

## SPECTRAL ANALYSIS OF AN ISOTROPIC STRATIFIED ELASTIC STRIP AND APPLICATIONS

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### 1. Introduction

The characteristics of an elastic isotropic stratified strip  $\Omega = \{\mathbf{x} = (x_1, x_2); x_2 \in (0, L)\} \subset \mathbb{R}^2$  are the density  $\rho$  and Lamé coefficients  $\lambda$  and  $\mu$  that we assume to be measurable functions, depending on  $x_2$  only, and bounded from below and above by two positive constants. We shall derive a limiting absorption principle (LAP) and a division theorem for the selfadjoint operator  $(\mathcal{D}(A), A)$  (see (2)) associated with  $\Omega$  with Dirichlet and free surface conditions on  $\{x_2 = L\}$  and  $\{x_2 = 0\}$ , respectively. These boundary conditions come from a model of a seismic problem. For other studies dealing with elasticity in different situations see for instance [7] and [12].

Roughly speaking, a LAP means that the resolvent operator  $z \rightarrow R_A(z) := (A - zI_d)^{-1}$  can be extended continuously to the essential spectrum (a part of the real axis) in suitable topologies. It is an important stage in scattering theory (cf. [1]). A division theorem enables to deal with a perturbed or a multistratified strip (cf. [3]).

A multiplication operator in  $\oplus^\infty L^2(\mathbb{R}) := \{(f^n)_{n \geq 1}; \sum^\infty \|f^n\|_{L^2(\mathbb{R})}^2 < \infty\}$  by a family of functions  $\mu_n, n \geq 1$ , is defined by

$$(1) \quad \begin{cases} \mathcal{D}(M) &= \{(f^n)_{n \geq 1} \in \oplus^\infty L^2(\mathbb{R}); (\mu_n f^n)_{n \geq 1} \in \oplus^\infty L^2(\mathbb{R})\} \\ M(f^n)_{n \geq 1} &= (\mu_n f^n)_{n \geq 1}. \end{cases}$$

As we will see  $A$  is unitarily equivalent to  $M$  with  $\mu_n$  being the dispersion curves  $\lambda_n$  of  $A$ . Such a result holds for other differential operators, in particular for the acoustic operator  $B$  (cf. (9)) studied in [4] and [5]. But it is the first time that the following original phenomena are proved (cf. [3], ch.5 and ch.7). They do not take place in the acoustic case.

$$(P) \quad \begin{cases} \bullet \text{ The functions } \lambda_n \text{ are not necessarily monotonic on } \mathbb{R}_+. \\ \bullet \text{ One can have } \lambda_n''(0) = \lambda_n'''(0) = 0 \text{ in addition to } \lambda_n'(0) = 0. \end{cases}$$

One of our objectives is to show the spectral consequences of these phenomena and illustrate the fundamental difference between the elastic and the acoustic cases.

Unlike the acoustic case the main difficulty is that there is little information about the functions  $\lambda_n$  so deriving a LAP and a division theorem was carried out by proving only some general properties for these functions. This suggests to study  $M$  in the abstract framework taking these properties as hypotheses. This can then be directly used in dealing with other differential operators, which is another objective of this paper. In this framework functions  $\mu_n$  are allowed to be non-monotonic on  $\mathbb{R}_+$  and  $\mu'_n$  can have roots with arbitrary orders. The richness of the elastic case, as  $(\mathcal{P})$  shows, justifies this framework and proves that it applies to concrete problems.

**1.1. Definition of  $A$  and dispersion curves  $\lambda_n, n \geq 1$**  The differential operator of linear elasticity in an isotropic medium in  $\mathbb{R}^2$  is given by  $(\mathcal{A}\mathbf{u})_i = -(1/\rho) \sum_{j=1}^2 \partial_{x_j} \sigma_{ij}(\mathbf{u})$  where

$$\sigma_{ij}(\mathbf{u}) = \lambda(\nabla \cdot \mathbf{u})\delta_{ij} + 2\mu\varepsilon_{ij}(\mathbf{u}), \quad \text{with} \quad \varepsilon_{ij}(\mathbf{u}) = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right), \quad 1 \leq i, j \leq 2,$$

and  $\mathbf{u} = (u_1, u_2)$  is the displacement field. The function  $\mathbf{u}$  belongs to  $H^1(\Omega, \mathbb{C}^2)$ , so that  $\sigma_{ij}(\mathbf{u}) \in L^2(\Omega)$ , and  $\mathcal{A}\mathbf{u}$  can then be understood in the sense of distributions.

It is well-known that the trace  $\mathbf{u}|_{x_2=L}$  belongs to  $H^{1/2}(\mathbb{R}, \mathbb{C}^2)$ . If in addition  $\mathcal{A}\mathbf{u} \in L^2(\Omega, \mathbb{C}^2)$  then  $(\sigma_{i1}(\mathbf{u}), \sigma_{i2}(\mathbf{u})) \in H(\Omega, \text{div}), i = 1, 2,$  (cf. [6], ch.9, p. 239) so  $\sigma_{i2}(\mathbf{u})|_{x_2=0}$  makes sense in  $H^{-1/2}(\mathbb{R})$ . Denote  $V = \{\mathbf{u} \in H^1(\Omega, \mathbb{C}^2); \mathbf{u}|_{x_2=L} = 0\}$  and  $H$  the Hilbert space  $L^2(\Omega, \mathbb{C}^2)$  with the inner product  $\langle \mathbf{u}, \mathbf{v} \rangle_H = \int_{\Omega} \sum_{i=1}^2 u_i \overline{v_i} \rho \, dx$ . The operator  $A$  is defined by

$$(2) \quad \begin{cases} \mathcal{D}(A) = \{\mathbf{u} \in V; \mathcal{A}\mathbf{u} \in H, \sigma_{12}(\mathbf{u})|_{x_2=0} = \sigma_{22}(\mathbf{u})|_{x_2=0} = 0\}, \\ \mathcal{A}\mathbf{u} = A\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A). \end{cases}$$

