Nakagawa, Y.
Osaka J. Math.
48 (2011), 497–513

ON GENERALIZED KÄHLER–RICCI SOLITONS

Dedicated to Professor Toshiki Mabuchi on his sixtieth birthday

YASUHIRO NAKAGAWA

(Received March 1, 2007, revised January 25, 2010)

Abstract

By definition, Kähler–Ricci solitons are defined on Fano manifolds. In this note, we shall generalize the notion of Kähler–Ricci solitons to the case of general polarized manifolds from the view point of K-energy, which are called "generalized Kähler–Ricci solitons". Moreover, "generalized Kähler–Ricci solitons" are also one of generalizations of constant scalar curvature Kähler metrics. Furthermore, we shall give a non-trivial example of a "generalized Kähler–Ricci soliton".

1. Introduction

The K-energy was originally introduced by Mabuchi for studying Einstein–Kähler metrics on Fano manifolds in [2] and [16] and its critical points are Einstein–Kähler metrics. The K-energy is easily generalized to the case of constant scalar curvature Kähler metrics ([16]). Moreover, in [11], Guan generalized the K-energy to the case of extremal Kähler metrics. In [21], Tian also generalized the K-energy to the case of Kähler–Ricci solitons. In Section 3, we shall modify Tian's definition of the K-energy associated to Kähler–Ricci solitons for some reason (see Section 3 for more details).

In the rest of this section, we shall introduce some notation and terminology. Let (M, L) be an n-dimensional *polarized manifold*, that is, M is a compact connected n-dimensional complex manifold and L an ample line bundle over M. If we can choose the anti-canonical line bundle K_M^{-1} of M as L, we call M a *Fano manifold*. For (M, L), we put

$$\beta_L := \frac{(c_1(M) \cup c_1(L)^{n-1})([M])}{c_1(L)^n([M])} \in \mathbb{Q}.$$

Since L is ample, we have a Kähler metric g whose Kähler form

$$\omega_g = \sqrt{-1} \sum_{i,j=1}^{n} g_{i\bar{j}}\, dz^i \wedge d\overline{z^j}$$

2000 Mathematics Subject Classification. Primary 32Q20; Secondary 53C55, 53C25.
Partly supported by JSPS Grant-in-Aid "Kiban-Kenkyu C No. 20540069".

represents $2\pi c_1(L) \in H^2(M;\mathbb{R})$, where $(z^1, z^2, \ldots, z^n)$ is a holomorphic local coordinate system for M. Moreover, by

$$\mathrm{Ric}_g = \mathrm{Ric}(\omega_g) = \sqrt{-1}\sum_{i,j=1}^{n} R_{i\bar{j}}\, dz^i \wedge d\overline{z^j} := \sqrt{-1}\,\overline{\partial}\partial \log\det(g_{\alpha\bar{\beta}}),$$

we denote the *Ricci form* of g. By means of the harmonic integration theory, there exists a real-valued C^∞ function $f_g \in C^\infty(M)_{\mathbb{R}}$ on M such that

$$s_g - n\beta_L = \Box_g\, f_g := \sum_{i,j=1}^{n} g^{\bar{j}i} \frac{\partial^2 f_g}{\partial z^i \partial \overline{z^j}}, \tag{1.1}$$

where $(g^{\bar{j}i})_{i,j=1,\ldots,n}$ is the inverse matrix of $(g_{i\bar{j}})_{i,j=1,\ldots,n}$ and

$$s_g = s(\omega_g) := \sum_{i,j=1}^{n} g^{\bar{j}i} R_{i\bar{j}}$$

the *scalar curvature* of g. If s_g is constant, then g is called a *constant scalar curvature Kähler metric*; in this case, $s_g = n\beta_L$. If there exists a constant $c \in \mathbb{R}$ such that $\mathrm{Ric}_g = c\omega_g$, we call g an *Einstein–Kähler metric*. If $c = 0$, then g is called a *Ricci-flat Kähler metric*. If $c \neq 0$, then $1/c = 1/\beta_L \in \mathbb{Z}$ and $L = K_M^{-1/c}$. When $L = K_M^k$ $(k \in \mathbb{Z} \setminus \{0\})$, g is an Einstein–Kähler metric with $c = -1/k$ if and only if g is a constant scalar curvature Kähler metric.

In general, for a complex-valued C^∞ function $\varphi \in C^\infty(M)_{\mathbb{C}}$ on M, we define a complex-valued vector field on M by

$$\mathrm{grad}_g\, \varphi := \frac{1}{\sqrt{-1}} \sum_{i,j=1}^{n} g^{\bar{j}i} \frac{\partial \varphi}{\partial \overline{z^j}} \frac{\partial}{\partial z^i}.$$

If $\mathrm{grad}_g\, s_g$ is a holomorphic vector field on M, then we call g an *extremal Kähler metric*, which was introduced by Calabi in [4]. By definition, constant scalar curvature Kähler metrics are extremal Kähler metrics.

When M is a Fano manifold and $L = K_M^{-1}$, if there exists a holomorphic vector field $X \in \mathfrak{X}_M := H^0(M; \mathcal{O}(T^{1,0}M))$ such that

$$\mathrm{Ric}_g - \omega_g = L_X \omega_g,$$

then g or the pair (g, X) is called a *Kähler–Ricci soliton*, where $\mathfrak{X}_M$ is the Lie algebra of holomorphic vector fields on M and $L_X\omega_g$ the Lie differentiation of ω_g with respect to X. If $X = 0$, then a Kähler–Ricci soliton g is nothing but an Einstein–Kähler metric with $c = 1$.

2. Preliminaries: Bott–Chern forms and K-energy

In this section, we shall recall some basic notions and facts concerning the K-energy according to [16], [20] and [21].

For an n-dimensional polarized manifold (M, L), we put

$$\mathcal{M}(M, L) := \{\omega\colon \text{Kähler form on } M \text{ such that } \omega \in 2\pi c_1(L)\}.$$

We fix a Kähler form $\omega_0 \in \mathcal{M}(M, L)$. Then for each $\omega \in \mathcal{M}(M, L)$, there exists $\varphi \in C^\infty(M)_{\mathbb{R}}$ such that

$$\omega = \omega_0(\varphi) := \omega_0 + \sqrt{-1}\,\partial\bar{\partial}\varphi.$$

We define a functional $\mu_L\colon \mathcal{M}(M, L) \to \mathbb{R}$ on $\mathcal{M}(M, L)$ by

$$\mu_L(\omega_0(\varphi)) := -\int_0^1 dt \int_M \dot{\varphi}_t(s(\omega_0(\varphi_t)) - n\beta_L)\left(\frac{\omega_0(\varphi_t)}{2\pi}\right)^n,$$

where $\{\varphi_t\}_{0\leqq t\leqq 1}$ is a path of real-valued C^∞-functions on M from $\varphi_0 \equiv 0$ to $\varphi_1 = \varphi$ with $\omega_0(\varphi_t) \in \mathcal{M}(M, L)$ $(0 \leqq t \leqq 1)$ and $\dot{\varphi}_t := \partial\varphi_t/\partial t$. We call μ_L the *K-energy* of (M, L) (associated to constant scalar curvature Kähler metrics). For μ_L, the following fact is well-known:

