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INDUCING CHARACTERS OF PRIME POWER DEGREE

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1. Introduction

Let G be a (finite) group and χ be an irreducible character for G. We consider the set of primitive characters that induce χ. In general, there is very little that can be said about this set other than the degrees of these characters must divide $\chi(1)$. When $\chi(1)$ is a power of some prime, this set often has more structure. For example, if p is an odd prime, G is p–solvable, and χ is monomial with $\chi(1)$ a power of p, then every primitive character inducing χ must be linear (Theorem 10.1 of [7]). Given any prime p, a p–solvable group G of p–length 1, and a character $\chi \in \mathrm{Irr}(G)$ where $\chi(1)$ is a power of p, it has been shown that every primitive character inducing χ has the same degree (Theorem A of [8]). It is easy to find examples of p–solvable groups that do not have p–length 1, but do have characters of prime power degree that are induced by primitive characters of different degrees. For example, $\mathrm{GL}_2(3)$ has a character of degree 4 that is induced by a linear character and a primitive character of degree 2. In [8], we construct an example where p is odd. The purpose of this note is to prove that such examples cannot occur for characters of p–power degree where this degree is "small." With this in mind, we have the following theorem.

Theorem A. *Let p be an odd prime, and let G be a p–solvable group. Let $\chi \in \mathrm{Irr}(G)$ be a character of p–power degree less than or equal to p^p. Then every primitive character inducing χ has the same degree.*

Note that the monomial character of degree 4 in $\mathrm{GL}_2(3)$ that is also induced by a primitive character of degree 2 shows that Theorem A is not necessarily true when we do not assume that p is odd. In [8], we find a p-solvable group that has character of degree p^{p+1} that is induced by primitive characters of different degrees where p is an odd prime. (The example in [8] has $p = 3$, but it is not difficult to find similar examples for many other primes.)

Using our methods, we also obtain an analogue to a result of Dade. The main theorem of [1] considers the following situation: G is a p–solvable group for some odd prime p, the character $\chi \in \mathrm{Irr}(G)$ is monomial and has p–power degree, and N is a subnormal subgroup. In this situation, he proved that if θ is an irreducible constituent of χ_N, then θ is monomial. In other words, he proved that θ and χ are induced by primitive characters of the same degree. We now have a similar result without assuming that χ is monomial.

Theorem B. *Let p be an odd prime, and let G be a p–solvable group. Let $\chi \in \mathrm{Irr}(G)$ be a character of p–power degree less than or equal to p^p. Suppose N is a subnormal subgroup of G and θ is an irreducible constituent of χ_N. Then the degree of a primitive character inducing θ divides the degree of a primitive character inducing χ.*

We would like to thank the referee for his careful reading of this paper and for the considerably simpler proofs of Lemmas 2.1 and 2.2.

2. Anisotropic modules.

Our proof of Theorem A models Isaacs' proof of super-monomiality in [7]. That proof relied on a very difficult result of Dade's regarding anisotropic modules (see [1]). In our result, we also need to examine the structure of anisotropic modules. We begin by outlining the theory of anisotropic modules that was developed by Dade in [1] and Isaacs in [5]. Throughout this discussion, $\mathcal{F}$ will be a finite field of characteristic p for some odd prime p, and G will be a p–solvable group. Given an $\mathcal{F}[G]$–module V, we often will associate a bilinear form $\langle \cdot, \cdot \rangle : V \times V \to \mathcal{F}$. (That is, $\langle \cdot, \cdot \rangle$ is an $\mathcal{F}$–linear transformation in each coordinate). We say that $\langle \cdot, \cdot \rangle$ is *nondegenerate* if the only element $v \in V$ with $\langle v, V \rangle = 0$ is $v = 0$. It is *alternating* if $\langle u, v \rangle = -\langle v, u \rangle$ for all elements $u, v \in V$. It is called *G–invariant* if $\langle u{\cdot}g, v{\cdot}g \rangle = \langle u, v \rangle$ for all elements $u, v \in V$ and $g \in G$. A finite dimensional module V is a *symplectic $\mathcal{F}[G]$–module* if it has a nondegenerate G–invariant alternating bilinear form. If U is a subspace of the symplectic $\mathcal{F}[G]$–module V, the *perpendicular subspace* of U with respect to $\langle \cdot, \cdot \rangle$ is $U^{\perp} = \{v \in V \mid \langle U, v \rangle = 0\}$. It is easy to see if U is an $\mathcal{F}[G]$–submodule then $U^{\perp}$ is also a submodule. We call U *isotropic* when $U \subseteq U^{\perp}$, and we define V to be *anisotropic* if 0 is its only isotropic submodule.