It is selfadjoint in  $H$ . Indeed, taking the boundary conditions satisfied by  $\mathbf{u} \in \mathcal{D}(A)$  into account and integrating by parts we obtain

$$(3) \quad \langle A\mathbf{u}, \mathbf{v} \rangle_H = \mathbf{a}(\mathbf{u}, \mathbf{v}) := \int_{\Omega} \left[ \lambda(\nabla \cdot \mathbf{u})(\overline{\nabla \cdot \mathbf{v}}) + 2\mu \sum_{ij} \varepsilon_{ij}(\mathbf{u}) \overline{\varepsilon_{ij}(\mathbf{v})} \right] dx, \quad \forall \mathbf{v} \in V.$$

One can build, as in [11], a suitable extension mapping from  $V$  into  $H^1(\mathbb{R}^2, \mathbb{C}^2)$  and deduce a Korn inequality:

$$\exists c > 0 \quad \text{such that} \quad \mathbf{a}(\mathbf{u}, \mathbf{u}) + \|\mathbf{u}\|_H^2 \geq c\|\mathbf{u}\|_V^2, \quad \forall \mathbf{u} \in V.$$

This, in other words, means that the bounded hermitian form  $\mathbf{a}(\cdot, \cdot)$  is coercive over  $V$ . Using (3) we prove that  $A$  is the selfadjoint operator associated with  $\mathbf{a}(\cdot, \cdot)$  (see [9], ch.6, 2.1; or [6], ch.6, p. 1205).

The partial Fourier transform  $\mathcal{F}$  is defined, for all  $\mathbf{u} \in H$ , by

$$(4) \quad \mathcal{F}\mathbf{u}(p, x_2) = \text{l.i.m.}(2\pi)^{-1/2} \int_{\mathbb{R}} \mathbf{u}(x_1, x_2)e^{-ipx_1} dx_1, \quad \text{a.e. } (p, x_2) \in \Omega,$$

and is unitary from  $H$  onto  $H$ . Since  $\rho$ ,  $\lambda$  and  $\mu$  depend only on  $x_2$ , the operator  $\widehat{A} := \mathcal{F}A\mathcal{F}^{-1}$ , which is unitarily equivalent to  $A$ , is the direct sum, as in [7], of the field of reduced operators:  $\widehat{A} = \int_{\mathbb{R}}^{\oplus} A_p dp$ . The operators  $A_p$  are defined as follows. Denote

$$A_p \mathbf{u} = -\frac{1}{\rho} \begin{pmatrix} -p^2(\lambda + 2\mu)u_1 + ip\lambda \frac{du_2}{dx_2} + \frac{d}{dx_2} \left( \mu(ipu_2 + \frac{d}{dx_2}u_1) \right) \\ -p^2\mu u_2 + ip\mu \frac{du_1}{dx_2} + \frac{d}{dx_2} \left( (\lambda + 2\mu) \frac{du_2}{dx_2} + ip\lambda u_1 \right) \end{pmatrix},$$

$\overline{V} = \{\mathbf{u} \in H^1((0, L), \mathbb{C}^2); \mathbf{u}(L) = 0\}$  and  $\overline{H}$  the Hilbert space  $L^2((0, L), \mathbb{C}^2)$  with the inner product  $(\mathbf{u}, \mathbf{v})_{\overline{H}} = \int_0^L \sum_1^2 u_i \overline{v_i} \rho dx_2$ . We have

$$\left\{ \begin{array}{l} \mathcal{D}(A_p) = \left\{ \mathbf{u} \in \overline{V}; A_p \mathbf{u} \in \overline{H}, \mu \left( \frac{du_1}{dx_2} + ipu_2 \right)_{|0} = 0, \left( i\lambda pu_1 + (\lambda + 2\mu) \frac{du_2}{dx_2} \right)_{|0} = 0 \right\}, \\ A_p \mathbf{u} = \mathcal{A}_p \mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A_p). \end{array} \right.$$

As for  $A$  we prove that  $A_p$  is selfadjoint. In fact, for  $\mathbf{u} \in \mathcal{D}(A_p)$  we have

$$(5) \quad \begin{aligned} \langle A_p \mathbf{u}, \mathbf{v} \rangle_{\overline{H}} &= a_p(\mathbf{u}, \mathbf{v}) \\ &:= \int_0^L \left[ \lambda \left( ipu_1 + \frac{du_2}{dx_2} \right) \overline{\left( i\overline{p}v_1 + \frac{dv_2}{dx_2} \right)} + 2\mu(ipu_1) \overline{(i\overline{p}v_1)} \right. \\ &\quad \left. + 2\mu \frac{du_2}{dx_2} \overline{\frac{dv_2}{dx_2}} + \mu \left( ipu_2 + \frac{du_1}{dx_2} \right) \overline{\left( i\overline{p}v_2 + \frac{dv_1}{dx_2} \right)} \right] dx_2, \quad \forall \mathbf{v} \in \overline{V}. \end{aligned}$$

Using inequality  $2ab \leq \varepsilon a^2 + b^2/\varepsilon$  we verify

$$(6) \quad \exists c > 0, \quad \|\mathbf{u}\|_{\overline{V}}^2 \leq c(1 + p^2)\|\mathbf{u}\|_{\overline{H}}^2 + a_p(\mathbf{u}, \mathbf{u}), \quad \forall \mathbf{u} \in \overline{V}.$$

Thus  $a_p(\cdot, \cdot)$  is coercive and we check, using (5), that  $A_p$  is the associated selfadjoint operator. The compactness of embedding  $\overline{V} \hookrightarrow \overline{H}$  implies that the resolvent  $(A_p - zI_d)^{-1}$  is compact from  $\overline{H}$  into  $\overline{H}$ .

