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PSEUDO-ANOSOV MAPS AND FIXED POINTS OF BOUNDARY HOMEOMORPHISMS COMPATIBLE WITH A FUCHSIAN GROUP

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Abstract

Let $\tilde{S}$ be a Riemann surface of type (p, n) with $3p - 3 + n > 0$. Let F be a pseudo-Anosov map of $\tilde{S}$ defined by two filling simple closed geodesics on $\tilde{S}$. Let $a \in \tilde{S}$, and $S = \tilde{S} - \{a\}$. For any map $f : S \to S$ that is generated by two simple closed geodesics and is isotopic to F on $\tilde{S}$, there corresponds to a configuration τ of invariant half planes in the universal covering space of $\tilde{S}$. We give a necessary and sufficient condition (with respect to the configuration) for those f to be pseudo-Anosov maps. As a consequence, we obtain infinitely many pseudo-Anosov maps f on S that are isotopic to F on $\tilde{S}$ as a is filled in.

1. Statement of results

Let $\tilde{S}$ be a Riemann surface of type (p, n), where p is the genus of $\tilde{S}$ and n is the number of punctures of $\tilde{S}$. Assume that $3p-3+n > 0$. Let $a \in \tilde{S}$, and $S = \tilde{S} - \{a\}$. Let F be a pseudo-Anosov map on $\tilde{S}$ in the sense that there exists a pair $(\mathcal{F}_+, \mathcal{F}_-)$ of transverse measured foliations of $\tilde{S}$ with $F(\mathcal{F}_+) = \lambda\mathcal{F}_+$ and $F(\mathcal{F}_-) = (1/\lambda)\mathcal{F}_-$ for some $\lambda > 1$. (See also FLP [7] and Penner [15].) In [10], Kra investigated the problem of finding pseudo-Anosov maps f on S so that f is isotopic to F on $\tilde{S}$ as a is filled in. He showed that if $\tilde{S}$ is compact with genus $p \geq 2$, then for some integer k, there is a pseudo-Anosov map f on S so that f is isotopic to F^k on $\tilde{S}$. In this article, we show that there always exist infinitely many pseudo-Anosov maps f on S so that f is isotopic to a pseudo-Anosov map F on $\tilde{S}$ that is obtained from Thurston's construction [17].

To illustrate, let $\tilde{\alpha}_1, \tilde{\alpha}_2 \subset \tilde{S}$ be two filling simple closed geodesics, that is, each component of $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ is a disk or an once punctured disk. Let $t_{\tilde{\alpha}_i}$ denote the positive Dehn twist along $\tilde{\alpha}_i$. It is well known [17] (see also [2, 12, 16] for some variations) that a finite product

$$\tilde{\zeta} = \prod_{i=1}^{N} t_{\tilde{\alpha}_2}^{r_i} \circ t_{\tilde{\alpha}_1}^{-s_i}, \quad N, r_i, s_i \in \mathbb{Z}^+ \tag{1.1}$$

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is isotopic to a pseudo-Anosov map F on $\tilde{S}$. Throughout the article we denote by $H_t(\,\cdot\,)\colon \tilde{S} \to \tilde{S}$, $0 \le t \le 1$, the isotopy between $\tilde{\zeta}$ and F. Note that $\tilde{\alpha}_1, \tilde{\alpha}_2$ can be viewed as curves on S (call them α_1 and α_2, respectively), and thus the maps $\tilde{\zeta}$ are also defined on S. Clearly, if $\tilde{S}$ is compact, $S - \{\alpha_1, \alpha_2\}$ consists of disks and only one once punctured disk. Hence $\tilde{\zeta}$ intimately represents a pseudo-Anosov mapping class on S that has the required property. However, if $\tilde{S}$ is non-compact and in particular, if $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ only consists of once punctured disks, then one component of $S - \{\alpha_1, \alpha_2\}$ is a twice punctured disk (enclosing the puncture a), which means that the map (1.1) on S does not represent a pseudo-Anosov mapping class.

A question arises as to whether or not we can take another geodesic α_2' on S as a substitution of α_2 so that $\{\alpha_1, \alpha_2'\}$ fills S and α_2' is still homotopic to α_2 on $\tilde{S}$ as a is filled in. In [19], we constructed such a geodesic α_2' with the required properties by utilizing topological methods. As a consequence, we showed that there exist infinitely many distinct pseudo-Anosov maps on S isotopic to on $\tilde{S}$.

Let $\mathcal{F}$ be the set of isotopy classes of maps of S that are isotopic to the identity on $\tilde{S}$. The main purpose of this article is to develop a tool to detect in general whether or not a pair $\{\alpha_1, \alpha_2'\}$, where $\alpha_2' = f(\alpha_2)$ for some $f \in \mathcal{F}$, fills S; we will give a necessary and sufficient condition for the pair $\{\alpha_1, \alpha_2\}$ of geodesics on S to fill S. To do this, we need to transform the view of Dehn twists on S to the view of some special fiber-preserving automorphisms on the Bers fiber space $F(\tilde{S})$. (See Bers [4] and Kra [10] for more details.)

Let $\mathbb{H} = \{z \in \mathbb{C}\colon \operatorname{Im} z > 0\}$ be a hyperbolic plane, and let $\varrho\colon \mathbb{H} \to \tilde{S}$ be a universal covering with the covering group G. It is well known [4, 6] that G is isomorphic (via an isomorphism φ^*, see Bers [4]) to $\mathcal{F}$. Further, φ^* naturally extends to an isomorphism (call φ^* also) of the group of fiber preserving automorphisms of $F(\tilde{S})$ onto the group of mapping classes on S fixing the puncture a.

Let $\hat{\alpha}_i \subset \mathbb{H}$, $i = 1, 2$, be geodesics such that $\varrho(\hat{\alpha}_i) = \tilde{\alpha}_i$. Let $\{D_i, D_i'\} = \mathbb{H} - \{\hat{\alpha}_i\}$. As we will explain in Section 3, the Dehn twist $t_{\tilde{\alpha}_i}\colon \tilde{S} \to \tilde{S}$ can be lifted to a quasiconformal map $\tau_i\colon \mathbb{H} \to \mathbb{H}$ with respect to D_i. The map τ_i determines a disjoint union of invariant half planes $D_i(j)$ with the property that the restriction of τ_i to the complement

$$H_i = \mathbb{H} - \bigcup_j D_i(j) \tag{1.2}$$

is the identity. Furthermore, the map τ_i induces a fiber-preserving automorphism $[\tau_i]$ of $F(\tilde{S})$ such that, if $\hat{\alpha}_i \in \{\varrho^{-1}(\tilde{\alpha}_i)\}$ (and hence D_i) is chosen properly, $\varphi^*([\tau_i]) = t_{\alpha_i}$ (see Lemma 3.3). Our main result is the following:

Theorem 1.1. *Let $\tilde{\alpha}_1, \tilde{\alpha}_2 \subset \tilde{S}$ be arbitrary two simple closed geodesics so that $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ only consists of once punctured disks. Let α_i, $i = 1, 2$, be two simple closed geodesics on S homotopic to $\tilde{\alpha}_i$ on $\tilde{S}$, and let τ_i be the corresponding lifts with*

$\varphi^*([\tau_i]) = t_{\alpha_i}$. *Then the map*

$$\prod_{i=1}^{N} t_{\alpha_2}^{r_i} \circ t_{\alpha_1}^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+, \tag{1.3}$$

represents a pseudo-Anosov mapping class on S (*and projects to* $\tilde{\zeta}$ *on* $\tilde{S}$) *if and only if the intersection* $H_1 \cap H_2$, *where* H_i *are defined as in* (1.2), *is empty.*

Given any pair $\{\tau_1, \tau_2\}$ of lifts of $t_{\tilde{\alpha}_1}$ and $t_{\tilde{\alpha}_2}$ with $H_1 \cap H_2 \neq \emptyset$, it is easy to replace τ_2 with a G-conjugation τ_2' so that $H_1 \cap H_2' = \emptyset$. Therefore, via the Bers isomorphism, we are able to construct infinitely many pairs $\{\alpha_1, \alpha_2'\}$ that fill S, where $\alpha_2' = f(\alpha_2)$ for some $f \in \mathcal{F}$. There are several applications of Theorem 1.1.