Modules of this form arise in character theory in the following manner. Let N and M be normal subgroups of a group G such that $N \subseteq M$ and M/N is an elementary abelian p–group for some prime p. We can view M/N as a module for G with coefficients in the prime field $\mathrm{GF}(p)$. (Note that the binary operation here is multiplication, instead of addition which is the usual operation for modules. Since the multiplication is commutative, this will not cause a problem.) If there is a G–invariant character $\varphi \in \mathrm{Irr}(N)$, then this module has an alternating G–invariant bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_{\varphi}$. This bilinear form has been constructed in a number of different places, but we will be using the definition found in [3]. It is proved that $\langle\langle \cdot, \cdot \rangle\rangle_{\varphi}$ is nondegenerate on M/N if and only if φ is fully ramified with respect to M/N, and M/N is isotropic with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\varphi}$ if and only if φ extends to M. In particular, the section M/N is anisotropic with respect to $\langle\langle \cdot, \cdot \rangle\rangle_{\varphi}$ as a module for G if and only if there is no normal subgroup K of G with $N < K < M$ where φ extends to K. When N is central in G and φ is the constituent of a character χ whose restriction to M is faithful, M/N will be anisotropic as a module for G if and only if there is no abelian subgroup of M that is normal in G and contains N as a proper subgroup. Similarly, if $N = \mathbf{Z}(M)$ and every abelian subgroup of M that is normal in G is central in G, then M/N is anisotropic as a module for G.

In order to extend Isaacs' results about super-monomiality, we need an analogue of Dade's powerful result about hyperbolic modules (Theorem 3.2 of [1]). In particular, we want a result that says: given an odd prime p, a p–solvable group G, a finite field $\mathcal{F}$ whose characteristic is p, a subgroup $H \subseteq G$ with index that is a power of p, and an anisotropic $\mathcal{F}[G]$–module V, then the restriction of V to an $\mathcal{F}[H]$–module is anisotropic. However, this is not true in general, but we will prove that it is true under the condition that p does not divide the degree of any irreducible Brauer character that is a constituent of the Brauer character afforded by V.

We consider an anisotropic $\mathcal{F}[G]$–module V. Now, we know that $\mathcal{F}$ has an algebraic closure $\mathcal{E}$ and that V determines an $\mathcal{E}[G]$–module $V \otimes \mathcal{E}$ (see Chapter 9 of [4]). Also, we know that $V \otimes \mathcal{E}$ affords a Brauer character φ of G (see Chapter 15 of [4]), and we say that φ is the Brauer character afforded by V. We prove if p does not divide the degree of any irreducible constituent of φ, then the restriction of V to H (written V_H) is an anisotropic module for H. We begin by looking at the restriction to subgroups with p–power index of modules that afford Brauer characters whose irreducible constituents have degrees not divisible by p.

Lemma 2.1. *Let p be an odd prime, G be a p–solvable group, and $\mathcal{F}$ be a finite field whose characteristic is p. Suppose that V is an irreducible $\mathcal{F}[G]$–module with the property that p does not divide the degree of any irreducible Brauer character that is a constituent of the Brauer character afforded by V. If the subgroup $H \subseteq G$ has p-power index, then V_H is an irreducible $\mathcal{F}[H]$–module. Furthermore, if U is another irreducible $\mathcal{F}[G]$–module with U_H isomorphic to V_H as $\mathcal{F}[H]$–modules, then U is isomorphic to V as $\mathcal{F}[G]$–modules.*