Thanks to (5) we extend the definition of  $a_p(\cdot, \cdot)$  to all  $p \in \mathbb{C}$  and obtain a selfadjoint holomorphic family of type (a) (cf. [9], ch.7). The associated family of operators  $A_p$ ,  $p \in \mathbb{C}$ , is thus selfadjoint holomorphic of type (B). Applying Remark 4.22 of ch.7 in [9] we infer the existence of two families of analytic functions on  $\mathbb{R}$ , namely  $p \rightarrow \lambda_n(p) \in \mathbb{R}$ ,  $p \rightarrow W_n(p) \in \overline{H}$ ,  $n \geq 1$ , such that, for all  $p \in \mathbb{R}$ ,  $\{\lambda_n(p); n \geq 1\}$  are the repeated eigenvalues of  $A_p$  and  $\{W_n(p); n \geq 1\}$  a corresponding orthonormal basis of eigenvectors.

For all  $\mathbf{u} \in \mathcal{D}(A_p)$  we then have

$$(7) \quad A_p \mathbf{u} = \sum_{n \geq 1} \lambda_n(p) \langle \mathbf{u}, W_n(p) \rangle_{\overline{H}} W_n(p),$$

and  $A$  is recovered this way since  $\mathcal{F}A\mathcal{F}^{-1} = \int_{\mathbb{R}}^{\oplus} A_p dp$ . In fact, consider the transformation  $F : H \rightarrow \oplus^{\infty} L^2(\mathbb{R})$  defined by  $F\mathbf{u} = (f^n)_{n \geq 1}$  where

$$(8) \quad f^n(p) = (2\pi)^{-1/2} \text{l.i.m.} \int_{\Omega} \sum_{j=1}^2 u_j(x_1, x_2) \overline{W_j^T(p, x_2)} e^{-ipx_1} \rho dx, \quad \text{a.e. } p \in \mathbb{R},$$

with  $W_n(p) = (W_n^1(p, \cdot), W_n^2(p, \cdot))$ . Using the fact that  $\mathcal{F}$  is unitary and that  $W_n(p)$ ,  $n \geq 1$ , is an orthonormal basis we check that  $F$  is unitary. On the other hand from (7) we verify that  $A = F^{-1}MF$ , with  $\mu_n = \lambda_n$  in (1). For details see [3], ch.4.

**1.2. Comparison with the acoustic case and consequences of properties (P)**

The approach is similar for the operator  $B$  associated with the acoustic strip:

$$(9) \quad \begin{cases} \mathcal{D}(B) = \{u \in H^1(\Omega); \nabla \cdot C^2 \nabla u \in L^2(\Omega), u|_{x_2=L} = 0, C^2 \partial_{x_2} u|_{x_2=0} = 0\}, \\ Bu = -\nabla \cdot C^2 \nabla u, \quad \forall u \in \mathcal{D}(B). \end{cases}$$

We verify that for every  $p$ , the reduced operator  $B_p$  is the selfadjoint operator associated with the hermitian form

$$a_{p,B}(u, v) := \int_0^L C^2 \frac{du}{dx_2} \frac{\overline{dv}}{dx_2} dx_2 + p^2 \int_0^L C^2 u \overline{v} dx_2.$$

The eigenvalues and eigenvectors of the operators  $B_p$  are represented by two families of analytic functions:  $(\lambda_{n,B})_{n \geq 1}$  and  $(W_{n,B})_{n \geq 1}$ , and we have (cf. [9], ch.7, 4.7)

$$\frac{d\lambda_{n,B}}{dp}(p) = a'_{p,B}(W_{n,B}(p), W_n(p)) := 2p \int_0^L C^2 |W_{n,B}|^2 dx_2, \quad \forall p \in \mathbb{R}.$$

Here  $a'_{p,B}(\mathbf{u}, \mathbf{u})$  is obtained by taking the derivative of  $p \rightarrow a_{p,B}(\mathbf{u}, \mathbf{u})$  for every fixed  $\mathbf{u}$ . It is thus clear that  $\lambda'_{n,B}(p) \geq 0$  for  $p \geq 0$ , and that  $\lambda''_{n,B}(0) \neq 0$  for all  $n$ . In other words, (P) does not hold in the acoustic case. In elasticity

$$(10) \quad \frac{d\lambda_n}{dp}(p) = a'_p(W_n(p), W_n(p)),$$

with

$$a'_p(\mathbf{u}, \mathbf{u}) = 2 \int_0^L \text{Re} \left( \lambda \frac{du_2}{dx_2} \overline{i u_1} + \mu \frac{du_1}{dx_2} \overline{i u_2} \right) dx_2 + 2p \int_0^L ((\lambda + 2\mu)|u_1|^2 + \mu|u_2|^2) dx_2,$$

and one can see that we have not necessarily  $\lambda'_n(p) \geq 0$ , for  $p \geq 0$ , nor  $\lambda''_n(0) \neq 0$ . In fact, we prove (P) in the homogeneous case (i.e.  $\rho, \lambda$  and  $\mu$  constants) with the

help of a mathematical formal-calculus software. For instance, with  $L = \rho = \mu = 1$  and  $\lambda = 7$  there exists  $n \geq 1$  such that  $\lambda'_n(0) = -10$  so  $\lambda_n$  is not monotonic on  $\mathbb{R}_+$  since, as we will see,  $\lambda_n(p) \rightarrow +\infty$ , as  $p \rightarrow +\infty$ .