We now assume that $\tilde{S}$ is an F-minimal surface in the sense that $\mathcal{F}_\pm$ are defined by a quadratic differential ϕ on $\tilde{S}$ (see Bers [5] for the definitions and terminology). If the genus $p \geq 2$, then by the Riemann-Roch theorem (see [9] for instance), there exists a finite number of zeros of ϕ on the compactification of $\tilde{S}$. Note that some zeros could be punctures of $\tilde{S}$.

If ϕ has non-puncture zeros z_0, we may choose $\hat{z}_0 \in \mathbb{H}$ with $\varrho(\hat{z}_0) = z_0$, and thereby a pair $\{\tau_1, \tau_2\}$ of configurations of invariant half planes under the lifts of the Dehn twists such that $H_1 \cap H_2 \neq \emptyset$ and $\hat{z}_0 \in H_1 \cap H_2$. This implies that the map

$$\zeta = \prod_{i=1}^{N} \tau_2^{r_i} \tau_1^{-s_i}, \quad r_i, s_i \in \mathbb{Z}^+ \tag{1.4}$$

fixes $\hat{z}_0 \in \mathbb{H}$. It is important to note that $\zeta\colon \mathbb{H} \to \mathbb{H}$ is a quasiconformal map compatible with G. It naturally extends to a map of $\overline{\mathbb{H}}$ onto itself, which is also denoted by ζ.

If ϕ has no non-puncture zeros, then some punctures (call z_0 also) must be zeros of ϕ. In this case, we can still choose a pair $\{\tau_1, \tau_2\}$ of lifts of the Dehn twists such that $H_1 \cap H_2 \neq \emptyset$ and ζ fixes $\hat{z}_0 \in \hat{\mathbb{R}}$.

Under certain conditions ζ can be replaced with a pseudo-Anosov map $\hat{F}$ so that $\varrho \circ \hat{F} = F \circ \varrho$, $\hat{F}(\hat{z}_0) = \hat{z}_0$ and $\hat{F}|_{\partial H} = \zeta|_{\partial H}$. Lemma 5.4 of Marden-Strebel [13] then asserts that ζ does not fix any other fixed points of G on $\hat{\mathbb{R}}$ (except for $\hat{z}_0$ in the second case). Consider the maps $h\zeta$ for $h \in G$. Unfortunately, the existence of fixed points of $h\zeta$ is not guaranteed, and a question arises as to whether $h\zeta$ fixes some fixed points of G on $\hat{\mathbb{R}}$. It is easy to show that for certain elements h of G, $h\zeta$ fix some points on $\hat{\mathbb{R}}$ that may not be fixed points of G. Our second result states:

Theorem 1.2. *Let* $\tilde{S}$ *be an* F*-minimal surface of genus* $p \geq 2$ *and* $n > 0$. *Let* z_0 *be a zero of the corresponding quadratic differential* ϕ *which may or may not be a puncture of* $\tilde{S}$. *Then associated to each* $\hat{z}_0 \in \overline{\mathbb{H}}$ *with* $\varrho(\hat{z}_0) = z_0$, *there exists a pair* $\{\tau_1, \tau_2\}$ *of lifts of the Dehn twists* $t_{\tilde{\alpha}_1}$ *and* $t_{\tilde{\alpha}_2}$ *with* $H_1 \cap H_2 \neq \emptyset$, *and hence a map* ζ *such that* $h_n\zeta$ *does not fix any fixed points of* G *on* $\hat{\mathbb{R}}$ *for an infinite sequence* $\{h_n\} \subset G$.

We call ζ_1 and ζ_2 with forms (1.4) are conjugate if there is an element $h \in G$ such that $\zeta_1 = h\zeta_2 h^{-1}$, which is equivalent to saying that $\varphi^*([\zeta_1])$ and $\varphi^*([\zeta_2])$ with forms (1.3) are conjugate if there is a map $f \in \mathcal{F}$ so that $\varphi^*([\zeta_1])$ is isotopic to $f \circ \varphi^*([\zeta_2]) \circ f^{-1}$. As a consequence of Theorem 1.1 and Theorem 1.2, we have:

Theorem 1.3. *Let $\tilde{S}$ be a Riemann surface of type (p, n) with $p \geq 2$, and $n > 0$. Let $\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ be a pair of filling simple closed geodesics on $\tilde{S}$. Let ζ be defined by* (1.4) *via an F-minimal surface and a pair $\{r_1, r_2\}$ with $H_1 \cap H_2 \neq \emptyset$. Then there are infinitely many mapping classes ω_j^* on S with these properties*:
(1) *all ω_j^* are pseudo-Anosov,*
(2) *every ω_j^* fixes a and projects to the mapping class represented by* (1.1) *as a is filled in,*
(3) *every ω_j^* is represented by two filling simple loops on S and is of form* (1.3).
If in addition we assume that z_0 is a non-puncture zero of ϕ so that $F(z_0) = z_0$ and the curve $H_t(z_0)$, $0 \leq t \leq 1$, is a trivial loop, then:
(4) *$\varphi^*(\zeta)$ is pseudo-Anosov if z_0 is a non-puncture zero of ϕ,*
(5) *$\varphi^*(\zeta)$ is not conjugate to any ω_j^*, and*
(6) *all ω_j^* lie in different conjugacy classes.*

This article is organized as follows. In Section 2, we establish a correspondence between the set of pseudo-Anosov maps of S (that are isotopic to $\tilde{\zeta}$ on $\tilde{S}$) and the set $\mathcal{L}$ of lifts of $\tilde{\zeta}$ that fix no fixed points of G. It follows from Lemma 5.4 of [13] (see Lemma 2.2 for a different approach) that elements in $\mathcal{L}$ that do not fix any parabolic fixed points of G must be pseudo-Anosov mapping classes on S. Details appear in Sections 3. Sections 4, 5, and 6 are devoted to the proofs of the results.

2. Notation and background

To establish notation and terminology, we begin with an overview of relevant Teichmüller theory. For more information, we refer to [4, 10].