Proof. Because G is p–solvable, it has a unique conjugacy class of p–complements. Write Q for a p–complement of G that is contained in H. Let $\mathcal{E} \supseteq \mathcal{F}$ be a splitting field for G. Since the Schur index of V is 1, we have $V \otimes \mathcal{E} = \oplus W_i$, where the W_i are distinct irreducible $\mathcal{E}[G]$–modules. Let φ_i be the irreducible Brauer character afforded by W_i. From the hypotheses, we know that p does not divide $\varphi_i(1)$. Theorem 8.1 of [6] states that φ_i lifts to an irreducible character χ_i of G whose restriction to Q is irreducible. This implies that $(\varphi_i)_H$ is irreducible, and $(W_i)_H$ is an irreducible $\mathcal{E}[H]$–module since $(W_i)_H$ affords $(\varphi_i)_H$. Moreover, the modules $(W_i)_H$ are distinct. Because the Galois group of $\mathcal{E}$ over $\mathcal{F}$ acts transitively on $\{W_i\}$, it acts transitively on the set $\{(W_i)_H\}$. Therefore, the module $V_H = \oplus(W_i)_H$ is an irreducible $\mathcal{F}[H]$–module.

The uniqueness in the second statement comes from the uniqueness in Theorem 8.1 of [6]. In particular, that result tells us that φ_i is the unique Brauer character whose restriction is $(\varphi_i)_H$, and so, φ_i is the unique Brauer character whose restriction is afforded by $(W_i)_H$. The uniqueness of V follows from the uniqueness of these characters. □

We continue in the scenario outlined in the beginning of this section, and we now work to prove that the restriction is still anisotropic.

Lemma 2.2. *Let p be an odd prime, G be a p–solvable group and $\mathcal{F}$ a finite field of characteristic p. Suppose that V is an anisotropic $\mathcal{F}[G]$–module where p does not divide the degree of any irreducible Brauer character that is a constituent of the Brauer character afforded by V. If $H \subseteq G$ is a subgroup with p–power index, then V_H is an anisotropic $\mathcal{F}[H]$–module.*

Proof. Suppose that V_H has an isotropic $\mathcal{F}[H]$–submodule W. We know that V_H is semi-simple (see Proposition 2.1 of [1]). Thus, we may use Lemma 2.1 to find an $\mathcal{F}[G]$–submodule U of V so that $U_H = W$. Note that the restriction of the bilinear form to U is the same as the restriction of the bilinear form to W. This implies that U is an isotropic $\mathcal{F}[G]$–submodule of V which implies that $U = 0$ since V is anisotropic, and thus $W = 0$ which yields the desired result. □

This next lemma is our main application of the theory of anisotropic modules to character theory.

Lemma 2.3. *Let p be an odd prime and G be a p–solvable group. Assume that the character $\chi \in \mathrm{Irr}(G)$ has p–power degree less than or equal to p^p. Suppose that E is a p–subgroup of G and R a p'–subgroup of G so that E and ER are normal subgroups of G, $[E, R] = E$, χ_E is faithful, and every abelian subgroup of E that is normal in G is central in G. Consider a subgroup $J \subseteq G$ and a character $\lambda \in \mathrm{Irr}(J)$ where $\lambda^G = \chi$ and λ is primitive. Then $E \subseteq J$.*