In general we prove, see Theorem 2.5, a “global symmetry” property: for every  $n$  there exists  $n'$  such that  $\lambda_n(-p) = \lambda_{n'}(p)$  for all  $p$ . One can easily check that the eigenvalues  $\lambda_n(0)$ ,  $n \geq 1$ , of  $A_0$  are at most double. If  $\lambda_n(0)$  is simple then  $\lambda_n$  is symmetric so  $\lambda'_n(0) = 0$ . For double eigenvalues we have  $\lambda'_n(0) = -\lambda'_{n'}(0)$ . If  $\lambda'_n(0) \neq 0$  then one of the functions  $\lambda_n$  or  $\lambda_{n'}$  is not monotonic on  $\mathbb{R}_+$ . In the homogeneous case we prove that this is the case if  $\lambda/(\lambda + 2\mu) \neq 1/4$ . Clearly, the associated dispersion curves intersect at  $p = 0$ .

The non-monotonicity is not only related to double eigenvalues of  $A_0$ . In fact, we prove numerically that  $\lambda_n$  may be non-monotonic on  $\mathbb{R}_+$  even though the eigenvalue  $\lambda_n(0)$  is simple, and that the dispersion curves can intersect for  $p \neq 0$ . For more details see [3], ch.5 and ch.7.

In the acoustic case the thresholds were defined as the eigenvalues  $\lambda_{n,B}(0)$ ,  $n \geq 1$ , of the reduced operator  $B_0$ . They are values where a LAP for  $B$  is not valid if the resolvent  $(B - zI_d)^{-1}$  is considered as an element of  $\mathcal{B}(L^2_{s_1}(\Omega), L^2_{-s_2}(\Omega))$  (cf. [4]), where  $L^2_s(\Omega)$  denotes the weighted space with  $\|u\|_{L^2_s(\Omega)} = \|(1 + x_1^2)^{s/2}u\|_{L^2(\Omega)}$ . On the other hand, the set of thresholds in the elastic case, i.e the set of critical values for obtaining a LAP, is

$$(11) \quad \Gamma = \{ \lambda_0 \in \mathbb{R}; \exists p_0 \in \mathbb{R} \text{ and } n \geq 1 \text{ such that } \lambda_0 = \mu_n(p_0) \text{ and } \mu'_n(p_0) = 0 \}$$

with  $(\mu_n) = (\lambda_n)$ . A first consequence of  $(\mathcal{P})$  is that  $\Gamma$  does not coincide with the set of eigenvalues of  $A_0$  unlike the acoustic case. Indeed, double eigenvalue of  $A_0$  are not always thresholds, and if  $\lambda'_n(p_0) = 0$ , with  $p_0 \neq 0$ , then  $\lambda_n(p_0)$  is not necessarily an eigenvalue of  $A_0$ .

The second consequence of  $(\mathcal{P})$  concerns a LAP at thresholds for the selfadjoint operators  $A^{SA}_{\Omega_k}$  and  $A^{AS}_{\Omega_k}$  (see (21) and (20)), needed in dealing with a multistratified strip (cf. [3], ch.4 and ch.6), associated with the half-strip

$$(12) \quad \Omega_k := \{ \mathbf{x} \in \Omega; (-1)^k x_1 > 0 \}, \quad k = 1, 2.$$

In fact, for the acoustic case the LAP, which is not valid at thresholds for the stratified strip, becomes valid at all thresholds for some multistratified strips (see [4] and [2]). This is the main result in [2] and the proof there uses the validity of a LAP at thresholds for the half acoustic strip, in which the property  $\lambda'_{n,B}(0) \neq 0$  is essential. In elasticity we prove the validity of a LAP at only some particular thresholds for  $A^{SA}_{\Omega_k}$  and  $A^{AS}_{\Omega_k}$ , and the non-validity at the whole set of thresholds unlike the acoustic case.

**1.3. Abstract framework** We shall say that a mapping  $z \rightarrow \widetilde{R}(z)$ , defined from  $\overline{\mathbb{C}^\pm} \setminus \widetilde{\Gamma}$  into a Banach space  $\widetilde{Y}$ , where  $\widetilde{\Gamma} \subset \mathbb{R}$ , is locally  $\delta$ -Hölder continuous

( $\delta$  will always be a positive number) if for all compact  $K \subset \mathbb{C} \setminus \widetilde{\Gamma}$  there exists a constant  $c > 0$  such that  $\|R(z') - \widetilde{R}(z)\|_{\widetilde{Y}} \leq c|z' - z|^\delta$ , for all  $z', z \in K \cap \overline{\mathbb{C}^\pm}$ .

For  $\mathbf{f} = (f^n)_{n \geq 1}$ ,  $\mathbf{g} = (g^n)_{n \geq 1}$  and a family of real functions  $\mu_n$ ,  $n \geq 1$ , put

$$(13) \quad \mathbf{b}_n(z) = \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - z} dp.$$

In deriving a LAP we shall prove a local Hölder continuous estimate for  $z \rightarrow R_M(z) := (M - zI_d)^{-1}$ . Since  $\langle R_M(z)\mathbf{f}, \mathbf{g} \rangle = \sum_{n=1}^\infty \mathbf{b}_n(z)$ , it amounts to deriving the Hölder continuous estimate for this infinite sum. To be able to use this part of the work in dealing with other operators it is advisable to carry it out in an abstract framework. The latter consists in supposing that

1.  $(\mu_n)_{n \geq 1}$  is a family of functions, not necessarily  $(\lambda_n)_{n \geq 1}$ , satisfying certain hypothesis **(H1)**.
2.  $\mathbf{f}$  and  $\mathbf{g}$  belong to suitable spaces  $Y^s$ , where the family  $Y^s$ ,  $s \geq 0$ , satisfies another hypothesis **(H2)**.