Let $\tilde{S}_1$ be a Riemann surface with the same type (p, n). A marking of $\tilde{S}_1$ is a homeomorphism $f_1 : \tilde{S} \to \tilde{S}_1$. By $(f_1 \colon \tilde{S} \to \tilde{S}_1)$ we denote a marked Riemann surface. The Teichmüller space $T(\tilde{S})$ is defined as a set of marked Riemann surfaces $(f_1 \colon \tilde{S} \to \tilde{S}_1)$ quotient by an equivalent relation "$\sim$", where $(f_1 \colon \tilde{S} \to \tilde{S}_1) \sim (f_2 \colon \tilde{S} \to \tilde{S}_2)$ if and only if there is a conformal map $h \colon \tilde{S}_1 \to \tilde{S}_2$ such that $h \circ f_1$ is isotopic to f_2.

We denote by $[f_1 \colon \tilde{S} \to \tilde{S}_1]$ the equivalence class of the marked surface $(f_1 \colon \tilde{S} \to \tilde{S}_1)$. Every marked surface $(f_1 \colon \tilde{S} \to \tilde{S}_1)$ defines a new conformal structure μ_1 on $\tilde{S}$ via pullbacks. Two conformal structures μ_1 and μ_2 are called equivalent if and only if $(f_1 \colon \tilde{S} \to \tilde{S}_1) \sim (f_2 \colon \tilde{S} \to \tilde{S}_2)$. Let $[\mu]$ denote the equivalence class of a conformal structure μ on $\tilde{S}$. By Ahlfors-Bers [1], every conformal structure μ on $\tilde{S}$ determines a quasiconformal mapping w^μ of $\mathbb{C}$ that fixes 0, 1 and is conformal on $\mathbb{H}^* = \{z \in \mathbb{C} \colon \operatorname{Im} z < 0\}$. The region $w^\mu(\mathbb{H})$ is a Jordan domain that only depends on $[\mu]$.

The Bers fiber space $F(\tilde{S})$ is defined as a collection $\{([\mu], z);\ [\mu] \in T(\tilde{S}),\ z \in w^{\mu}(\mathbb{H})\}$ of pairs endowed with a product structure. The natural projection $\pi\colon F(\tilde{S}) \to T(\tilde{S})$ defined by sending each point $([\mu], z)$ to $[\mu]$ is holomorphic. From Theorem 9 of Bers [4], There is an isomorphism $\varphi\colon F(\tilde{S}) \to T(S)$ such that

$$\pi = \iota \circ \varphi, \tag{2.1}$$

where $\iota\colon T(S) \to T(\tilde{S})$ is the natural forgetful map.

The group of isotopy classes of self-maps f of $\tilde{S}$ is the mapping class group $\mathrm{Mod}_{\tilde{S}}$, which naturally acts on $T(\tilde{S})$ as holomorphic automorphisms. Let mod $\tilde{S}$ denote the full group of fiber preserving holomorphic automorphisms of $F(\tilde{S})$ that projects to $\mathrm{Mod}_{\tilde{S}}$. Elements of mod $\tilde{S}$ are of forms $[\hat{f}]$, where $\hat{f}\colon \mathbb{H} \to \mathbb{H}$ is a lift of a self-map f of $\tilde{S}$. $[\hat{f}]$ only depends on the boundary values $\hat{f}|_{\hat{\mathbb{R}}}$. The Bers isomorphism $\varphi\colon F(\tilde{S}) \to T(S)$ induces an isomorphism φ^* of mod $\tilde{S}$ onto a group Mod_S^a of mapping classes of S fixing the puncture a.

An element $\theta \in \mathrm{Mod}_S^a$ is called a reducible mapping class if there is a curve system $\mathcal{C} = \{c_1, \ldots, c_s\}$, $s \geq 1$, of independent and disjoint simple closed geodesics on S with $f(\{c_1, \ldots, c_s\}) = \{c_1, \ldots, c_s\}$ for certain representative f of θ. There is a smallest positive integer K such that f^K maps each loop in $\mathcal{C}$ to itself and the restriction of f^K to each component of $S - \{c_1, \ldots, c_s\}$ is either the identity or a pseudo-Anosov map. θ is called pure if $K = 1$.

We now assume that θ is reducible and projects to a pseudo-Anosov mapping class $\tilde{\theta}$ on $\tilde{S}$ that is induced by a map F. By Lemma 5.1 and 5.2 of [18], the curve system $\mathcal{C}$ consists of only one curve c_1 that bounds a twice punctured disk enclosing a and another puncture of $\tilde{S}$, which is equivalent to that c_1 is peripheral on $\tilde{S}$. If we write $\varphi^{*-1}(\theta) = [\hat{f}]$, then $\hat{f}\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ fixes a parabolic fixed point of G. Conversely, each element $[\hat{f}]$ fixing the fixed point of a parabolic element of G corresponds to a reducible mapping class in Mod_S^a which is reduced by a single simple closed geodesic that is trivial on $\tilde{S}$. For hyperbolic fixed points, we have

Lemma 2.1 (Marden-Strebel [13]). *Assume that $\tilde{S}$ is F-minimal. Let z_0 be a zero of ϕ, and let $\hat{z}_0 \in \overline{\mathbb{H}}$ be such that $\varrho(\hat{z}_0) = z_0$. Suppose that $\hat{f}\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ fixes $\hat{z}_0$. Then $\hat{f}$ does not fix any hyperbolic fixed point of G.*

To proof our theorems, we need a slightly general version of the lemma that states:

Lemma 2.2. *Let $\hat{f}\colon \overline{\mathbb{H}} \to \overline{\mathbb{H}}$ be any lift of a pseudo-Anosov map $F\colon \tilde{S} \to \tilde{S}$. Then $\hat{f}$ does not fix any hyperbolic fixed point of G.*

REMARK 2.1. If $\tilde{S}$ is F-minimal, the lemma is covered by the argument of Lemma 5.4 of [13]. Our approach however does not assume that $\tilde{S}$ is F-minimal.

Proof of Lemma 2.1. Suppose that $\hat{f}(x) = x$ for a fixed point of a primitive hyperbolic element g of G. Let $\sigma : S \to S$ be a map that induces the mapping class $\varphi^*([\hat{f}])$. There are three cases to consider.

CASE 1. $g \in G$ is simple hyperbolic (the axis of g projects to a simple closed geodesic on $\tilde{S}$). We claim that σ is reduced by a pair $\{\alpha, \beta\}$ of geodesics which bounds a punctured cylinder enclosing a (throughout α and β are called parallel geodesics). Note that $g' = \hat{f} g \hat{f}^{-1}$ is also an element of G that fixes x. g' cannot be parabolic. For otherwise, g' and g would share the same fixed point x, it would follow that $\langle g, g' \rangle \subset G$ is not discrete. We see that g' is also hyperbolic. If g and g' share only one fixed point x, by Theorem 4.3.5 of Beardon [3], the commutator $[g, g']$ is parabolic whose fixed point is x. From Theorem 5.1.2 of [3], $\langle g, [g, g'] \rangle \subset G$ is not discrete. This is a contradiction. We conclude that g and g' share both fixed points. It follows that $g' = g^k$, where $k = \pm 1$ since $g' = \hat{f} g \hat{f}^{-1}$ and g is primitive in G.