Fact 2.1 (Mabuchi [16, Theorems (2.4) and (3.2)]). (1) *μ_L is independent of the choice of a path $\{\varphi_t\}_{0\leqq t\leqq 1}$ and therefore well-defined*;
(2) *An element ω_g of $\mathcal{M}(M, L)$ is a critical point of μ_L if and only if g is a constant scalar curvature Kähler metric.*

We assume that, for a complex Lie subgroup G of the holomorphic automorphism group $\mathrm{Aut}(M)$ of M, L is G-equivariant. Then for an element Y of the Lie algebra $\mathfrak{g} := \mathrm{Lie}(G)$ $(\subset \mathfrak{X}_M = \mathrm{Lie}(\mathrm{Aut}(M)))$ of G, the following formula was also proved ([16, Theorem (5.3)]):

$$\left.\frac{d}{dt}\right|_{t=0} \mu_L(\exp tY_{\mathbb{R}}^*\omega_0) = 2\,\Re\mathrm{e}(\sqrt{-1}F_M^L(Y)),$$

where $Y_{\mathbb{R}} := Y + \overline{Y}$ and $\Re\mathrm{e}(\alpha)$ is the real part of a complex number $\alpha \in \mathbb{C}$. Here $F_M^L\colon \mathfrak{X}_M \to \mathbb{C}$ is the *Bando–Calabi–Futaki character* of (M, L), which was introduced as an obstruction to the existence of constant scalar curvature Kähler metrics in $\mathcal{M}(M, L)$ by Bando ([1]), Calabi ([5]) and Futaki ([6], [7]) and defined by

$$F_M^L(V) := \frac{1}{\sqrt{-1}}\int_M (Vf_g)\left(\frac{\omega_g}{2\pi}\right)^n, \quad V \in \mathfrak{X}_M.$$

It is well-known that F_M^L is independent of the choice of $\omega_g \in \mathcal{M}(M, L)$.

Next, according to Tian ([20]), we shall give an interpretation of the K-energy in terms of Bott–Chern forms. For an Hermitian metric h of L, we denote the Hermitian connection of h by ∇^h and its curvature by $\Theta(h)$, that is, $\Theta(h) := \overline{\partial}\partial \log h$. For a polynomial (or a power series) ϕ of one-variable and two Hermitian metrics h_0 and h_1 of L, we put

$$\begin{aligned}\mathrm{BC}^{\phi}(L; h_0, h_1) &:= \int_0^1 \phi'(\Theta(h_t))\dot{h}_t h_t^{-1}\, dt \\ &\in \left(\bigoplus_k A^{k,k}(M)\right) \Big/ \operatorname{Im}(\partial) + \operatorname{Im}(\overline{\partial}),\end{aligned}$$

where $A^{p,q}(M)$ $(0 \leqq p, q \leqq n)$ is the space of (p, q)-forms on M, $\{h_t\}_{0\leqq t\leqq 1}$ a path of Hermitian metrics of L from h_0 to h_1, $\dot{h}_t := \partial h_t/\partial t$ and ϕ' the differentiation of ϕ. Then $\mathrm{BC}^{\phi}(L; h_0, h_1)$ is independent of the choice of a path $\{h_t\}_{0\leqq t\leqq 1}$, hence well-defined and called a *Bott–Chern form* of $(L; h_0, h_1)$ associated to ϕ (see [20] for more details). Moreover we put

$$\mathcal{BC}^{\phi}(L; h_0, h_1) := \int_M \mathrm{BC}^{\phi}(L; h_0, h_1).$$

Now we assume that, for Hermitian metrics h_0 and h_1 of L, both $\sqrt{-1}\Theta(h_0)$ and $\sqrt{-1}\Theta(h_1)$ are Kähler forms on M and we denote their associated Kähler metrics by $g(h_0)$ and $g(h_1)$, respectively. Then we have

$$\begin{aligned}&\sum_{j=0}^{n}(-1)^j \binom{n}{j} \mathcal{BC}^{c_1^{n+1}}(K_M^{-1} \otimes L^{n-2j}; \det g(h_0)\cdot h_0^{n-2j}, \det g(h_1)\cdot h_1^{n-2j}) \\ &- 2^n n!\, n\beta_L \mathcal{BC}^{c_1^{n+1}}(L; h_0, h_1) \\ &= 2^n(n+1)!\, \frac{\sqrt{-1}}{2\pi}\mu_L(\omega_0(\varphi)),\end{aligned}$$

where $h_1 = e^{-\varphi}h_0$, $\omega_0 = \omega_{g(h_0)} = \sqrt{-1}\Theta(h_0)$, $\det g(h_0)\cdot h_0^{n-2j}$ and $\det g(h_1)\cdot h_1^{n-2j}$ are the induced Hermitian metrics of $K_M^{-1} \otimes L^{n-2j}$ and $c_1^k(T) = ((\sqrt{-1}/2\pi)T)^k$.

In the rest of this section, we shall recall the equivariant Bott–Chern forms according to [21]. Let (M, L) be an n-dimensional polarized manifold. In general, we assume that L is G-equivariant for a complex Lie subgroup G of $\mathrm{Aut}(M)$. Then, for each holomorphic vector field $V \in \mathfrak{g} \subset \mathfrak{X}_M$ on M and $p = 0, 1, \ldots, 2n$, we can define the *holomorphic action* (see [3])

$$\Lambda_V^{L,p}\colon A^p(M; L) \to A^p(M; L),$$

of V on L, that is, $\Lambda_V^{L,p}$ satisfies the following properties:

(i) $\Lambda_V^{L,p}$ is a $\mathbb{C}$-linear map;
(ii) For all $\psi \in A^p(M)$ and $s \in A^0(M; L)$,

$$\Lambda_V^{L,p}(\psi s) = (L_V \psi)s + \psi \Lambda_V^{L,0} s,$$

where $L_V \psi$ is the Lie differentiation of ψ with respect to V;
(iii) $\Lambda_V^{L,p}$ commutes with $\overline{\partial}$, that is, $\overline{\partial} \circ \Lambda_V^{L,p} = \Lambda_V^{L,p+1} \circ \overline{\partial}$, for $p = 0, 1, \ldots, 2n-1$.
Here, for $p = 0, 1, \ldots, 2n$, we denote by $A^p(M)$ and $A^p(M; L)$ the space of C^∞ complex-valued p-forms and L-valued p-forms on M, respectively. For an Hermitian metric h of L, we put

$$\mathcal{L}_V^{(L,h)} := \nabla_V^h - \Lambda_V^{L,0} \in C^\infty(M)_{\mathbb{C}}.$$