In the applications we are interested (see the last section) in a class of selfadjoint differential operators with analytic and real dispersion curves, this is why we assume that  $\mu_n$  are real and regular. On the other hand, assuming that  $|\mu_n|$  goes uniformly to infinity as  $n \rightarrow \infty$ , permits to treat separately the terms  $\mathbf{b}_n(z)$  instead of their infinite sum. Also, supposing that, for every  $n$ ,  $|\mu_n(p)| \rightarrow \infty$ , as  $p \rightarrow \infty$ , and that the roots of  $\mu'_n$  is a discrete set implies that  $\Gamma$  is also discrete. It is then natural to put

$$(H1) \quad \left\{ \begin{array}{l} \bullet \lim_{n \rightarrow \infty} (\inf_{p \in \mathbb{R}} |\mu_n(p)|) = \infty; \text{ and } \lim_{p \rightarrow \infty} (|\mu_n(p)|) = \infty, \quad \forall n \geq 1. \\ \bullet \forall n \geq 1, \text{ the roots of } \mu'_n \text{ have finite orders and form a discrete set.} \end{array} \right.$$

The fact that the roots of  $\mu'_n$  have finite orders permits to define appropriate spaces for obtaining a LAP at thresholds. Hypothesis **(H1)** is obviously fulfilled in the acoustic case knowing that  $\lambda_{n,B}(p) \geq \lambda_{n,B}(0)$  for all  $p$ , and  $\lambda_{n,B}(0) \rightarrow +\infty$ , as  $n \rightarrow +\infty$ . The proof, as we will see, is much less easy in the elastic case.

For the local Hölder continuous estimate of (13) we need local Hölder continuity of  $p \rightarrow f^n(p)\overline{g^n(p)}$ . This is why we introduce the second hypothesis below which enables to use some interesting properties of the elements of Sobolev spaces  $H^s(\mathbb{R})$  such as Hölder continuity.

$$(H2) \quad \left\{ \begin{array}{l} \bullet \forall s' \geq s \geq 0, \text{ the embeddings } Y^{s'} \hookrightarrow Y^s \hookrightarrow \oplus^\infty L^2(\mathbb{R}) \\ \text{are continuous with dense ranges.} \\ \bullet \forall \phi \in C_0^\infty(\mathbb{R}), n \geq 1, \text{ the mappings} \\ \quad (f_k)_{k \geq 1} \mapsto \phi f_n \text{ and } f \mapsto (\delta_n^k \phi f)_{k \geq 1}, \\ \text{defined respectively from } Y^s \text{ into } H^s(\mathbb{R}), \text{ and from} \\ \quad H^s(\mathbb{R}) \text{ into } Y^s, \text{ are continuous.} \end{array} \right.$$

In the proof we will indicate more precisely how **(H1)** and **(H2)** are needed.

Denote  $H_s$  the weighted space with the norm  $\|\mathbf{u}\|_{H_s} := \|(1 + x_1^2)^{s/2}\mathbf{u}\|_H$ . For the elasticity  $Y^s$  will be  $F(H_s)$  (cf. (8)), with  $\|\mathbf{f}\|_{Y^s} := \|F^{-1}\mathbf{f}\|_{H_s}$ , and  $(\mu_n) = (\lambda_n)$ . We will see that **(H1)** and **(H2)** are satisfied, so the Hölder continuous estimate for  $z \rightarrow R_A(z)$  follows since  $\|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}, H_{-s_2})} = \|R_M(z') - R_M(z)\|_{\mathcal{B}(Y_{s_1}, Y_{-s_2})}$ .

This paper is organized as follows.

In Section 2 we prove the validity of **(H1)** and **(H2)** in elasticity, and give a “global symmetry” property satisfied by  $\lambda_n$  and  $W_n$ ,  $n \geq 1$ . We study in Section 3 the operator  $M$  in an abstract framework. In Section 4 we apply these results to  $A$  and propose a procedure to study other differential operators. We finally prove the validity (at some thresholds) of a LAP for  $A_{\Omega_k}^{SA}$  and  $A_{\Omega_k}^{AS}$ , as well as its failure.

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**2. Validity of hypotheses (H1) and (H2) in the elastic case and some symmetry properties**

In Theorem 2.2 (resp. Theorem 2.3) we prove that the family  $(\lambda_n)_{n \geq 1}$  (resp.  $F(H_s)$ ,  $s \geq 0$ .) satisfies **(H1)** (resp. **(H2)**). Then we define the operators  $A_{\Omega_k}^{SA}$  and  $A_{\Omega_k}^{AS}$  and give a property in Theorem 2.4. In Theorem 2.5 we prove a symmetry property satisfied by  $(\lambda_n)_{n \geq 1}$  and  $(W_n)_{n \geq 1}$ .

We need the following lemma.

**Lemma 2.1.** *If a continuous function  $f$  satisfies*

$$(14) \quad f(p) \leq ap^2 + bp + \varepsilon \int_0^p f(\tau) d\tau, \quad \forall p \in \mathbb{R}_+,$$

where  $a, b, \varepsilon \geq 0$ , then

$$(15) \quad f(p) \leq \frac{b}{\varepsilon}(e^{\varepsilon p} - 1) + 2a \left( \frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2}(e^{\varepsilon p} - 1) \right), \quad \forall p \in \mathbb{R}_+.$$

The proof uses Gronwall’s lemma.

**Theorem 2.2.** *The family  $(\lambda_n)_{n \geq 1}$  satisfies **(H1)**.*

**Proof.** We first prove that there exists a constant  $c_{R-} > 0$  such that  $\lambda_n(p) \geq c_{R-}^2 p^2$  for all  $p \in \mathbb{R}$ ,  $n \geq 1$ , and after that  $\inf_{p \in \mathbb{R}} \lambda_n(p) \rightarrow +\infty$ , as  $n \rightarrow +\infty$ . This will complete the proof knowing that  $\lambda_n$  is analytic.