Let $h^* : S \to S$ denote a map that induces the mapping class $\varphi^*(h)$ for an element $h \in G$. From Theorem 2 of [10] or Theorem 2 of [14], we can write $g^* = t_\beta^{-1} \circ t_\alpha$, where α, β are parallel geodesics. Hence $g'^* = g^{k*} = t_\beta^{-k} \circ t_\alpha^k$. Recall that $g' = \hat{f} g \hat{f}^{-1}$, we thus obtain

$$t_\beta^{-k} \circ t_\alpha^k = \sigma \circ (t_\beta^{-1} \circ t_\alpha) \circ \sigma^{-1} = t_{\sigma(\beta)}^{-1} \circ t_{\sigma(\alpha)}.$$

This means that $\sigma(\{\alpha, \beta\}) = \{\alpha, \beta\}$, which says that σ is reduced by $\{\alpha, \beta\}$.

Observe that both α and β project to a non-trivial geodesic $\tilde{\alpha}$ on $\tilde{S}$ as a is filled in. θ projects to $\tilde{\theta}$ that is reduced by $\tilde{\alpha}$. Hence $\tilde{\theta}$ is reducible, contradicting the hypothesis.

REMARK 2.2. Conversely, if σ is reduced by a pair $\{\alpha, \beta\}$ of parallel geodesics, then we claim that $\hat{f}$ fixes a hyperbolic fixed point of G. In fact, σ commutes with $t_\beta^{-1} \circ t_\alpha$. From Theorem 2 of [10] or Theorem 2 of [14], there is a simple hyperbolic element $g \in G$ so that $g^* = t_\beta^{-1} \circ t_\alpha$. We see that $\hat{f}$ commutes with g. That is,

$$g = \hat{f} g \hat{f}^{-1}. \tag{2.2}$$

Denote $\{x, y\}$ the attracting and repelling fixed points of g. It follows from (2.2) that $\hat{f}(\{x, y\}) = \{x, y\}$. If $\hat{f}(x) = y$, then by (2.2) again, for any integer k,

$$g^k(\hat{f}(z)) = \hat{f}(g^k(z)) \tag{2.3}$$

for a $z \in \mathbb{H}$. As $k \to +\infty$, $g^k(\hat{f}(z)) \to x$ and $g^k(z) \to x$. It follows that $\hat{f}(g^k(z)) \to y$. This contradicts to (2.3).

CASE 2. g is essential hyperbolic (the axis of g projects to a filling geodesic on $\tilde{S}$). Then by Theorem 2 of [10], g^* is pseudo-Anosov. Using the same argument as in Case 1, we have $\hat{f} g \hat{f}^{-1} = g^k$ for $k = \pm 1$.

If $k = 1$, then $\hat{f}$ commutes with g. So σ commutes with g^*. Suppose that σ is pseudo-Anosov. Since g^* is pseudo-Anosov, by Theorem 7.5.A of [8], there are integers i, j such that $\sigma^i = g^{*j}$. This implies that σ^i projects to the trivial mapping class on $\tilde{S}$. But σ^i projects to the pseudo-Anosov mapping class represented by the map (1.1). This is impossible. Suppose that σ is reduced by a simple loop c on S which is peripheral on $\tilde{S}$. Recall that $\hat{f} = g\hat{f}g^{-1}$. We obtain $\sigma = g^* \circ \sigma \circ g^{*-1}$. This implies that σ is also reduced by a unique loop $g^*(c)$. It follows that $g^*(c) = c$, which says g^* is reducible. This is also a contradiction.

If $k = -1$, then we have $g = \hat{f}^2 g \hat{f}^{-2}$ instead of (2.2). That is, $\hat{f}^2$ commutes with g. The similar argument as above can be applied in this case.

CASE 3. $g \in G$ is a non-simple and non-essential hyperbolic element. By Theorem 2 of [10], g^* is a pure mapping class with a single component R on which g^* is pseudo-Anosov. Write $g^* = f_R$. If $g = \hat{f} g \hat{f}^{-1}$, then $f_R = \sigma \circ f_R \circ \sigma^{-1} = f_{\sigma(R)}$. We conclude that σ keeps R invariant. Since σ is reduced by only one loop c which bounds a twice punctured disk Δ, c is the only boundary of R. That is, $R = S - \Delta$. Both f_R and σ restrict to commuting mapping classes on R. By Theorem 7.5.A of [8] again, there are integers i, j such that $f_R^i = \sigma^j$. That is, σ^j projects to the trivial mapping class on $\tilde{S}$. But σ projects to the pseudo-Anosov mapping class represented by (1.1). This is also impossible. The case that $g^{-1} = \hat{f} g \hat{f}^{-1}$ can be handled in the same way. $\square$

3. Special cases

In this section, we consider those elements in mod $\tilde{S}$ that come from some special mapping classes on $\tilde{S}$. We assume that $\tilde{S}$ contains some punctures.

For $i = 1, 2$, let $\hat{\alpha}_i \subset \mathbb{H}$ be a geodesic with $\varrho(\hat{\alpha}_i) = \tilde{\alpha}_i$, where $\tilde{\alpha}_i$ are filling simple closed geodesics on $\tilde{S}$ as introduced in Section 1. Let D_i, D_i' be the components of $\mathbb{H} - \hat{\alpha}_i$. The Dehn twist $t_{\tilde{\alpha}_i}$ can be lifted to a quasiconformal mapping τ_i of $\mathbb{H}$ with respect to D_i. The construction is as follows. Let $g_i \in G$ be the primitive simple hyperbolic element keeping both D_i and D_i' invariant. Throughout we assume that g_i is oriented as shown in Fig. 1.

In the figure, the arrow on $\hat{\alpha}_i$ indicates the orientation of g_i that points from the repelling fixed point to the attracting fixed point of g_i. We take an earthquake g_i-shift on D_i and leave D_i' fixed. Then we define $\tau_i : \mathbb{H} \to \mathbb{H}$ via G-invariance, which gives rise to a collection $\mathcal{U}_i$ of layered half planes in $\mathbb{H}$ in a partial order. In Fig. 1, the arrow underneath $\hat{\alpha}_i$ points to the direction of the motion of τ_i on D_i.

There are infinitely many disjoint maximal elements $D_i(j)$ of $\mathcal{U}_i$ each of which is invariant under τ_i (D_i is just one of them). Recall that H_i is defined as in (1.2). From the definition, the restriction $\tau_i|_{H_i} = \mathrm{id}$. Since τ_i defined in this way is quasiconformal, it extends continuously to act on $\overline{\mathbb{H}}$. In particular, $\tau_i|_{\hat{\mathbb{R}}}$ is quasisymmetric if we normalize so that "∞" lies outside of all maximal elements of $\mathcal{U}_i$.

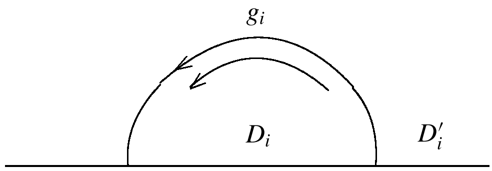

Fig. 1.

Lemma 3.1. *Let $x \in \hat{\mathbb{R}}$ be fixed by a parabolic element of G. There are only finitely many elements of $\mathcal{U}_i$ that cover x.*

Proof. Let $D_i(0)$ be the maximal element of $\mathcal{U}_i$ that covers x. Pick a point z in H_i, and draw a geodesic ray $\hat{\Gamma}$ connecting z to x. $\hat{\Gamma}$ projects to a geodesic ray $\tilde{\Gamma}$ connecting $\varrho(z)$ to the puncture b of $\tilde{S}$ corresponding to x.