Then $\mathcal{L}_V^{(L,h)}$ satisfies $\overline{\partial}\mathcal{L}_V^{(L,h)} = -i(V)\Theta(h)$, where $i(V)$ is the interior product with respect to V. For $V \in \mathfrak{g} \subset \mathfrak{X}_M$ and a polynomial (or a power series) ϕ of one-variable, we put

$$\mathcal{C}_L^\phi(V) := \int_M \phi(\Theta(h) + \mathcal{L}_V^{(L,h)}).$$

Then $\mathcal{C}_L^\phi(V)$ is independent of the choice of an Hermitian metric h of L and hence well-defined. By $\phi^G(L) \in H_G^{2\deg\phi}(M)$, we denote the G-equivariant characteristic class of L associated to ϕ. Then we have

$$\mathcal{C}_L^\phi(V) = (\varpi_*^G(\phi^G(L)))(V), \quad V \in \mathfrak{g},$$

where $\varpi_*^G \colon H_G^*(M) \to H_G^{*-2n}(\{*\})$ is the G-equivariant Gysin map induced by the trivial G-equivariant map ϖ of M to a single point $\{*\}$. Here we identify $H_G^{2k}(\{*\})$ with $I^k(G)$, which is the space of holomorphic G-invariant polynomial of degree k on $\mathfrak{g}$ for $k = 0, 1, 2, \ldots$ (see, e.g., [8] for more details). The following is well-known ([19, Lemma 6.1], see also [17] and [18]):

$$\begin{aligned} F_M^L(V) &= -\frac{2\pi}{2^n(n+1)!}\sum_{j=0}^{n}\binom{n}{j}\mathcal{C}_{K_M^{-1}\otimes L^{n-2j}}^{c_1^{n+1}}(V) + \frac{2\pi n\beta_L}{n+1}\mathcal{C}_L^{c_1^{n+1}}(V) \\ &= 2\pi\left(\varpi_*^G\left(-c_1^G(K_M^{-1})c_1^G(L)^n + \frac{n\beta_L}{n+1}c_1^G(L)^{n+1}\right)\right)(V), \end{aligned}$$

for $V \in \mathfrak{g}$. Hence if M is a Fano manifold and $L = K_M^{-1}$, then we have

$$F_M^{K_M^{-1}}(V) = -\frac{2\pi}{n+1}(\varpi_*^G(c_1^G(K_M^{-1})^{n+1}))(V),$$

for $V \in \mathfrak{X}_M$.

Now, we assume that a holomorphic vector field $X \in \mathfrak{X}_M$ on M generates the holomorphic S^1-action on M. In this case, there exists a real Lie subgroup K of $\mathrm{Aut}(M)$ such that K is isomorphic to S^1 and its Lie algebra generated by $X_{\mathbb{R}}$. Furthermore, we assume that L is K-equivariant. Then the real part of $\mathcal{L}_X^{(L,h)}$ is constant, for a K-invariant Hermitian metric h of L such that $\omega_g = \sqrt{-1}\Theta(h) \in \mathcal{M}(M, L)$ (see, e.g., [13, Theorem 4.4 (p. 94)]). For a polynomial (or a power series) ϕ of one-variable and two K-invariant Hermitian metrics h_0 and h_1 of L, we put

$$\mathrm{BC}_K^{\phi}(L; h_0, h_1) := \int_0^1 \phi'(\Theta(h_t) + \mathcal{L}_X^{(L,h_t)})\dot{h}_t h_t^{-1}\, dt$$
$$\in \left(\bigoplus_k A_K^{k,k}(M)\right) \Big/ \mathrm{Im}(\partial_K) + \mathrm{Im}(\overline{\partial}_K),$$

where $\{h_t\}_{0\leqq t\leqq 1}$ is a path of K-invariant Hermitian metrics of L from h_0 to h_1 and $A_K^{p,q}(M)$ the space of C^∞ K-equivariant (p, q)-forms on M (see [21] for the definitions of $A_K^{p,q}(M)$, ∂_K and $\overline{\partial}_K$). Then we can prove that $\mathrm{BC}_K^{\phi}(L; h_0, h_1)$ is independent of the choice of a path $\{h_t\}_{0\leqq t\leqq 1}$; hence $\mathrm{BC}_K^{\phi}(L; h_0, h_1)$ is well-defined and called a *K-equivariant Bott–Chern form* of $(L; h_0, h_1)$ associated to ϕ (see [21] for more details). Moreover we put

$$\mathcal{BC}_K^{\phi}(L; h_0, h_1) := \int_M \mathrm{BC}_K^{\phi}(L; h_0, h_1).$$

For a complex Lie subgroup G ($\supset K$) of $\mathrm{Aut}(M)$, if L is G-equivariant and K commutes with $\exp tY_{\mathbb{R}}$ ($t \in \mathbb{R}$) for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$, then we have

$$\begin{aligned}&\left.\frac{d}{dt}\right|_{t=0} \mathcal{BC}_K^{\phi}(L; h_0, \exp tY_{\mathbb{R}}^* h_0)\\ &= \int_M 2\,\Re\mathrm{e}(\mathcal{L}_Y^{(L,h_0)})\phi'(\Theta(h_0) + \mathcal{L}_X^{(L,h_0)}).\end{aligned}$$

Moreover, we also have

$$\mathcal{C}_L^{\phi}(Z; Y) = \int_M \mathcal{L}_Y^{(L,h)}\phi'(\Theta(h) + \mathcal{L}_Z^{(L,h)}), \tag{2.2}$$

for $Y, Z \in \mathfrak{g}$ and a polynomial (or a power series) ϕ of one-variable. Here, for a function Φ on a vector space $\mathfrak{V}$, we put

$$\Phi(S; T) := \left.\frac{d}{dt}\right|_{t=0} \Phi(S + tT), \quad S, T \in \mathfrak{V},$$

that is, $\Phi(S; \cdot\,) \in \mathfrak{V}^*$ is the differentiation of Φ at $S \in \mathfrak{V}$.

3. Generalized Kähler–Ricci solitons

In this section, first of all, we shall explain the K-energy associated to Kähler–Ricci solitons according to Tian ([21]). Next, we shall give some modified version of it.