Proof. Since χ has p–power degree, J has p–power index in G. Thus, J contains some p–complement of G. By replacing R with an appropriate conjugate if necessary, we may assume that $R \subseteq J$. When E is abelian, the hypotheses imply that E is central in G, and the result follows (see Problem 5.12 of [4]). Thus, we need only consider the possibility that E is not abelian. Applying Satz III.13.6 of [2], it follows that E is special (in particular, $\mathbf{Z}(E)$ is elementary abelian). As $\mathbf{Z}(E)$ is central in G, there exists a character $\varphi \in \mathrm{Irr}(\mathbf{Z}(E))$ so that $\chi_{\mathbf{Z}(E)} = \chi(1)\varphi$. Because χ_E is faithful, φ must be faithful and $\mathbf{Z}(E)$ must be cyclic. This can happen only if $\mathbf{Z}(E)$ has order p. Therefore, E is extra special. By Fitting's theorem, the fact that $E = [E, R]$ implies that $\mathbf{C}_{E/\mathbf{Z}(E)}(R) = 1$. Let $H/\mathbf{Z}(E) = \mathbf{N}_{G/\mathbf{Z}(E)}(R\mathbf{Z}(E)/\mathbf{Z}(E))$. It is not difficult to prove that $G = HE$ and $H \cap E = \mathbf{Z}(E)$, for example see Lemma 4.3 of [10]. Observe that φ is fully ramified with respect to $E/\mathbf{Z}(E)$. In particular, we have the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$ on $E/\mathbf{Z}(E)$, and using this form, we define $(J \cap E)^\perp$ to be the preimage of $((J \cap E)/\mathbf{Z}(E))^\perp$.

We would like to apply Lemma 7.3 of [9] to this situation. Thus, we must see that the hypotheses of that lemma are satisfied. It is not difficult to see that in the terminology of [9] $(G, E, \mathbf{Z}(E), \epsilon, \varphi)$ is a controlled abelian fully-ramified configuration with stabilizing complement H where ϵ is the unique irreducible constituent of φ^E. Also, $|E : \mathbf{Z}(E)|$ is a power of the odd prime p; so $|E : \mathbf{Z}(E)|$ is odd. We know that the restriction of λ to $J \cap E$ is homogeneous. The remaining hypothesis that we need to satisfy is that $JE \cap H$ is admissible. The term admissible is defined in [9] just prior to Lemma 7.3. Looking at the definition of admissible, we see that it suffices to show $R \subseteq J$. Since this is the case, we may apply Lemma 7.3 of [9] to see that $(J \cap E)^\perp \subseteq J \cap E$. It follows that $(J \cap E)^\perp/\mathbf{Z}(E)$ is a totally isotropic J–submodule of $E/\mathbf{Z}(E)$.

As we noted earlier, the fact that every abelian subgroup of E that is normal in G is central in G implies that $E/\mathbf{Z}(E)$ is an anisotropic module for G over the field of p elements with respect to the bilinear form $\langle\langle \cdot, \cdot \rangle\rangle_\varphi$. Let ϵ be the unique irreducible constituent of φ^E. Since φ is fully ramified and linear, we know that $|E : \mathbf{Z}(E)| = \epsilon(1)^2$. Let V be an irreducible submodule of $E/\mathbf{Z}(E)$ for G. By Proposition 2.1 of [1], we know that V is anisotropic. Thus, the dimension of V over the field of p elements is the even integer $2c$ which satisfies $p^{2c} \leq \epsilon(1)^2 \leq \chi(1)^2 \leq p^{2p}$. Thus, $c \leq p$ and if $c = p$, then $V = E/\mathbf{Z}(E)$. Recall that the degree of any irreducible constituent of the Brauer character afforded by V must divide the dimension of V. If $c < p$, then p does not divide the degree of any irreducible constituent of the Brauer character afforded by V. For now, we assume that $E/\mathbf{Z}(E)$ is not an irreducible module of dimension $2p$. It follows that p does not divide the degree of any irreducible constituent of the Brauer character afforded by $E/\mathbf{Z}(E)$ viewed as a module. In light of Lemma 2.2, $E/\mathbf{Z}(E)$ is anisotropic as a module for J over the field of p elements. Since $(J \cap E)^\perp/\mathbf{Z}(E)$ is totally isotropic as a module for J, we conclude that $(J \cap E)^\perp = \mathbf{Z}(E)$, and hence, $J \cap E = E$. We now have $E \subseteq J$, in this case.