Let $\tilde{U}$ be a punctured disk around b. $\tilde{U}$ is chosen so small that is disjoint from $\tilde{\alpha}_i$ and $\tilde{U} \cap \tilde{\Gamma}$ is a single ray. Observe that $\tilde{\Gamma} \cap (\tilde{S} - \tilde{U})$ has finite hyperbolic length. It intersects $\tilde{\alpha}_i$ only finitely many times. So $\tilde{\Gamma}$ intersects $\tilde{\alpha}_i$ only finitely many times. This implies that $\hat{\Gamma}$ meets finite number of elements $D_i(0), \ldots, D_i(m)$ of $\mathcal{U}_i$ and the horodisk U at x that corresponds to $\tilde{U}$ is included in all $D_i(j)$. $\square$

For each parabolic fixed point $z \in \hat{\mathbb{R}}$, by Lemma 3.1, let $D_i(0) \supset D_i(1) \supset \cdots \supset D_i(m)$, $D_i(j) \in \mathcal{U}_i$, cover z. Let g_{ik}, $k = 0, 1, \ldots, m$, denote the primitive simple hyperbolic elements of G that keep $D_i(j)$ invariant and take the same orientation as g_{i0} (here we refer to Fig. 1 for $D_i = D_i(0)$ and $g_i = g_{i0}$). Then $\tau_i(z)$ is defined as

$$\tau_i(z) = g_{i0} g_{i1} \cdots g_{im}(z). \tag{3.1}$$

For each $z \in \hat{\mathbb{R}}$ not covered by any element of $\mathcal{U}_i$, $\tau_i(z) = z$. Let $x \in \hat{\mathbb{R}}$ be arbitrary. Since the set of parabolic fixed points of G is dense on $\hat{\mathbb{R}}$, we choose a sequence $\{x_j\}$ of parabolic fixed points so that $x_j \to x$. We see that

$$\tau_i(x) = \lim_{j \to \infty} \tau_i(x_j). \tag{3.2}$$

We summarize some additional properties of τ_i which are derived from the definition:

(1) If τ_i is with respect to D_i, then $\tau'_i = g_i^{-1} \tau_i = \tau_i g_i^{-1}$ is also a lift of $t_{\tilde{\alpha}_i}$ and τ'_i is with respect to D'_i.

(2) For any point x covered by a maximal element D_i of $\mathcal{U}_i$, $\tau_i^m(x)$ and $\tau_i^{-m}(x)$, $m \to \infty$, tend to the attracting and repelling fixed point of g_{i0}, respectively, and if g_{i0} is oriented as in Fig. 1, we have

$$\tau_i^{m+1}(x) < \tau_i^m(x), \quad \text{for} \quad m \geq 1.$$

(3) For any $x, y \in \hat{\mathbb{R}}$, $x \leq y$ implies $\tau_i(x) \leq \tau_i(y)$, and $\tau_i(x) = x$ if and only if x does not lie in the interior of any maximal element of $\mathcal{U}_i$.
(4) For each hyperbolic element $h \in G$ and each maximal element D_i of $\mathcal{U}_i$, $h(D_i) \in \mathcal{U}_i$ if the repelling fixed point of h does not lie in D_i; and $h(\mathbb{H} - D_i) \in \mathcal{U}_i$ if D_i covers the repelling but not the attracting fixed point of h. Furthermore, $h(D_i)$ is also a maximal element of $\mathcal{U}_i$ if D_i does not contain any fixed points of h.

We observe that the map τ_i determines a fiber-preserving automorphism $[\tau_i]$ of the Bers fiber space $F(\tilde{S})$. Let $\Delta \subset \mathbb{H}$ denote a fundamental region of G such that $\Delta \cap \hat{\alpha}_i \neq \emptyset$. Let $\hat{a} = \varrho^{-1}(a) \cap \Delta$. Since a Bers isomorphism φ: $F(\tilde{S}) \to T(\tilde{S} - \{a\})$ is defined by picking up any point $a \in \tilde{S}$, we may choose a point $a \in \tilde{S}$ so that $\hat{a} \in D_i'$. Under the isomorphism φ we then obtain a mapping class $\varphi^*([\tau_i]) \in \mathrm{Mod}_S^a$.

Lemma 3.2. (1) $\varphi^*([\tau_i])$ *is represented by the Dehn twist* t_{α_i}, *where* α_i *is homotopic to* $\tilde{\alpha}_i$ *on* $\tilde{S}$ *as* a *is filled in.*
(2) *For any simple closed geodesic* α_i *on* S, *let* $\tilde{\alpha}_i \subset \tilde{S}$ *be the geodesic homotopic to* α_i *on* $\tilde{S}$. *Then a geodesic* $\hat{\alpha}_i$ *in* $\{\varrho^{-1}(\tilde{\alpha}_i)\}$, *and thus a component* D_i *of* $\mathbb{H} - \hat{\alpha}_i$ *can be selected so that the map* τ_i *with respect to* D_i *satisfies the condition that* $\varphi^*([\tau_i]) = t_{\alpha_i}$.

Proof. For simplicity, we denote $\tau = \tau_i$ and $g = g_i$. Since $\varphi^*([\tau])$ is a mapping class, we denote by $f: S \to S$ the map that represents $\varphi^*([\tau])$. By construction, τ commutes with g. Thus $\varphi^*([\tau])$ commutes with $g^* = \varphi^*(g)$. By Theorem 2 of [10] or Theorem 2 of [14], $g^* = \varphi^*(g)$ is represented by $t_\beta^{-1} \circ t_\alpha$, where $\{\alpha, \beta\}$ bounds a punctured cylinder P containing a. we obtain

$$f \circ ({t_\beta}^{-1} \circ t_\alpha) \circ f^{-1} = {t_\beta}^{-1} \circ t_\alpha.$$

That is,

$$t_{f(\beta)}{}^{-1} \circ t_{f(\alpha)} = {t_\beta}^{-1} \circ t_\alpha. \tag{3.3}$$

From (3.3) we conclude that $f(P) = P$, i.e., f keeps $\{\alpha, \beta\}$ invariant.

Let $\tilde{f}: \tilde{S} \to \tilde{S}$ be the map isotopic to f as a is filled in. Since P is a cylinder containing a, it projects to a simple geodesic $\tilde{\alpha}$. $\tilde{\alpha}$ is the projection of the axis of g. It follows that $\tilde{f}$ keeps $\tilde{\alpha}$ invariant. Thus it defines a map $\tilde{f}_0$ on $\tilde{S} - \{\tilde{\alpha}\}$.

On the other hand, by (2.1), we know that f projects to the Dehn twist along $\tilde{\alpha}$. So $\tilde{f} = t_{\tilde{\alpha}}$. That is, $\tilde{f}_0 = \mathrm{id}$, which says that $f|_{S-P}$ is isotopic to the identity. In particular, this implies that $f(\alpha) = \alpha$ and $f(\beta) = \beta$. Hence, f can be written as $t_\beta^{-k+1} \circ t_\alpha^k$, where we may assume that $k \geq 1$.