Let (M, L) be an n-dimensional polarized manifold. We assume that a holomorphic vector field $X \in \mathfrak{X}_M$ on M generates the holomorphic S^1-action (K $(\cong S^1) \subset \mathrm{Aut}(M)$) on M. Furthermore, we assume that L is K-equivariant and put

$$\mathcal{M}(M,\, L)^K := \{\omega \in \mathcal{M}(M,\, L) \colon \omega \text{ is } K\text{-invariant}\}.$$

By the equation

$$(x - y - n\beta_L + n + 1)e^y = \sum_{k=0}^{\infty} \sum_{m=0}^{k} p_{m,k}(x + my)^k, \tag{3.1}$$

we can determine constants $p_{m,k} \in \mathbb{R}$ ($m = 0, 1, \ldots, k$; $k = 0, 1, \ldots$). For examples,

$$p_{0,0} = n + 1 - n\beta_L, \quad p_{0,1} = n\beta_L + 1 - n, \quad p_{1,1} = n - n\beta_L.$$

Tian defined a functional τ_L^X on $\mathcal{M}(M,\, L)^K$ by

$$\begin{aligned}
\tau_L^X(\omega_0(\varphi)) &:= \sum_{k=0}^{\infty} \sum_{m=0}^{k} p_{m,k} \mathcal{B}\mathcal{C}_K^{c_1^k}(K_M^{-1} \otimes L^m; \det g(h_0) \cdot h_0^m, \det g(h_1) \cdot h_1^m) \\
&= -\int_0^1 dt \int_M \frac{\sqrt{-1}}{2\pi} \dot{\varphi}_t \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(L,h_t)}}}{n!} \\
&\qquad \times \Bigg\{ s(\omega_0(\varphi_t)) - n\beta_L - \frac{\sqrt{-1}}{2\pi} \mathcal{L}_X^{(L,h_t)} - 2\frac{\sqrt{-1}}{2\pi} \,\Box_{g_t}\, \mathcal{L}_X^{(L,h_t)} \\
&\qquad\qquad - \left(\partial\left(\frac{\sqrt{-1}}{2\pi} \mathcal{L}_X^{(L,h_t)} \right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi} \mathcal{L}_X^{(L,h_t)}} \right) \right)_{g_t} \Bigg\} \left(\frac{\omega_0(\varphi_t)}{2\pi} \right)^n,
\end{aligned}$$

where $\{\varphi_t\}_{0 \leqq t \leqq 1}$ is a path of K-invariant real-valued C^∞-functions on M from $\varphi_0 \equiv 0$ to $\varphi_1 = \varphi$, $\omega_0 = \sqrt{-1}\Theta(h_0)$ and $h_t = e^{-\varphi_t} h_0$ with $\omega_{g_t} = \sqrt{-1}\Theta(h_t) \in \mathcal{M}(M,\, L)^K$ $(0 \leqq t \leqq 1)$. Here by $(\,\cdot\,,\,\cdot\,)_{g_t}$, we denote the Hermitian metric of the holomorphic vector bundle $T^{1,0}M^*$ of cotangent vectors of type (1, 0) on M induced by g_t, that is, for $\omega_g \in \mathcal{M}(M,\, L)$ and $\varphi_1, \varphi_2 \in C^\infty(M)_{\mathbb{C}}$,

$$(\partial\varphi_1,\, \partial\varphi_2)_g = \sum_{i,j=1}^{n} g^{\bar{j}i} \frac{\partial\varphi_1}{\partial z^i} \frac{\partial\overline{\varphi_2}}{\partial\overline{z^j}}.$$

The Euler–Lagrange equation for τ_L^X is

$$
\begin{aligned}
&s_g - n\beta_L - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)} - 2\frac{\sqrt{-1}}{2\pi}\,\Box_g\,\mathcal{L}_X^{(L,h)} \\
&\quad - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}}\right)\right)_g = 0,
\end{aligned}
\tag{3.2}
$$

where $\omega_g = \sqrt{-1}\Theta(h)$ and h is K-invariant. If $X = 0$, then $\mathcal{L}_X^{(L,h)} = 0$ and hence a solution g for the Euler–Lagrange equation (3.2) is a constant scalar curvature Kähler metric.

If M is a Fano manifold and $L = K_M^{-1}$, then we have $\beta_{K_M^{-1}} = 1$ and

$$
\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)} + \frac{\sqrt{-1}}{2\pi}\,\Box_g\,\mathcal{L}_X^{(K_M^{-1},h)} + \left(\partial f_g, \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}}\right)\right)_g \equiv 0.
$$

Hence, by (1.1), the Euler–Lagrange equation (3.2) becomes

$$
\begin{aligned}
&\Box_g\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right) \\
&\quad + \left(\partial\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}}\right)\right)_g = 0,
\end{aligned}
\tag{3.3}
$$

where $\omega_g = \sqrt{-1}\Theta(h)$ and h is K-invariant. Therefore, by the maximum principle, a solution for the equation (3.3) satisfies

$$
f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)} \equiv \text{constant},
$$

where $\omega_g = \sqrt{-1}\Theta(h)$. In this case, we have

$$
\begin{aligned}
\mathrm{Ric}_g - \omega_g &= \sqrt{-1}\,\partial\bar{\partial} f_g = -\frac{1}{2\pi}\,\partial\bar{\partial}\mathcal{L}_X^{(K_M^{-1},h)} = -\frac{1}{2\pi}\,d\bar{\partial}\mathcal{L}_X^{(K_M^{-1},h)} \\
&= -\frac{\sqrt{-1}}{2\pi}\,di(X)\omega_g = -\frac{\sqrt{-1}}{2\pi}L_X\omega_g = L_{X'}\omega_g,
\end{aligned}
$$

where $X' := -(\sqrt{-1}/2\pi)X$. Hence, (g, X') is a Kähler–Ricci soliton and we can regard $\tau_{K_M^{-1}}^X$ as the K-energy of $(M; X')$ associated to Kähler–Ricci solitons. Therefore a solution for the equation (3.2) could be regarded as a generalization of a Kähler–Ricci soliton to the case where the polarization L is general.

For a complex Lie subgroup G ($\supset K$) of $\mathrm{Aut}(M)$, if L is G-equivariant and K commutes with $\exp tY_{\mathbb{R}}$ ($t \in \mathbb{R}$) for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$, then we have

$$\begin{aligned}
&\frac{d}{dt}\Big|_{t=0} \tau_L^X(\exp tY_{\mathbb{R}}^* \omega_0) \\
&= \int_M \frac{\sqrt{-1}}{2\pi} \Re\mathrm{e}(\mathcal{L}_Y^{(L,h_0)}) \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(L,h_0)}}}{n!} \\
&\qquad \times \Bigg\{ s(\omega_0) - n\beta_L - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h_0)} - 2\frac{\sqrt{-1}}{2\pi}\,\Box_{g_0}\,\mathcal{L}_X^{(L,h_0)} \\
&\qquad\qquad - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h_0)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h_0)}}\right)\right)_{g_0} \Bigg\} \left(\frac{\omega_0}{2\pi}\right)^n,
\end{aligned}$$

for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$. Hence,

$$\begin{aligned}
T_L^X(Y) := \int_M &\frac{\sqrt{-1}}{2\pi}\mathcal{L}_Y^{(L,h)} \frac{e^{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}}}{n!} \\
&\times \Bigg\{ s_g - n\beta_L - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)} - 2\frac{\sqrt{-1}}{2\pi}\,\Box_g\,\mathcal{L}_X^{(L,h)} \\
&\qquad - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}}\right)\right)_g \Bigg\} \left(\frac{\omega_g}{2\pi}\right)^n,
\end{aligned}$$