We now assume that $E/\mathbf{Z}(E)$ is an irreducible module of dimension $2p$. It follows that $p^{2p} = |E : \mathbf{Z}(E)| = \epsilon(1)^2$, and $p^p = \epsilon(1) = \chi(1)$. In particular, we have that $\chi_E = \epsilon$. We claim that this forces χ to be primitive, and we obtain $J = G$ which yields the desired result. Let T be any subgroup so that there is a character $\tau \in \mathrm{Irr}(T)$ so that $\tau^G = \chi$. We know that $G = ET$ and $\epsilon = (\tau_{E \cap T})^T$ (see Problems 5.2 and 5.7 of [4]). As $\mathbf{Z}(E)$ is central in G, we have $\mathbf{Z}(E) \subseteq T$ by Problem 5.12 of [4]. Finally, since $E/\mathbf{Z}(E)$ is abelian and since E is normal, it follows that $G = ET$ normalizes $E \cap T$. On the other hand, from the fact that $E/\mathbf{Z}(E)$ irreducible as a module for G, we have $E \cap T = E$ or $E \cap T = \mathbf{Z}(E)$. Since $|G : T|$ divides $\chi(1) = p^p$ and $|E : \mathbf{Z}(E)| = p^{2p}$, we conclude that $T = G$. This forces χ to be primitive. □

3. Primitive characters inducing characters of p–power degree.

In this section, we present the argument that underlie Theorems A and B. To prove these, we combine various ideas found in [1] and [7]. The key result in this section (Theorem 3.2) mirrors Theorem 7.1 of [1]. We begin with an easy lemma that addresses a situation that arises in Theorem 3.2.

Lemma 3.1. *Let p be a prime and G be a group with a normal subgroup N and a character $\chi \in \mathrm{Irr}(G)$ with the property that $\chi(1)$ is a p–power and χ_N is faithful. Let $P = \mathbf{O}_p(N)$ and $Q = \mathbf{O}_{p'}(N)$. If all the abelian subgroups of N that are normal in G are central in G, then $\mathbf{O}_{p',p}(N) = P \times Q$ and Q is central in G.*

Proof. We begin by noting that the degrees of all the irreducible constituents of χ_Q must divide both $\chi(1)$ and $|Q|$. Since these two values are relatively prime, the constituents of χ_Q are linear. It follows that $[Q, Q] \subseteq \ker(\chi_Q) \subseteq \ker(\chi_N) = 1$ (χ_N is faithful); so Q is abelian and normal in G. By applying the hypotheses, we determine that Q is central in G. Let $Y = \mathbf{O}_{p',p}(N)$, and observe that Q and P are subgroups of Y. Use R to denote a Sylow p–subgroup of Y so that $P \subseteq R$ and $Y = QR$. Because Q is central, R must be normal in Y, and thus, R is normal in N (this uses the fact that Y is normal in N and R is characteristic in Y). Therefore, we must have $R = P$. This proves $Y = Q \times P$. □

Let G be a group and $\chi \in \mathrm{Irr}(G)$ be a character. Define $a(\chi)$ to be the smallest degree of any character inducing χ. Observe that there exists a subgroup $J \subseteq G$ and a character $\lambda \in \mathrm{Irr}(J)$ so that $\lambda^G = \chi$ and $\lambda(1) = a(\chi)$. Furthermore, we see that λ must be primitive. The next result shows in the situation of Theorems A and B that this value is preserved by the induction coming from Clifford's theorem (Theorem 6.11 of [4]). Recall that $\mathrm{GL}_2(3)$ has an monomial irreducible character of degree 4 that is induced from a primitive character of degree 2 of a normal subgroup. Thus, the hypothesis that p be odd is necessary.

Theorem 3.2. *Let p be an odd prime and G be a p–solvable group. Suppose that there is a character $\chi \in \mathrm{Irr}(G)$ with $\chi(1)$ a power of p less than or equal to p^p. Consider a normal subgroup N of G and a character $\theta \in \mathrm{Irr}(N)$ that is a constituent of χ_N. Take T to be the stabilizer of θ in G, and write $\gamma \in \mathrm{Irr}(T|\theta)$ for the Clifford correspondent of χ with respect to θ (thus, $\gamma^G = \chi$). Then $a(\gamma) = a(\chi)$.*

Proof. Assume that the theorem is false, and choose G to be a group that contradicts the theorem with $\chi(1)$ and then $|G|$ as small as possible.