To show that $k = 1$, we consider $\tau' = g^{-1}\tau$. By Property (1), τ' is with respect to D', and is also a lift of $t_{\tilde{\alpha}}$. By the same argument as above, $\varphi^*([\tau'])$ is represented by $t_\beta^m \circ t_\alpha^{-m+1}$ for $m \geq 1$. Thus $\varphi^*([\tau'^{-1}])$ is represented by $t_\beta^{-m} \circ t_\alpha^{m-1}$. Since $\tau'^{-1}\tau$ coincides with g on $\partial\mathbb{R}$, $\varphi^*([\tau'^{-1}\tau])$ is represented by $t_\beta^{-m-k+1} \circ t_\alpha^{m+k-1}$. Once again, by Theorem 2 of [10] or Theorem 2 of [14], $\varphi^*(g)$ is represented by $t_\beta^{-1} \circ t_\alpha$. We see that

$$t_\beta^{-m-k+1} \circ t_\alpha^{m+k-1} = t_\beta^{-1} \circ t_\alpha^1.$$

It follows that $m + k - 1 = 1$. Since $m \geq 1$ and $k \geq 1$, we conclude that $m = k = 1$. This proves (1).

From (1), we see that either $\varphi^*([\tau])$ or $\varphi^*([g^{-1}\tau])$ is represented by a Dehn twist t_{α_1} along a simple closed geodesic α_1 on S for which there is an element $\zeta \in \mathcal{F}$ such that $\zeta(\alpha_1) = \alpha$. Since $\varphi^*(G) = \mathcal{F}$, there is an element $h \in G$ such that $\varphi^*(h) = \zeta$. Now it is easy to see that $h(\hat{\alpha}) \subset \mathbb{H}$ is the desired geodesic, and thus either $h\tau h^{-1}$ or $hg^{-1}\tau h^{-1}$ is the desired lift of $t_{\tilde{\alpha}}$. This proves (2). $\square$

Lemma 3.3. *Let τ_1 and τ_2 be any lifts of $t_{\tilde{\alpha}_1}$ and $t_{\tilde{\alpha}_2}$ with $H_1 \cap H_2 = \emptyset$. Then for sufficiently large integers r, s, the map $\tau_2^r \tau_1^{-s}$ does not fix any parabolic fixed points of G.*

Proof. Suppose that $x \in \partial\mathbb{H} = \hat{\mathbb{R}}$ is a parabolic fixed point that is fixed by $\tau_2^r \tau_1^{-s}$. There is a parabolic element $T \in G$ so that $T(x) = x$.

Notice that H_i is closed. Thus $H_1 \cap H_2$ is also closed. If x lies outside of any maximal elements of $\mathcal{U}_1$ and $\mathcal{U}_2$ (in the sense that x does not belong to any closed half plane in $\mathcal{U}_1$ and $\mathcal{U}_2$), then x lies in the closure $(H_1 \cap H_2) \cap \hat{\mathbb{R}}$. There is a fundamental region $\Delta \subset \mathbb{H}$ that takes x as a cusp and has an overlap with $H_1 \cap H_2$. This in particular implies that $H_1 \cap H_2$ is not empty. This is a contradiction.

Assume that $x \in D_2$ for a maximal element D_2 of $\mathcal{U}_2$. If x does not lie in any maximal elements of $\mathcal{U}_1$, then $\tau_1^{-s}(x) = x$. Thus $\tau_2^r \tau_1^{-s}(x) = \tau_2^r(x) \neq x$. If $x \in D_1$ for a maximal element D_1 of $\mathcal{U}_1$, but not lie in any maximal elements of $\mathcal{U}_2$, we use the same argument to prove that $(\tau_2^r \tau_1^{-s})^{-1}(x) \neq x$.

For any half plane D in $\mathcal{U}_1$ or $\mathcal{U}_2$, let ∂D denote the boundary of D in $\mathbb{H}$. Let $h \in G$ be a simple hyperbolic element so that $h(D) = D$. If x is a vertex of D, i.e., $x \in \partial D \cap \mathbb{H}$, then T and h would share a common fixed point x, and this would contradict to that G is discrete.

By the above discussion, we are left with the possibility that $x \in D_2 \cap D_1$ for a maximal element D_1 of $\mathcal{U}_1$ and a maximal element D_2 of $\mathcal{U}_2$. If ∂D_2 intersects ∂D_1, the intersection point is in $\mathbb{H}$. It follows that $H_1 \cap H_2 \neq \emptyset$, contradicting the hypothesis.

Now we assume that $D_2 \subset D_1$. Let $g_i \in G$ be hyperbolic such that $g_i(D_i) = D_i$, $i = 1, 2$. Since $x \in D_2$, from (3.2),

$$g_1^{-s}(x) \leq \tau_1^{-s}(x). \tag{3.4}$$

From Property (4), we know that $g_1^{-s}(D_2) \in \mathcal{U}_2$ is also maximal, and Property (2) says that τ_2 keeps $g_1^{-s}(D_2)$ invariant. Thus $\tau_2^r g_1^{-s}(x) \in g_1^{-s}(D_2)$. Since ∂D_2 projects to a simple closed geodesic $\tilde{\alpha}_2$, $D_2 \cap g_1^{-s}(D_2) = \emptyset$. We assert that

$$x < \tau_2^r g_1^{-s}(x). \tag{3.5}$$

By Property (3), τ_i is monotonic. It follows from (3.4) and (3.5) that

$$x < \tau_2^r g_1^{-s}(x) \leq \tau_2^r \tau_1^{-s}(x).$$

In particular, $\tau_2^r \tau_1^{-s}(x) \neq x$. If $x \in D_1 \cap D_2$ and $D_1 \subset D_2$, by using the same argument above, we conclude that the inverse $(\tau_2^r \tau_1^{-s})^{-1}$ does not fix any parabolic fixed point of G, which is equivalent to that $\tau_2^r \tau_1^{-s}(x) \neq x$. Finally, we assume that $x \in D_1 \cap D_2$ where $\partial D_1 \cap \partial D_2 = \emptyset$ and neither $D_1 \subset D_2$ nor $D_2 \subset D_1$. In this case, we can use Lemma 3.1 to prove that for large integers r and s, $\tau_2^r \tau_1^{-s}(s) \in D_1 \cap D_2$ is covered by more elements of $\mathcal{U}_1$ than x is, from which we derive $\tau_2^r \tau_1^{-s}(x) \neq x$. Details are omitted. See [20] for more information. The lemma is proved. □

4. Proof of Theorem 1.1

For the sufficient condition, suppose that $H_1 \cap H_2 \neq \emptyset$. Choose a point $\hat{z} \in H_1 \cap H_2$ and let $z = \varrho(\hat{z})$, where $\varrho : \mathbb{H} \to \tilde{S}$ is the universal covering. Then z belongs to a component of $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$. By hypothesis, $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ consists of once punctured disks $\{Q_1, \ldots, Q_k\}$.

Assume that $z \in Q_1$, say. Let x_0 be the puncture of Q_1. In Q_1, we connect z and the puncture x_0 by an arc γ that avoids $\tilde{\alpha}_1$ and $\tilde{\alpha}_2$. Obviously, γ can be lifted to an arc $\hat{\gamma} \subset \mathbb{H}$ connecting $\hat{z}$ and a parabolic cusp $\hat{x}_0$. Since $\gamma \subset Q_1$, $\hat{\gamma} \subset H_1 \cap H_2$. But $\zeta|_{H_1 \cap H_2} = \text{id}$. It follows that $\zeta|_{\hat{\gamma}} = \text{id}$. Since ζ has a continuous extension to $\overline{\mathbb{H}}$, we see that $\zeta(\hat{x}_0) = \hat{x}_0$. Therefore, according to the discussion in Section 2, $\varphi^*([\zeta])$ is reducible by a single reduced loop on S that is the boundary of a twice punctured disk enclosing a.