($Y \in \mathfrak{g} \subset \mathfrak{X}_M$) is an obstruction to the existence of solutions for the equation (3.2), where $\omega_g = \sqrt{-1}\Theta(h)$, $X \in \mathfrak{g}$ and we do not have to assume that X generates a holomorphic S^1-action on M. If M is a Fano manifold, then K_M^{-1} is $\mathrm{Aut}(M)$-equivariant and $T^X_{K_M^{-1}}$ can be defined on $\mathfrak{X}_M$. Moreover, we have

$$\begin{aligned}
T^X_{K_M^{-1}}(Y) &= \int_M \frac{\sqrt{-1}}{2\pi}\mathcal{L}_Y^{(K_M^{-1},h)} \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(K_M^{-1},h)}}}{n!} \\
&\qquad \times \Bigg\{ \Box_g\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right) \\
&\qquad\qquad + \left(\partial\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}}\right)\right)_g \Bigg\} \left(\frac{\omega_g}{2\pi}\right)^n \\
&= \int_M \frac{\sqrt{-1}}{2\pi} Y\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right) \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(K_M^{-1},h_0)}}}{n!} \left(\frac{\omega_g}{2\pi}\right)^n,
\end{aligned}$$

for $Y \in \mathfrak{X}_M$, where $\omega_g = \sqrt{-1}\Theta(h)$. Therefore $T^X_{K_M^{-1}}$ coincides with the obstruction to the existence of Kähler–Ricci solitons introduced by Tian and Zhu ([22], [23] and [24]).

By the way, when M is a Fano manifold, $L = K_M^{-k}$ $(k = 2, 3, \dots)$ and g a solution for the equation (3.2), however, unless $X = 0$, $(1/k)g$ does not give a Kähler–Ricci soliton in general. Because of this fact, we shall modify τ^X_L a little. Instead of (3.1), by the equation

$$(x - \beta_L y + \beta_L)e^y = \sum_{k=0}^{\infty} \sum_{m=0}^{k} q_{m,k}(x + my)^k \tag{3.4}$$

we define constants $q_{m,k} \in \mathbb{R}$ $(m = 0, 1, \dots, k;\ k = 0, 1, 2, \dots)$. Moreover we define a functional $\tilde{\tau}^X_L$ on $\mathcal{M}(M, L)^K$ by

$$\begin{aligned}
\tilde{\tau}^X_L(\omega_0(\varphi)) &:= \sum_{k=0}^{\infty} \sum_{m=0}^{k} q_{m,k} \mathcal{BC}^{c_1^k}_K(K_M^{-1} \otimes L^m; \det g(h_0) \cdot h_0^m, \det g(h_1) \cdot h_1^m) \\
&= -\int_0^1 dt \int_M \frac{\sqrt{-1}}{2\pi} \dot{\varphi}_t \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(L,h_t)}}}{n!} \\
&\qquad \times \left\{ s(\omega_0(\varphi_t)) - n\beta_L - \beta_L \frac{\sqrt{-1}}{2\pi} \mathcal{L}_X^{(L,h_t)} - 2\frac{\sqrt{-1}}{2\pi} \Box_{g_t} \mathcal{L}_X^{(L,h_t)} \right. \\
&\qquad \left. - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h_t)}\right), \partial\overline{\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h_t)}\right)} \right)_{g_t} \right\} \left(\frac{\omega_0(\varphi_t)}{2\pi}\right)^n,
\end{aligned}$$

where $\{\varphi_t\}_{0 \leqq t \leqq 1}$ is a path of K-invariant real-valued C^∞-functions on M from $\varphi_0 \equiv 0$ to $\varphi_1 = \varphi$, $\omega_0 = \sqrt{-1}\Theta(h_0)$ and $h_t = e^{-\varphi_t}h_0$ with $\omega_{g_t} = \sqrt{-1}\Theta(h_t) \in \mathcal{M}(M, L)^K$ $(0 \leqq t \leqq 1)$. Note that if M is a Fano manifold and $L = K_M^{-1}$, then we have $\tilde{\tau}^X_{K_M^{-1}} = \tau^X_{K_M^{-1}}$. The Euler–Lagrange equation for $\tilde{\tau}^X_L$ is

$$\begin{aligned}
&s_g - n\beta_L - \beta_L \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)} - 2\frac{\sqrt{-1}}{2\pi} \Box_g \mathcal{L}_X^{(L,h)} \\
&- \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}\right), \partial\overline{\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}\right)} \right)_g = 0,
\end{aligned} \tag{3.5}$$

where $\omega_g = \sqrt{-1}\Theta(h)$ and h is K-invariant. If $X = 0$, then a solution g for the Euler–Lagrange equation (3.5) is also a constant scalar curvature Kähler metric.

If M is a Fano manifold and $L = K_M^{-k}$ $(k = 1, 2, \dots)$, then we have $\beta_{K_M^{-k}} = 1/k$ and

$$\frac{1}{k}\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-k},h)} + \frac{\sqrt{-1}}{2\pi} \Box_g \mathcal{L}_X^{(K_M^{-k},h)} + \left(\partial f_g, \partial\overline{\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-k},h)}\right)} \right)_g \equiv 0.$$

Hence the Euler–Lagrange equation (3.5) also becomes the equation (3.3) and $((1/k)g, X'')$ is a Kähler–Ricci soliton, where $X'' = -k(\sqrt{-1}/2\pi)X$. Moreover, we can prove that $\omega_g \in \mathcal{M}(M, L)^K$ is a critical point of $\tilde{\tau}_X^L$ if and only if $\omega_{kg} = k\omega_g \in \mathcal{M}(M, L^k)^K$ is that of $\tilde{\tau}_{(1/k)X}^{L^k}$ for general L. Therefore a solution for the equation (3.5) could also be regarded as a generalization of a Kähler–Ricci soliton to the case where the polarization L is general.

For a complex Lie subgroup G of $\mathrm{Aut}(M)$, if L is G-equivariant, then

$$\begin{aligned}\tilde{T}_L^X(Y) := \int_M \frac{\sqrt{-1}}{2\pi}\mathcal{L}_Y^{(L,h)} &\frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(L,h)}}}{n!}\\ &\times \left\{ s_g - n\beta_L - \beta_L \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)} - 2\frac{\sqrt{-1}}{2\pi}\,\Box_g\, \mathcal{L}_X^{(L,h)} \right.\\ &\qquad \left. - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(L,h)}}\right)\right)_g \right\} \left(\frac{\omega_g}{2\pi}\right)^n,\end{aligned}$$

$(Y \in \mathfrak{g} \subset \mathfrak{X}_M)$ is an obstruction to the existence of solutions for the equation (3.5), where $\omega_g = \sqrt{-1}\Theta(h)$, $X \in \mathfrak{g}$ and we do not have to assume that X generates a holomorphic S^1-action on M.