STEP 1. χ is faithful.

Proof 1. Let K be the kernel of χ and $\bar{}$ be the natural homomorphism $G \to G/K$. Observe that $\chi \in \mathrm{Irr}(\bar{G})$. Because γ induces χ, we use Lemma 5.11 of [4] to see that $K \subseteq T$ and γ is a character in $\mathrm{Irr}(\bar{T})$. We also know that $K \subseteq \ker(\theta)$; so we may view $\theta \in \mathrm{Irr}(\bar{N})$. Observe that $\bar{T}$ is the stabilizer in $\bar{G}$ of θ. Thus, $\bar{G}$ is a group that satisfies the hypotheses of the theorem. If $K > 1$, then $|\bar{G}| < |G|$, and the choice of counterexample yields $a(\gamma) = a(\chi)$. This is to a contradiction of the choice of counterexample; so we must have $K = 1$ and χ is faithful. □

Choose a subgroup $J \subseteq G$ and a character $\lambda \in \mathrm{Irr}(J)$ so that $\lambda^G = \chi$ and $\lambda(1) = a(\chi)$. This implies that λ is primitive.

STEP 2. Every abelian subgroup of N that is normal in G is central in G.

Proof 2. Let A be a subgroup of N that is abelian and normal in G. By Lemma 4.1 of [1], we may replace (J, λ) by a pair with the same properties and $A \subseteq J$. Since λ is primitive, λ_A has a unique irreducible constituent α. There is an element $g \in G$ so that α^g is a constituent of θ_A. Replacing (J, λ) by (J^g, λ^g), we may assume that α is a constituent of θ_A. Let S be the stabilizer of α in G, and note that $J \subseteq S$ and $\lambda^S \in \mathrm{Irr}(S|\alpha)$. Write $\hat{\theta} \in \mathrm{Irr}(S \cap N|\alpha)$ for the Clifford correspondent of θ with respect to α. Observe that $S \cap N$ is a normal subgroup of S and any element of S that stabilizes $\hat{\theta}$ must stabilize θ. On the other hand, all the elements in $S \cap T$ stabilize both α and θ, and because these two characters uniquely determine $\hat{\theta}$, they must stabilize it as well. Therefore, $S \cap T$ is the stabilizer in S of $\hat{\theta}$. We use $\hat{\gamma} \in \mathrm{Irr}(S \cap T|\hat{\theta})$ to denote the Clifford correspondent for λ^S with respect to $\hat{\theta}$. Also, we have $\hat{\gamma}^G = (\hat{\gamma}^S)^G = (\lambda^S)^G = \chi$; so it follows that $\hat{\gamma}^T \in \mathrm{Irr}(T|\theta)$. From the fact that γ is uniquely determined by lying in $\mathrm{Irr}(T|\theta)$ and inducing χ, we conclude that $\hat{\gamma}^T = \gamma$. If $S < G$, then S is an example that satisfies the hypotheses of the theorem with $\lambda^S(1) < \chi(1)$. (Since λ^S induces χ, $\lambda^S(1)$ is a p–power less than or equal to p^p.) By the choice of counterexample, we conclude that $a(\hat{\gamma}) = a(\lambda^S) = \lambda(1)$. We now obtain $a(\chi) = \lambda(1) = a(\hat{\gamma}) \geq a(\gamma) \geq a(\chi)$. Equality must hold, and this contradicts the choice of counterexample. Therefore, $S = G$ and $A \subseteq \mathbf{Z}(\chi) = \mathbf{Z}(G)$. □

By Lemma 3.1, $\mathbf{O}_{p'}(N)$ is central in G and $\mathbf{O}_{p',p}(N) = \mathbf{O}_{p'}(N) \times \mathbf{O}_p(N)$. Note that any abelian subgroup of $\mathbf{O}_p(N)$ that is normal in G must be central in G. This is sufficient to see that $\mathbf{O}_p(N)$ is nilpotent of class at most 2. In fact, we may use Satz III.13.6 of [2] to see that $\mathbf{O}_p(N)$ is either cyclic or extra-special.