For the necessary condition, we assume that $H_1 \cap H_2 = \emptyset$. By Lemma 3.2, the mapping class $\varphi^*([\tau_i])$ is induced by the Dehn twist t_{α_i}, where α_i is a geodesic on S homotopic to $\tilde{\alpha}_i$ on $\tilde{S}$. It follows that $\varphi^*([\zeta]) \in \text{Mod}_S^a$ is represented by (1.3).

From Lemma 3.3, $\tau_2^r \tau_1^{-s}$ does not fix any parabolic fixed point of G for large r and s, which says that if $\varphi^*([\tau_2^r \tau_1^{-s}])$ is reducible, it must be reduced by a loop c that is also non-trivial on $\tilde{S}$. It follows that $\varphi^*([\tau_2^r \tau_1^{-s}])$ projects to a reducible map F_0 that is reduced by $\tilde{c}$. But since τ_i is a lift of $t_{\tilde{\alpha}_i}$, F_0 is isotopic to $t_{\tilde{\alpha}_2}^r \circ t_{\tilde{\alpha}_1}^{-s}$. By hypothesis, $\{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ fills $\tilde{S}$. Thus $t_{\tilde{\alpha}_2}^r \circ t_{\tilde{\alpha}_2}^{-s}$ is isotopic to a pseudo-Anosov map. It follows that F_0 can not be reducible. This is a contradiction.

We conclude that $\varphi^*([\tau_2^r \tau_1^{-s}])$ is pseudo-Anosov. Hence $\{\alpha_1, \alpha_2\}$ fills S. Now by the Theorem of [17, 2, 12], for any integers $N, r_i, s_i \in \mathbb{Z}^+$,

$$\zeta^* = \prod_{i=1}^{N} t_{\alpha_2}^{r_i} \circ t_{\alpha_1}^{-s_i}$$

is pseudo-Anosov. This completes the proof of Theorem 1.1. □

5. Proof of Theorem 1.2

Let $\{\tau_1, \tau_2\}$ be such that $H_1 \cap H_2 \neq \emptyset$. Let $D_i \in \mathcal{U}_i$ be maximal half planes such that $\partial D_1 \cap \partial D_2 \neq \emptyset$. Let $D_1' \in \mathcal{U}_1$ be another maximal element that is disjoint from both D_1 and D_2. From Lemma 5.3.8 of Beardon [3], we can choose a hyperbolic element $h \in G$ whose repelling fixed point lies in D_2 and whose attracting fixed point lies in D_1'. For $j \geq 1$, $h^j(D_2)$ is a maximal half plane for $h^j \tau_2 h^{-j}$ and the complement of $h^j(D_2)$ is contained in D_1'. Thus for large $j \geq 1$, all pairs $(\tau_1,\ h^j \tau_2 h^{-j})$ satisfies the condition of Theorem 1.1. From the theorem we conclude that

$$\varphi^*\left(\left[\prod_i h^j \tau_2^{r_i} h^{-j} \tau_1^{-s_i}\right]\right) \tag{5.1}$$

is a pseudo-Anosov mapping class projecting to the class represented by $\tilde{\zeta}$. This is equivalent to that

$$\prod_{i=1}^{N} h^j \tau_2^{r_i} h^{-j} \tau_1^{-s_i} \tag{5.2}$$

does not fix any parabolic fixed point of G on $\hat{\mathbb{R}}$.

REMARK 5.1. To understand the mapping class (5.1) in topological term, we notice that the map that represents (5.1) is generated by the two geodesics α_1 and $f(\alpha_2)$ where $f \in \mathcal{F}$ is determined by an element h of G. To see how the curve α_2 is altered to $f(\alpha_2)$, we refer to Theorem 2 of Kra [10]. For example, if h is a simple hyperbolic, then $f = \varphi^*(h)^j$ is a multiple of a spin map, written as $t_c^j \circ t_{c_0}^{-j}$, where both c and c_0 are homotopic to $\tilde{c}$, the projection of the axis of h. if h is parabolic, then f is an ordinary power of the Dehn twist along the boundary of a twice punctured disk on S enclosing a.

Note that τ_i determines an isomorphism $\chi_i : G \to G$ that is defined by

$$\tau_i h = \chi_i(h)\tau_i. \tag{5.3}$$

It follows from (5.3) that (5.2) can be written as $g_j \zeta$ for $g_j \in G$.

We claim that for sufficiently large j,

$$g_{j+1}\zeta \neq g_j \zeta.$$

Indeed, as discussed above, for large j, the complement of $D_3 = h^j(D_2)$ is contained in D_1'. This implies that $D_3 \supset D_1$. Let $D_4 = h(D_3)$. We have $D_4 \supset D_3 \supset D_1$ and D_4 is

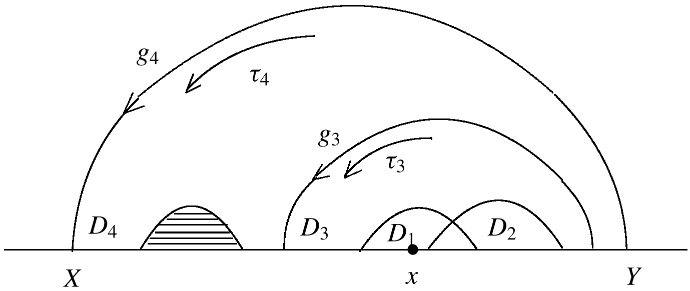

Fig. 2.

a maximal element for the map $h^{j+1}\tau_2 h^{-j-1}$. For simplicity, we denote $\tau_3 = h_j\tau_2 h_j^{-1}$ and $\tau_4 = h_{j+1}\tau_2 h_{j+1}^{-1}$. Let

$$\omega_j = \prod_{i=1}^{N} (\tau_3^{r_i}\tau_1^{-s_i})$$

and

$$\omega_{j+1} = \prod_{i=1}^{N} (\tau_4^{r_i}\tau_1^{-s_i}).$$

Pick any point $x \in D_1 \cap \hat{\mathbb{R}}$. Since D_1 is maximal for τ_1, $y = \tau_1^{-s_1}(x) \in D_1 \cap \hat{\mathbb{R}}$. Note also that D_3 is maximal for τ_3. Since $\tilde{\alpha}_1$ is simple, by Property (3), τ_3 is monotonic on $D_3 \cap \hat{\mathbb{R}}$. By (3.2), we conclude that for all positive integers r_1, $\tau_3^{r_1}(y) \in (D_3 - D_1) \cap \hat{\mathbb{R}}$. By induction process, one can show that $\omega_j(x) \in (D_3 - D_1) \cap \hat{\mathbb{R}}$.

On the other hand, since $D_4 \supset D_3$ is a maximal element for τ_4 and hence is invariant under the action of τ_4. Denote $\partial D_4 \cap \hat{\mathbb{R}} = \{X, Y\}$ as shown in Fig. 2. Let $g_4 \in G$ be the element that keeps D_4 invariant and takes the same orientation as in Fig. 1.