When L is G-equivariant for a complex Lie subgroup G of $\mathrm{Aut}(M)$, we shall study T_L^X and $\tilde{T}_L^X$. In this case, by the identities (2.2), (3.1) and (3.4), we have

$$\begin{aligned} T_L^X(Y) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} p_{m,k}\mathcal{C}^{c_1^k}_{K_M^{-1}\otimes L^m}(X;Y)\\ &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} p_{m,k}(\varpi_*^G c_1^G(K_M^{-1}\otimes L^m)^k)(X;Y)\\ &= \left(\varpi_*^G\left(\sum_{k=0}^{\infty}\sum_{m=0}^{k} p_{m,k}(c_1^G(K_M^{-1}) + mc_1^G(L))^k\right)\right)(X;Y)\\ &= (\varpi_*^G((c_1^G(K_M^{-1}) - c_1^G(L) - n\beta_L + n + 1)e^{c_1^G L}))(X;Y)\\ &= (\varpi_*^G((c_1^G(K_M^{-1}) - c_1^G(L) - n\beta_L + n + 1)\,\mathrm{ch}^G(L)))(X;Y),\\ \tilde{T}_L^X(Y) &= \sum_{k=0}^{\infty}\sum_{m=0}^{k} q_{m,k}\mathcal{C}^{c_1^k}_{K_M^{-1}\otimes L^m}(X;Y)\\ &= (\varpi_*^G((c_1^G(K_M^{-1}) - \beta_L c_1^G(L) + \beta_L)\,\mathrm{ch}^G(L)))(X;Y),\end{aligned}$$

for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$, where $\mathrm{ch}^G(L)$ is the G-equivariant Chern character for L. Moreover, if $X = 0$, then we obtain

$$T_L^X(Y)|_{X=0} = \tilde{T}_L^X(Y)|_{X=0} = -\frac{1}{2\pi n!} F_M^L(Y),$$

for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$.

REMARK 3.6. By a direct calculation without using the identities (3.1) and (3.4), we can also show the formulae stated above:

$$\begin{aligned}
T_L^X(Y) &= (\varpi_*^G((c_1^G(K_M^{-1}) - c_1^G(L) - n\beta_L + n + 1)\,\mathrm{ch}^G(L)))(X; Y),\\
\tilde{T}_L^X(Y) &= (\varpi_*^G((c_1^G(K_M^{-1}) - \beta_L c_1^G(L) + \beta_L)\,\mathrm{ch}^G(L)))(X; Y),
\end{aligned}$$

for $Y \in \mathfrak{g} \subset \mathfrak{X}_M$.

In particular, if M is a Fano manifold, then we have the following:

Theorem 3.7. *Let M be a Fano manifold; in this case K_M^{-1} is* Aut(M)*-equivariant; in this case, $T_{K_M^{-1}}^X$ and $\tilde{T}_{K_M^{-1}}^X$ can be defined on $\mathfrak{X}_M$. Then we have*

$$\begin{aligned}
T_{K_M^{-1}}^X(Y) &= \int_M \frac{\sqrt{-1}}{2\pi} Y\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-1},h)}\right) \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(K_M^{-1},h)}}}{n!}\left(\frac{\omega_g}{2\pi}\right)^n\\
&= (\varpi_*^G(\mathrm{ch}^G(K_M^{-1})))(X; Y),
\end{aligned}$$

for $Y \in \mathfrak{X}_M$, where $\omega_g = \sqrt{-1}\Theta(h)$. Furthermore, we also have

$$\begin{aligned}
\tilde{T}_{K_M^{-k}}^X(Y) &= \int_M \frac{\sqrt{-1}}{2\pi} Y\left(f_g - \frac{\sqrt{-1}}{2\pi}\mathcal{L}_X^{(K_M^{-k},h)}\right) \frac{e^{(\sqrt{-1}/2\pi)\mathcal{L}_X^{(K_M^{-k},h)}}}{n!}\left(\frac{\omega_g}{2\pi}\right)^n\\
&= \frac{1}{k}(\varpi_*^G(\mathrm{ch}^G(K_M^{-k})))(X; Y),
\end{aligned}$$

for $Y \in \mathfrak{X}_M$ and $k = 1, 2, \ldots$, where $\omega_g = \sqrt{-1}\Theta(h)$.

For a general G-equivariant polarization L, where G is a complex Lie subgroup of Aut(M), we define functions $\mathcal{T}_L$ and $\tilde{\mathcal{T}}_L$ on $\mathfrak{g} = \mathrm{Lie}(G) \subset \mathfrak{X}_M$ by

$$\begin{aligned}
\mathcal{T}_L &:= \varpi_*^G((c_1^G(K_M^{-1}) - c_1^G(L) - n\beta_L + n + 1)\,\mathrm{ch}^G(L)),\\
\tilde{\mathcal{T}}_L &:= \varpi_*^G((c_1^G(K_M^{-1}) - \beta_L c_1^G(L) + \beta_L)\,\mathrm{ch}^G(L)).
\end{aligned}$$

Then T_L^X and $\tilde{T}_L^X$ are the differentiations of $\mathcal{T}_L$ and $\tilde{\mathcal{T}}_L$ at $X \in \mathfrak{g}$, respectively. Therefore we have the following:

Theorem 3.8. *If $X \in \mathfrak{g}$ is not a critical point of $\mathcal{T}_L$ (resp. $\tilde{\mathcal{T}}_L$), then the equation* (3.2) (*resp.* (3.5)) *does not admit any solutions. Here, by fixing the holomorphic vector field X, we consider the equations* (3.2) *and* (3.5) *as those for the Kähler metric g.*

REMARK 3.9. When M is a Fano manifold and $L = K_M^{-1}$, Tian and Zhu proved that

$$\mathcal{T}_{K_M^{-1}} = \tilde{\mathcal{T}}_{K_M^{-1}} = \varpi_*^G(\mathrm{ch}^G(K_M^{-1}))$$

is a proper convex function on $\mathfrak{X}_M^{\mathrm{red}}$ and hence admits a unique critical point on $\mathfrak{X}_M^{\mathrm{red}}$ ([24, Lemma 2.2]). Here the Chevalley decomposition allows us to write the identity component $\mathrm{Aut}^\circ(M)$ of $\mathrm{Aut}(M)$ as a semidirect product

$$\mathrm{Aut}^\circ(M) = H \ltimes U_M,$$

where U_M is the unipotent radical of $\mathrm{Aut}^\circ(M)$ and H a reductive algebraic subgroup of $\mathrm{Aut}^\circ(M)$, and $\mathfrak{X}_M^{\mathrm{red}}$ $(\subset \mathfrak{X}_M)$ is the Lie algebra of H (see for instance [9]).

