbf{M}$ is symmetric with respect to its canonical measure m and it satisfies (I) and (II). The boundary point r_i of I is classified into four classes: *regular boundary*, *exit boundary*, *entrance boundary and natural boundary* ([15, Chapter 5]):

(a) If r_2 is a regular or exit boundary, then $\lim_{x\to r_2} R_1 1(x) = 0$.

(b) If r_2 is an entrance boundary, then $\lim_{r\to r_2} \sup_{x\in(r_1,r_2)} R_1 1_{(r,r_2)}(x) = 0$.

(c) r_2 is a natural boundary, then $\lim_{x\to r_2} R_1 1_{(r,r_2)}(x) = 1$ and thus $\sup_{x\in(r_1,r_2)} R_1 1_{(r,r_2)}(x) = 1$.

Therefore, (III) is satisfied if and only if no natural boundaries are present. As a corollary of the equation (5.8), If r_2 is entrance, for any $\lambda > 0$ there exists $r_1 < r < r_2$ such that

$$\sup_{x>r} \mathbb{E}_x(\exp(\lambda\sigma_r)) < \infty,$$

where σ_r is the first hitting time of $\{r\}$. The statement above implies a uniqueness of quasi-stationary distributions ([3]).

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