STEP 3. $N > \mathbf{O}_{p',p}(N)$.

Proof 3. Suppose that $N = \mathbf{O}_{p',p}(N)$; so $N = \mathbf{O}_{p'}(N) \times \mathbf{O}_p(N)$. Since $\mathbf{O}_{p'}(N)$ is abelian and $\mathbf{O}_p(N)$ has nilpotence class at most 2, we see that N is nilpotent with class at most 2. Let μ be the unique irreducible constituent of $\theta_{\mathbf{Z}(N)}$. Because χ is faithful and $\mathbf{Z}(N)$ is cyclic and central in G, we know that μ is a faithful character of $\mathbf{Z}(N)$ that is invariant in G. If $\mathbf{O}_p(N)$ is abelian, then $N = \mathbf{Z}(N)$. If $\mathbf{O}_p(N)$ is extra-special, then μ is fully ramified with respect to $N/\mathbf{Z}(N)$. In either case, the fact that μ is G–invariant implies that θ is G–invariant. It follows that $T = G$, and we have already mentioned that this contradicts the choice of counterexample. Therefore, we conclude that $N > \mathbf{O}_{p',p}(N)$. □

Since J has index in G that is a power of p, we know that J contains a p–complement R of $\mathbf{O}_{p',p,p'}(N)$. Define $E = [\mathbf{O}_p(N), R] \subseteq \mathbf{O}_p(N)$. By Lemma 5.2 of [9], E and ER are normal subgroups of G. If E is abelian, then E is central in G and $1 = [E, R] = [\mathbf{O}_p(N), R, R] = [\mathbf{O}_p(N), R] = E$. This implies that R is a normal subgroup of G. It follows that $\mathbf{O}_{p',p,p'}(N) = R \times \mathbf{O}_p(N) = \mathbf{O}_{p',p}(N)$ which contradicts Step 3. Therefore, E is not abelian. We apply Lemma 2.3 to see that $E \subseteq J$. Let φ be the unique irreducible constituent of $\chi_{\mathbf{Z}(E)}$. Because χ is faithful, φ is faithful and is G–invariant. On the other hand, E is not abelian, but every subgroup of E that is abelian and normal in G is central in G. From these two facts it is not difficult to show that E has nilpotence class 2, and hence φ is fully ramified with respect to $E/\mathbf{Z}(E)$ (combine Corollary 2.30 and Theorem 2.31 with Problem 6.3, all of [4]). Write ϵ for the unique irreducible constituent of φ^E. It is easy to show $\lambda \in \mathrm{Irr}(J|\epsilon)$.

By Theorem 11.28 of [4], we know that there is a character triple isomorphism $(*, \hat{}) : (G, E, \epsilon) \to (G^*, E^*, \hat{\epsilon})$ where E^* is central in G^*. We know that $\hat{\chi}(1) = \hat{\chi}(1)/\hat{\epsilon}(1) = \chi(1)/\epsilon(1) < \chi(1)$. Thus, $\hat{\chi}(1)$ is a p–power less than or equal to p^p. Furthermore, it is easy to see that T^* is the stabilizer of $\hat{\theta}$ in G^* and $\hat{\gamma}$ is the Clifford correspondent for $\hat{\chi}$. By the inductive hypothesis, we have that $a(\hat{\gamma}) = a(\hat{\chi})$. On the other hand, if $X \subseteq G^*$ and $\xi \in \mathrm{Irr}(X)$ so that $\xi^{G^*} = \hat{\chi}$, then E^* central implies that $E^* \subseteq X$ (this is Problem 5.12 of [4], once again) and $\hat{\epsilon}$ is a constituent of ξ_{E^*}. It follows that there is a subgroup I with $E \subseteq I$ and a character $\nu \in \mathrm{Irr}(I|\epsilon)$ so that $I^* = X$ and $\hat{\nu} = \xi$. Furthermore, since $\xi^{G^*} = \hat{\chi}$, we use the character triple isomorphism to see that $\nu^G = \chi$. We have $\xi(1) = \nu(1)/\epsilon(1) \geq a(\chi)/\epsilon(1) = \lambda(1)/\epsilon(1) = \hat{\lambda}(1)$. It follows that $a(\hat{\chi}) = \hat{\lambda}(1)$, and $a(\hat{\gamma}) = a(\chi)/\epsilon(1)$. It is easy to see that $a(\gamma) \leq a(\hat{\gamma})\epsilon(1)$; so we have $a(\gamma) \leq (a(\chi)/\epsilon(1))\epsilon(1) = a(\chi)$. Since γ induces χ, the other inequality is immediate, and we conclude that $a(\chi) = a(\gamma)$ in contradiction to the choice of counterexample. This proves the theorem. □