Now $y = \tau_1^{-s_1}(x) \in D_1 \cap \hat{\mathbb{R}} \subset D_3 \cap \hat{\mathbb{R}}$. Hence we get that $g_4^{r_1}(y) \in g_4^{r_1}(D_3)$. Observe that $g_4^{r_1}(D_3)$ is disjoint from D_3 (the shaded region in Fig. 2).

It is obvious that $g_4^{r_1}(D_3)$ contains $g_4^{r_1}(D_1)$. By Property (4), $g_4^{r_1}(D_1)$ is a maximal element of $\mathcal{U}_1$, which means that $\tau_1^{-s_2}$ keeps $g_4^{r_1}(D_1)$ invariant. It follows that

$$\tau_1^{-s_2} g_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}. \tag{5.4}$$

On the other hand, (3.2) and (3.2) along with Property (2) yield

$$X < \tau_4^{r_1}(y) < g_4^{r_1}(y).$$

Hence by Property (3) again, we obtain

$$X < \tau_1^{-s_2}\tau_4^{r_1}(y) < \tau_1^{-s_2} g_4^{r_1}(y).$$

It follows from (5.4) that $\tau_1^{-s_2}\tau_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$ and that $\tau_4^{r_2}\tau_1^{-s_2}\tau_4^{r_1}(y) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$. That is, $\tau_4^{r_2}\tau_1^{-s_2}\tau_4^{r_1}\tau_1^{-s_1}(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$.

By induction process, we can show that

$$(\tau_4^{r_N}\tau_1^{-s_N}) \cdots (\tau_4^{r_2}\tau_1^{-s_2})(\tau_4^{r_1}\tau_1^{-s_1})(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}.$$

That is, $\omega_{j+1}(x) \in (D_4 - D_3) \cap \hat{\mathbb{R}}$. In particular, we conclude that $\omega_{j+1}(x) \neq \omega_j(x)$. Similar argument yields that $\omega_{j+k}(x) \neq \omega_{j+l}(x)$ for $k \neq l$ and $k, l \geq 0$. This completes the proof of Theorem 1.2. □

6. Proof of Theorem 1.3

From Theorem 1.2, there are infinitely many elements $h_j \in G$ so that $\omega_j = h_j\zeta$ do not fix any fixed points of G. Hence all $\omega_j^* = \varphi^*([\omega_j])$ are pseudo-Anosov mapping class of S projecting to (1.1). From the construction, each ω_j^* is induced by a map with from (1.3). By the argument of Theorem 1.2, there are infinitely many distinct elements in the sequence $\{\omega_j^*\}$. This proves (1)–(3) of Theorem 1.3.

To prove (4)–(6) of Theorem 1.3, we choose a point in the Teichimüller space $T(\tilde{S})$ represented F-minimal surface denoted by $\tilde{S}$. Let G be the Fuchsian group so that $\mathbb{H}/G = \tilde{S}$. Let z_0 be a zero of ϕ (ϕ is defined by the pseudo-Anosov map F), so that $F(z_0) = z_0$ and $H_t(z_0)$, $0 \leq t \leq 1$, is trivial if z_0 is not a puncture. It may or may not be a puncture of $\tilde{S}$.

Associated to each $\hat{z}_0 \in \overline{\mathbb{H}}$ with $\varrho(\hat{z}_0) = z_0$, there is a map ζ defined by (1.3). To see that $\varphi^*([\zeta])$ is pseudo-Anosov if z_0 is a non-puncture zero, we refer to [18] and outline the proof as follows. Let l denote the (unique) Teichmüller geodesic in $T(\tilde{S})$ determined by $\tilde{\zeta}$ defined as (1.1). Let $\hat{l} \subset F(\tilde{S})$ be a lift of l defined by

$$\hat{l} = \{([t\mu], w^{t\mu}(\hat{z}_0)),\ t \in (-1, 1)\} \subset F(\tilde{S}).$$

Clearly, $\hat{l}$ is a line in $F(\tilde{S})$ passing through $\hat{z}_0$. By [10], $\varphi(\hat{l}) \subset T(S)$ is a Teichmüller geodesic in $T(S)$. By the argument of Proposition 3 and Corollary 2 of [10], $\hat{l}$ is invariant under a lift $\hat{F}$ of F with $\hat{F}(\hat{z}_0) = \hat{z}_0$. From the assumption, $H_t(z_0)$ is a trivial loop. This implies that $\zeta|_{\hat{\mathbb{R}}} = \hat{F}|_{\hat{\mathbb{R}}}$. Therefore, $\varphi(\hat{l})$ is invariant under the action of $\varphi^*([\zeta])$. So by Bers [5], $\varphi^*([\zeta])$ is pseudo-Anosov.

We need to prove that all ω_n^*, for large j, are not conjugate to $\varphi^*([\zeta])$.

Suppose that for some $j \geq 1$, there is $h_0 \in G$ such that

$$\prod_{i=1}^{N}(h^j\tau_2h^{-j})^{r_i}\tau_1^{-s_i} = h_0\zeta h_0^{-1}, \tag{6.1}$$

where ζ is defined in (1.4). Note that $\{\tau_1, \tau_2\}$ possesses the property that $H_1 \cap H_2 \neq \emptyset$. If $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ consists of once punctured disks only, ζ fixes a parabolic fixed point of

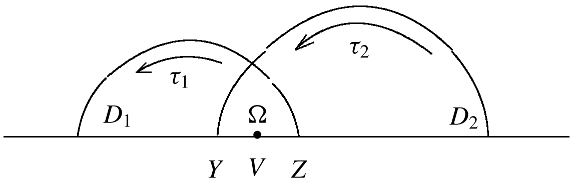

Fig. 3.

G on $\mathbb{R}$. However, if $\tilde{S} - \{\tilde{\alpha}_1, \tilde{\alpha}_2\}$ contains some disk components, for some ζ defined as (1.4), the existence of fixed points of ζ that are also fixed by elements of G is not clear. In this case, we use a different approach as follows.

For $i = 1, 2$, let D_i be maximal elements of $\mathcal{U}_i$ so that ∂D_1 intersects with ∂D_2. Let $\Omega = D_1 \cap D_2$ and $\Lambda = \Omega \cap \hat{\mathbb{R}}$. See Fig. 3.

Now we consider the action of ζ on Λ. The endpoint Y is moved to the right while the point Z is moved to the left. Since the action of ζ on Λ is continuous, there is a point $V \in \Lambda$ so that $\zeta(V) = V$ (according to Lemma 2.2, V is not a hyperbolic fixed point of G, but it could be a parabolic fixed point of G). Thus $h_0 \zeta h_0^{-1}$ fixes $h_0(V) \in \hat{\mathbb{R}}$.

On the other hand, $\{\tau_1, h^j \tau_2 h^{-j}\}$ has the property that $H_1 \cap H_2 = \emptyset$. By the same argument of Lemma 3.3,

$$\prod_{i=1}^{N} (h^j \tau_2 h^{-j})^{r_i} \tau_1^{-s_i}$$

does not fix any point on $\hat{\mathbb{R}}$. This contradicts to (6.1). The same methods can be used to prove that all ω_j^* lie in different conjugacy classes. Details are omitted. □

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