We use spectral properties of an elastic homogeneous half-space in  $\mathbb{R}^2$  (with constant density  $\rho_0$  and Lamé coefficients  $\lambda_0$  and  $\mu_0$ ) with a free surface condition

on  $x_2 = 0$ . The corresponding reduced operators  $\tilde{A}_p$ ,  $p \in \mathbb{R}$ , are defined exactly as  $A_p$ : it suffices to replace  $L$  by  $+\infty$  and  $(\rho, \lambda, \mu)$  by  $(\rho_0, \lambda_0, \mu_0)$ . For fixed  $p$ ,  $\tilde{A}_p$  is the selfadjoint operator, in the Hilbert space  $L^2(\mathbb{R}_+, \mathbb{C}^2, \rho_0 dx_2)$  equipped with  $\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\mathbb{R}_+, \mathbb{C}^2, \rho_0 dx_2)} = \int_{\mathbb{R}_+} \sum_1^2 u_i \bar{v}_i \rho dx_2$ , associated with a hermitian and coercive form  $\tilde{a}_p(\cdot, \cdot)$  defined on  $H^1(\mathbb{R}_+, \mathbb{C}^2)$ . The spectrum of  $\tilde{A}_p$  (cf. [7]) is  $\{c_{R, \rho_0, \lambda_0, \mu_0}^2 p^2\} \cup [\mu_0 p^2 / \rho_0, +\infty[$  where  $c_{R, \rho_0, \lambda_0, \mu_0} > 0$  is the propagation velocity of Rayleigh waves.

The extension  $\tilde{\mathbf{u}}$  of  $\mathbf{u} \in \tilde{V}$  by zero belongs to  $H^1(\mathbb{R}_+, \mathbb{C}^2)$  and we have  $a_p(\mathbf{u}, \mathbf{u}) \geq \tilde{a}_{p, \lambda_-, \mu_-}(\tilde{\mathbf{u}}, \tilde{\mathbf{u}})$  and  $\|\mathbf{u}\|_{\tilde{H}} \leq \|\tilde{\mathbf{u}}\|_{L^2(\mathbb{R}_+, \mathbb{C}^2, \rho_+ dx_2)}$ , where  $\rho_+ = \text{ess sup } \rho$ ,  $\lambda_- = \text{ess inf } \lambda$  and  $\mu_- = \text{ess inf } \mu$ . Denoting  $c_{R-} := c_{R, \rho_+, \lambda_-, \mu_-}$  we have

$$\frac{a_p(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|_{\tilde{H}}^2} \geq \frac{\tilde{a}_{p, \lambda_-, \mu_-}(\tilde{\mathbf{u}}, \tilde{\mathbf{u}})}{\|\tilde{\mathbf{u}}\|_{L^2(\mathbb{R}_+, \mathbb{C}^2, \rho_+ dx_2)}^2} \geq c_{R-}^2 p^2,$$

hence

$$(16) \quad \lambda_n(p) \geq \inf_{\mathbf{u} \in \tilde{V}, \mathbf{u} \neq 0} \frac{a_p(\mathbf{u}, \mathbf{u})}{\|\mathbf{u}\|_{\tilde{H}}^2} \geq c_{R-}^2 p^2, \quad \forall p \in \mathbb{R}, n \geq 1.$$

It remains to prove that

$$(17) \quad \forall m > 0, \quad \exists N_m \in \mathbb{N} \text{ such that } n \geq N_m \Rightarrow \lambda_n(p) > m, \quad \forall p \in \mathbb{R}.$$

We shall use (10). Thanks to inequality  $2ab \leq \varepsilon|a|^2 + 1/\varepsilon|b|^2$  we verify that  $|a'_p(\mathbf{u}, \mathbf{u})| \leq \varepsilon\|\mathbf{u}\|_{\tilde{V}}^2 + (c|p| + c'_\varepsilon)\|\mathbf{u}\|_{\tilde{H}}^2$ . Moreover, from (6) and inequality (16) we get  $\|\mathbf{u}\|_{\tilde{V}}^2 \leq c_1\|\mathbf{u}\|_{\tilde{H}}^2 + c_2 a_p(\mathbf{u}, \mathbf{u})$ . The combination of the last two inequalities (we suppose  $c_2 = 1$  since  $\varepsilon$  can be replaced by  $\varepsilon/c_2$ ) yields  $|a'_p(\mathbf{u}, \mathbf{u})| \leq \varepsilon a_p(\mathbf{u}, \mathbf{u}) + (c|p| + c_\varepsilon)\|\mathbf{u}\|_{\tilde{H}}^2$ . Replacing  $\mathbf{u}$  by  $W_n(p)$  in (10) we obtain  $|\lambda'_n(p)| \leq c|p| + c_\varepsilon + \varepsilon\lambda_n(p)$  so that

$$\begin{aligned} |\lambda_n(p) - \lambda_n(0)| &\leq \int_0^p \left| \frac{d\lambda_n(\tau)}{d\tau} \right| d\tau \\ &\leq \frac{c}{2} p^2 + (c_\varepsilon + \varepsilon\lambda_n(0)) p + \varepsilon \int_0^p |\lambda_n(\tau) - \lambda_n(0)| d\tau. \end{aligned}$$

Applying Lemma 2.1 to  $f(p) = |\lambda_n(p) - \lambda_n(0)|$ , with  $a = c/2$  and  $b = c_\varepsilon + \varepsilon\lambda_n(0)$ , we get

$$(18) \quad |\lambda_n(p) - \lambda_n(0)| \leq \lambda_n(0) (e^{\varepsilon p} - 1) + \frac{c_\varepsilon}{\varepsilon} (e^{\varepsilon p} - 1) + c \left( \frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2} (e^{\varepsilon p} - 1) \right).$$