Our proof of Theorem A is based on the ideas found in the proof of Theorem 10.1 of [7]. Let G be a group and let T be a subgroup of G. We say that $\tau \in \mathrm{Irr}(T)$

with $\psi = \tau^G \in \mathrm{Irr}(G)$ is a *Clifford induction* if there is a normal subgroup N of G and a character $\theta \in \mathrm{Irr}(N)$ so that T is the stabilizer of θ in G and τ is a constituent of θ^T. In particular, τ is the Clifford correspondent for ψ with respect to θ. In particular, the graph used in this proof was originally defined in Section 8 of [7].

Proof of Theorem A. Let $\mathcal{C}(\chi)$ be the graph whose vertices are pairs (A, α) where $A \subseteq G$, $\alpha \in \mathrm{Irr}(A)$, and $\alpha^G = \chi$, and there is an edge between (A, α) and (B, β) if either $A \subseteq B$ and $\alpha^B = \beta$ is a Clifford induction or $B \subseteq A$ and $\beta^A = \alpha$ is a Clifford induction. In Theorem 8.8 of [7], Isaacs proved that $\mathcal{C}(\chi)$ is a connected graph. For a pair (A, α) in $\mathcal{C}(\chi)$, we know that α induces χ and $\alpha(1)$ must be a p–power less than or equal to p^p. If (A, α) and (B, β) are adjacent vertices in $\mathcal{C}(\chi)$, then we apply Theorem 3.2 to see that $a(\alpha) = a(\beta)$. For a primitive character $\lambda \in \mathrm{Irr}(J)$ that induces χ, there is a path in $\mathcal{C}(\chi)$ from (J, λ) to (G, χ). We have proved that a is preserved along this path; so $\lambda(1) = a(\lambda) = a(\chi)$. Therefore, all the primitive characters that induce χ have degree equal to $a(\chi)$. □

Proof of Theorem B. We work by induction on $|G|$. Observe that $\theta(1)$ must divide $\chi(1)$; so both θ and χ have p–power degree less than or equal to p^p. By Theorem A, we know that the primitive characters inducing χ all have degree $a(\chi)$ and those inducing θ have degree $a(\theta)$. To prove the theorem, we must prove that $a(\theta)$ divides $a(\chi)$. Since $\chi(1)$ and $\theta(1)$ are powers of p that $a(\chi)$ and $a(\theta)$ divide, it follows that $a(\chi)$ and $a(\theta)$ are powers of p. If $N = G$, then $\theta = \chi$ and the result is immediate. Thus, we may assume that $N < G$, and there is a subgroup M so that N is subnormal in M and M is a maximal normal subgroup of G. Take ψ to be an irreducible constituent of χ_M with θ a constituent of ψ_N. We know that either $\chi_M = \psi$ or $\psi^G = \chi$ (Corollary 6.19 of [4]). In the first case, it is easy to see that $a(\psi)$ divides $a(\chi)$ (Lemma 8.1 of [9]). In the second case, we are in the situation of Theorem 3.2 where the stabilizer of ψ is M. Using that result, we deduce that $a(\psi) = a(\chi)$. In either case, $\psi(1)$ is a power of p that is less than or equal to p^p. By the inductive hypothesis, we determine that $a(\theta)$ divides $a(\psi)$, and we conclude that $a(\theta)$ divides $a(\chi)$. □

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