Now, in view of the equation (3.5), we can introduce the notion of a generalized Kähler–Ricci soliton as follows:

DEFINITION 3.10. Let M be a compact connected n-dimensional complex manifold, which may not be projective. If a Kähler metric g on M and a holomorphic vector field $X \in \mathfrak{X}_M$ on M satisfy the following equality:

$$\begin{aligned} s_g - n\beta - \beta \frac{\sqrt{-1}}{2\pi}\theta_X - 2\frac{\sqrt{-1}}{2\pi}\, \Box_g\, \theta_X \\ - \left(\partial\left(\frac{\sqrt{-1}}{2\pi}\theta_X\right), \partial\overline{\left(\frac{\sqrt{-1}}{2\pi}\theta_X\right)}\right)_g = 0, \end{aligned}$$

for some $\theta_X \in C^\infty(M)_{\mathbb{C}}$ satisfying $\sqrt{-1}\,\bar\partial\theta_X = -i(X)\omega_g$, where

$$\beta := \frac{2\pi(c_1(M) \cup [\omega_g]^{n-1})([M])}{[\omega_g]^n([M])} \in \mathbb{R},$$

then we call (g, X), or simply g, a *generalized Kähler–Ricci soliton* on M.

4. An example

In this section, we shall give a non-trivial example of a generalized Kähler–Ricci soliton, which is a generalization of an Einstein–Kähler metric constructed by Koiso and Sakane ([15]).

EXAMPLE 4.1. Let $p\colon E \to N$ be a holomorphic line bundle over a compact connected $(n-1)$-*dimensional* complex manifold N, h an Hermitian metric of E and $s\colon E^0 \to (0, +\infty)$ the corresponding norm functions, where $E^0 := E \setminus$ (0-section). We assume the following:
(i) There exists a compactification $\hat{E}$, such that $\hat{E} \setminus E$ is disjoint union of two complex submanifolds of $\hat{E}$;
(ii) s extends to a continuous function $s\colon \hat{E} \to [0, +\infty]$.
Let S be a vector field on E^0, which generates the standard S^1-action on E^0, and put $H := -JS$, where J is the standard complex structure of E^0. For a monotone increasing diffeomorphism $\tau\colon (0, +\infty) \to (0, R)$, we put $t := \tau \circ s\colon \hat{E} \to [0, R]$. For a one-parameter family $\{g_t\}_{t\in\mathbb{R}}$ of Riemannian metrics on N, we consider the following Riemannian metric

$$g := p^* g_t + dt^2 + (dt \circ J)^2 \tag{4.2}$$

on E^0. We put $u(t) := \sqrt{g(H, H)}$ and $b := (1/2) \int_0^R u(x)\, dx$, and define a function $U\colon [0, R] \to [-b, b]$ by

$$U(w) := -b + \int_0^w u(x)\, dx.$$

Then, g is a Kähler metric on E^0 if and only if g_0 is a Kähler metric on N and $g_t = g_0 - U(t)B$, where B is a 2-tensor associated to the curvature form of h. Furthermore, we assume the following:
(iii) The eigenvalues of B and Ric_{g_0} with respect to g_0 are constant.
Under these assumptions, we put

$$\begin{aligned}
&\varphi(U(t)) := u(t)^2 = g(H, H),\\
&Q(U(t)) := \det(g_0^{-1} g_t) = \det(I - U(t) g_0^{-1} B),\\
&G(U(t)) := \mathrm{tr}_{g_t}(\mathrm{Ric}_{g_0}),\\
&\Delta(U) := Q(U)G(U).
\end{aligned}$$

Then, in view of [12, Lemma 3.1], the scalar curvature of g is given by

$$s_g = \frac{\Delta(U)}{Q(U)} - \frac{1}{2Q(U)} (Q\varphi)''(U).$$

Since $-U$ is a Hamiltonian function of the holomorphic vector field $S + \sqrt{-1}H$ with respect to ω_g, i.e., $-\bar{\partial} U = i(S + \sqrt{-1}H)\omega_g$, we put $M := \hat{E}$ and $-(\sqrt{-1}/2\pi)\theta_X :=$

$\alpha_1 U + \alpha_2$, where α_1 and α_2 are real constants. Then we have

$$-\frac{\sqrt{-1}}{2\pi}\,\Box_g\,\theta_X = \frac{\alpha_1}{2Q}(Q\varphi)'(U),$$
$$\left(\partial\left(\frac{\sqrt{-1}}{2\pi}\theta_X\right), \partial\left(\overline{\frac{\sqrt{-1}}{2\pi}\theta_X}\right)\right)_g = \frac{\alpha_1^2}{2}\varphi(U).$$

Hence, the equation (3.5) becomes the ordinary differential equation

$$\frac{\Delta(U)}{Q(U)} - \frac{1}{2Q(U)}(Q\varphi)''(U) - m + \frac{m}{n}(\alpha_1 U + \alpha_2)$$
$$+ \frac{\alpha_1}{Q(U)}(Q\varphi)'(U) - \frac{\alpha_1^2}{2}\varphi(U) = 0,$$

where we put $m := n\beta$. Therefore, we have

$$\varphi(U) = \frac{e^{\alpha_1 U}}{Q(U)}\left\{\int_{-b}^{U}\Psi(x)e^{-\alpha_1 x}(U-x)\,dx + C_1 U + C_2\right\},$$

where C_1 and C_2 are constants and we put

$$\Psi(U) := 2\Delta(U) - 2mQ(U) + \frac{2m}{n}Q(U)(\alpha_1 U + \alpha_2).$$

In view of [12, Theorem 5], this defines a Kähler metric on $M = \hat{E}$ if and only if the following two conditions hold:
(a) $\varphi > 0$ on $(-b, b)$, $\varphi(\pm b) = 0$ and φ extends smoothly over $\pm b$;
(b) $\varphi'(-b) = 2$ and $\varphi'(b) = -2$.
The conditions $\varphi(-b) = 0$ and $\varphi'(-b) = 2$ imply $C_2 = 0$ and $C_1 = 2e^{b\alpha_1}Q(-b)$. Moreover, by the condition $\varphi(b) = 0$, we have

$$\alpha_2 = \frac{(n/m)(b\delta_0 - bmq_0 + (m/n)b\alpha_1 q_1 - \delta_1 + mq_1 - (m/n)\alpha_1 q_2 + 2be^{b\alpha_1}Q(-b))}{q_1 - bq_0},$$

where, for $i = 0, 1,\ j = 0, 1, 2$, we put

$$\delta_i := \int_{-b}^{b} x^i \Delta(x)e^{-\alpha_1 x}\,dx, \quad q_j := \int_{-b}^{b} x^j Q(x)e^{-\alpha_1 x}\,dx.$$

The positivity and extendability of φ can be proved similarly to the case of [12, Section 6], since

$$\frac{d^2}{dU^2}(Q(U)\varphi(U)e^{-\alpha_1 U})$$
$$= 2e^{-\alpha_1 U}Q(U)\left(G(U) + \frac{2m\alpha_1}{n}U + \frac{2m\alpha_2}{n} - 2m\right).$$