In view of (16) we can suppose, in order to prove (17), that  $0 \leq p \leq P_0 := m^{1/2}/c_{R-} + 1$ . Take  $\varepsilon > 0$  small enough that  $e^{\varepsilon P_0} - 1 < 1/2$ . We have

$$\left| \frac{c_\varepsilon}{\varepsilon} (e^{\varepsilon p} - 1) + c \left( \frac{-p}{\varepsilon} + \frac{1}{\varepsilon^2} (e^{\varepsilon p} - 1) \right) \right| \leq m', \quad \forall p \in [0, P_0].$$

Then inequality (18) gives

$$\lambda_n(p) \geq \lambda_n(0) - |\lambda_n(p) - \lambda_n(0)| \geq \frac{\lambda_n(0)}{2} - m', \quad \forall p \in [0, P_0],$$

which ends the proof since  $\lambda_n(0) \rightarrow +\infty$ , as  $n \rightarrow +\infty$ . □

**Theorem 2.3.** *The family  $F(H_s)$ ,  $s \geq 0$ , satisfies (H2).*

Proof. We will only prove the continuity for the mapping  $\mathbf{f} = (f^k)_{k \geq 1} \mapsto \phi f^n$ . Let  $\mathbf{f} = F\mathbf{u}$  with  $\mathbf{u} = \sum_{i=1}^N g_i(x_1)\mathbf{u}_i(x_2) \in \mathcal{S} \otimes \overline{H}$  where  $\{\mathbf{u}_i; i = 1, \dots, N\}$ , is an orthonormal family in the Hilbert space  $\overline{H}$  and  $\mathcal{S} = \mathcal{S}(\mathbb{R})$  is the Schwartz class. Clearly

$$(19) \quad \|\mathbf{f}\|_{F(H_s)} = \|\mathbf{u}\|_{H_s} = \left( \sum_1^N \|g_i\|_{L^2_s(\mathbb{R})}^2 \right)^{1/2} = \left( \sum_1^N \|\mathcal{F}g_i\|_{H^s(\mathbb{R})}^2 \right)^{1/2}.$$

For all  $n \geq 1$ , we have

$$f^n(p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \sum_{i=1}^N g_i(x_1) \langle \mathbf{u}_i, W_n(p) \rangle_{\overline{H}} e^{-ipx_1} dx_1 = \sum_{i=1}^N \langle \mathbf{u}_i, W_n(p) \rangle_{\overline{H}} (\mathcal{F}g_i)(p).$$

It is well-known (cf. [13], Proposition 25.1) that for all  $\psi \in \mathcal{S}$  and  $h \in H^s(\mathbb{R})$  the product  $\psi h$  belongs to  $H^s(\mathbb{R})$  and we have, with  $k = E(s) + 2$  and  $E(s)$  denoting the smallest integer such that  $E(s) \leq s \leq E(s) + 1$ , the following inequality  $\|\psi h\|_{H^s(\mathbb{R})} \leq c_s \|\psi\|_{H^k(\mathbb{R})} \|h\|_{H^s(\mathbb{R})}$ . Combining this inequality and equation (19), with  $\psi = \phi \langle \mathbf{u}_i, W_n(p) \rangle_{\overline{H}}$  and  $h = \mathcal{F}g_i$ , yields

$$\begin{aligned} \|\phi f^n\|_{H^s(\mathbb{R})} &\leq \sum_{i=1}^N \|\phi \langle \mathbf{u}_i, W_n(p) \rangle_{\overline{H}} \mathcal{F}g_i\|_{H^s(\mathbb{R})} \\ &\leq c_s \underbrace{\left( \sum_{i=1}^N \|\phi \langle \mathbf{u}_i, W_n(p) \rangle_{\overline{H}}\|_{H^k(\mathbb{R})}^2 \right)^{1/2}}_c \underbrace{\left( \sum_{i=1}^N \|\mathcal{F}g_i\|_{H^s(\mathbb{R})}^2 \right)^{1/2}}_{\|\mathbf{f}\|_{F(H_s)}}. \end{aligned}$$

Using the fact that  $\mathbf{u}_i, i = 1, \dots, N$ , is orthonormal we verify that  $c$  is independent from  $\mathbf{f}$ . Thanks to the density of  $\mathcal{S} \otimes \overline{H}$  in  $H_s$ , the last estimate is valid for all  $\mathbf{f} \in F(H_s)$ . □

Denote  $S^{\text{SA}}$  and  $S^{\text{AS}}$  the unitary transformations on  $\overline{H}$  defined by  $S^{\text{SA}}(u_1, u_2) = (u_1, -u_2)$  and  $S^{\text{AS}} = -S^{\text{SA}}$ . Denote also  $H^{\text{SA}}$  (resp.  $H^{\text{AS}}$ ) the subspace of functions  $(u_1, u_2)$  such that  $u_1$  is symmetric (resp. anti-symmetric) and  $u_2$  is anti-symmetric (resp. symmetric). The orthogonal projections  $\mathbf{u}^1$  on  $H^{\text{SA}}$  and  $\mathbf{u}^2$  on  $H^{\text{AS}}$  of  $\mathbf{u}$  are

given by

$$\mathbf{u}^1(x_1, \cdot) = \frac{\mathbf{u}(x_1, \cdot) + S^{\text{SA}}\mathbf{u}(-x_1, \cdot)}{2}, \quad \text{and} \quad \mathbf{u}^2(x_1, \cdot) = \frac{\mathbf{u}(x_1, \cdot) - S^{\text{SA}}\mathbf{u}(-x_1, \cdot)}{2}.$$