The condition $\varphi'(b) = -2$ becomes

$$\delta_0 - mq_0 + \frac{m}{n}\alpha_1 q_1 + \frac{m}{n}\alpha_2 q_0 + e^{b\alpha_1} Q(-b) + e^{-b\alpha_1} Q(b) = 0. \tag{4.3}$$

Finally, by this equation, we want to determine the constant α_1. However, in a general situation, we have not been able to settle this problem yet. Henceforth, we consider a special setting, that is, we put $N = \mathbb{P}^1(\mathbb{C})$, $E = \mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(k)$, $M = \hat{E} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(k) \oplus \mathcal{O}_{\mathbb{P}^1(\mathbb{C})})$, $\omega_{g_0} = a\omega$, $\sqrt{-1}\Theta(h) = k\omega$ and $\mathrm{Ric}_{g_0} = \kappa\omega$, where $k \in \mathbb{Z}$, a is a positive constant satisfying $a \pm kb > 0$, h an Hermitian metric of E and ω the Fubini–Study form on $\mathbb{P}^1(\mathbb{C})$ such that $[\omega] = 2\pi c_1(\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(1))$. In this case, each Kähler class on M is represented by a Kähler form of a Kähler metric of the form as in (4.2) (cf. [12, Section 4]) and we have

$$Q(U) = 1 - \frac{k}{a}U, \quad G(U) = \frac{\kappa}{a - kU}.$$

Then a simple calculation shows that

$$\lim_{\alpha_1 \to \pm\infty} (\text{the left hand side of (4.3)}) = \mp\infty.$$

Therefore, we can conclude that the equation (4.3) has a solution. Hence, for an arbitrary Kähler class, we have a generalized Kähler–Ricci soliton on $M = \hat{E} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1(\mathbb{C})}(k) \oplus \mathcal{O}_{\mathbb{P}^1(\mathbb{C})})$.

REMARK 4.4. The Kähler metric of the type in Example 4.1 was originally used by Koiso and Sakane in [15] to construct examples of non-homogeneous Einstein–Kähler metrics. For Kähler–Ricci solitons, Koiso constructed examples of this type in [14]. (In [14], Koiso called Kähler–Ricci solitons as quasi-Einstein metrics.) Moreover, Hwang ([12]) and Guan ([10]) constructed examples of this type for extremal Kähler metrics and generalized quasi-Einstein metrics (see [10] for the definition of a generalized quasi-Einstein metric), respectively.

References

[1] S. Bando: *An obstruction for Chern class forms to be harmonic*, Kodai Math. J. **29** (2006), 337–345.

[2] S. Bando and T. Mabuchi: *Uniqueness of Einstein Kähler metrics modulo connected group actions*; in Algebraic Geometry, Sendai, 1985, Adv. Stud. Pure Math. **10** North-Holland, Amsterdam, 11–40, 1987.

[3] R. Bott: *A residue formula for holomorphic vector-fields*, J. Differential Geometry **1** (1967), 311–330.

[4] E. Calabi: *Extremal Kähler metrics*; in Seminar on Differential Geometry, Ann. of Math. Stud. **102**, Princeton Univ. Press, Princeton, NJ, 259–290, 1982.

[5] E. Calabi: *Extremal Kähler metrics*, II; in Differential Geometry and Complex Analysis, Springer, Berlin, 95–114, 1985.

[6] A. Futaki: *An obstruction to the existence of Einstein Kähler metrics*, Invent. Math. **73** (1983), 437–443.

[7] A. Futaki: *On compact Kähler manifolds of constant scalar curvatures*, Proc. Japan Acad. Ser. A Math. Sci. **59** (1983), 401–402.

[8] A. Futaki: Kähler–Einstein Metrics and Integral Invariants, Lecture Notes in Mathematics **1314**, Springer, Berlin, 1988.

[9] A. Futaki and T. Mabuchi: *Bilinear forms and extremal Kähler vector fields associated with Kähler classes*, Math. Ann. **301** (1995), 199–210.

[10] D.Z.-D. Guan: *Quasi-Einstein metrics*, Internat. J. Math. **6** (1995), 371–379.

[11] D. Guan: *On modified Mabuchi functional and Mabuchi moduli space of Kähler metrics on toric bundles*, Math. Res. Lett. **6** (1999), 547–555.

[12] A.D. Hwang: *On existence of Kähler metrics with constant scalar curvature*, Osaka J. Math. **31** (1994), 561–595.

[13] S. Kobayashi: Transformation Groups in Differential Geometry, Springer, New York, 1972.

[14] N. Koiso: *On rotationally symmetric Hamilton's equation for Kähler–Einstein metrics*; in Recent Topics in Differential and Analytic Geometry, Adv. Stud. Pure Math. **18-I**, Academic Press, Boston, MA, 327–337, 1990.

[15] N. Koiso and Y. Sakane: *Nonhomogeneous Kähler–Einstein metrics on compact complex manifolds*; in Curvature and Topology of Riemannian Manifolds (Katata, 1985), Lecture Notes in Math. **1201**, Springer, Berlin, 165–179, 1986.

[16] T. Mabuchi: *K-energy maps integrating Futaki invariants*, Tohoku Math. J. (2) **38** (1986), 575–593.

[17] Y. Nakagawa: *Bando–Calabi–Futaki characters of Kähler orbifolds*, Math. Ann. **314** (1999), 369–380.

[18] Y. Nakagawa: *The Bando–Calabi–Futaki character and its lifting to a group character*, Math. Ann. **325** (2003), 31–53.

[19] G. Tian: *Kähler–Einstein metrics on algebraic manifolds*; in Transcendental Methods in Algebraic Geometry (Cetraro, 1994), Lecture Notes in Math. **1646**, Springer, Berlin, 143–185, 1996.

[20] G. Tian: *Bott–Chern forms and geometric stability*, Discrete Contin. Dynam. Systems **6** (2000), 211–220.

[21] G. Tian: *An equivariant version of the K-energy*, Acta Math. Sin. (Engl. Ser.) **21** (2005), 1–8.

[22] G. Tian and X. Zhu: *Uniqueness of Kähler–Ricci solitons on compact Kähler manifolds*, C.R. Acad. Sci. Paris Sér. I Math. **329** (1999), 991–995.

[23] G. Tian and X. Zhu: *Uniqueness of Kähler–Ricci solitons*, Acta Math. **184** (2000), 271–305.

[24] G. Tian and X. Zhu: *A new holomorphic invariant and uniqueness of Kähler–Ricci solitons*, Comment. Math. Helv. **77** (2002), 297–325.

[25] X.-J. Wang and X. Zhu: *Kähler–Ricci solitons on toric manifolds with positive first Chern class*, Adv. Math. **188** (2004), 87–103.

School of Mathematics and Physics
College of Science and Engineering
Kanazawa University
Kakuma-machi, Kanazawa, 920–1192
Japan
e-mail: yasunaka@kenroku.kanazawa-u.ac.jp