The symmetry of our problem makes that  $A$  is reduced by  $H^{\text{SA}}$  and  $H^{\text{AS}}$  (cf. [9], ch.5, 3.9), and it is thus the direct sum of the operators  $A^{\text{SA}}$  and  $A^{\text{AS}}$  defined as follows:

$$\begin{aligned} A^{\text{SA}}\mathbf{u} &= A\mathbf{u}, & \forall \mathbf{u} \in \mathcal{D}(A^{\text{SA}}) &= H^{\text{SA}} \cap \mathcal{D}(A), \\ A^{\text{AS}}\mathbf{u} &= A\mathbf{u}, & \forall \mathbf{u} \in \mathcal{D}(A^{\text{AS}}) &= H^{\text{AS}} \cap \mathcal{D}(A). \end{aligned}$$

Thus for all  $z \in \mathbb{C}^\pm$  we have

$$\|R_A(z)\mathbf{u}\|_H^2 = \|R_{A^{\text{SA}}}(z)\mathbf{u}^1\|_H^2 + \|R_{A^{\text{AS}}}(z)\mathbf{u}^2\|_H^2.$$

We shall now see that  $A^{\text{SA}}$  and  $A^{\text{AS}}$  are, respectively, unitarily equivalent to the operators  $A_{\Omega_k}^{\text{SA}}$  and  $A_{\Omega_k}^{\text{AS}}$  defined as follows. Recall that  $\Omega_k$  is the half-strip defined in (12). Denote

$$\begin{aligned} V_{\Omega_k}^{\text{SA}} &= \{\mathbf{u} = (u_1, u_2) \in H^1(\Omega_k, \mathbb{C}^2); \mathbf{u}|_{x_2=L} = u_2|_{x_1=0} = 0\}, \\ V_{\Omega_k}^{\text{AS}} &= \{\mathbf{u} = (u_1, u_2) \in H^1(\Omega_k, \mathbb{C}^2); \mathbf{u}|_{x_2=L} = u_1|_{x_1=0} = 0\}. \end{aligned}$$

One sets

$$\begin{aligned} (20) \quad & \left\{ \begin{aligned} \mathcal{D}(A_{\Omega_k}^{\text{SA}}) &= \{\mathbf{u} \in V_{\Omega_k}^{\text{SA}}; A\mathbf{u} \in L^2(\Omega_k, \mathbb{C}^2), \sigma_{12}(\mathbf{u})|_{x_2=0} = \sigma_{22}(\mathbf{u})|_{x_2=0} = \sigma_{11}(\mathbf{u})|_{x_1=0} = 0\}, \\ A_{\Omega_k}^{\text{SA}}\mathbf{u} &= A\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A_{\Omega_k}^{\text{SA}}), \end{aligned} \right. \\ (21) \quad & \left\{ \begin{aligned} \mathcal{D}(A_{\Omega_k}^{\text{AS}}) &= \{\mathbf{u} \in V_{\Omega_k}^{\text{AS}}; A\mathbf{u} \in L^2(\Omega_k, \mathbb{C}^2), \sigma_{12}(\mathbf{u})|_{x_2=0} = \sigma_{22}(\mathbf{u})|_{x_2=0} = \sigma_{21}(\mathbf{u})|_{x_1=0} = 0\}, \\ A_{\Omega_k}^{\text{AS}}\mathbf{u} &= A\mathbf{u}, \quad \forall \mathbf{u} \in \mathcal{D}(A_{\Omega_k}^{\text{AS}}). \end{aligned} \right. \end{aligned}$$

These operators are selfadjoint in the Hilbert space  $H_{\Omega_k} = L^2(\Omega_k, \mathbb{C}^2)$  equipped with the inner product  $\langle \mathbf{u}, \mathbf{v} \rangle_{H_{\Omega_k}} = \int_{\Omega_k} \sum_1^2 u_i \bar{v}_i \rho \, d\mathbf{x}$ .

**Theorem 2.4.** *The operator  $A_{\Omega_k}^{\text{SA}}$  (resp.  $A_{\Omega_k}^{\text{AS}}$ ) is unitarily equivalent to  $A^{\text{SA}}$  (resp.  $A^{\text{AS}}$ ).*

*Proof.* For a given function  $\mathbf{u}$  defined on  $\Omega_k$  denote by  $\mathcal{U}_k^{\text{SA}}\mathbf{u}$  (resp.  $\mathcal{U}_k^{\text{AS}}\mathbf{u}$ ) the function equal to  $\mathbf{u}/\sqrt{2}$  on  $\Omega_k$ , and to  $S^{\text{SA}}\mathbf{u}(-x_1, x_2)/\sqrt{2}$  (resp.  $S^{\text{AS}}\mathbf{u}(-x_1, x_2)/\sqrt{2}$ ) on  $\Omega_{k'}$  with  $k' = 3 - k$ . We verify that  $\mathcal{U}_k^{\text{SA}}$  (resp.  $\mathcal{U}_k^{\text{AS}}$ ) is a unitary transformation defined from  $H_{\Omega_k}$  onto  $H^{\text{SA}}$  (resp.  $H^{\text{AS}}$ ) and that

$$(22) \quad A_{\Omega_k}^{\text{SA}} = (\mathcal{U}_k^{\text{SA}})^{-1} A^{\text{SA}} \mathcal{U}_k^{\text{SA}}, \quad (\text{resp. } A_{\Omega_k}^{\text{AS}} = (\mathcal{U}_k^{\text{AS}})^{-1} A^{\text{AS}} \mathcal{U}_k^{\text{AS}}). \quad \square$$

Let us now give the consequence on  $(\lambda_n)_{n \geq 1}$  and  $(W_n)_{n \geq 1}$  of the symmetry of our differential problem. The operators  $A_p$  and  $A_{-p}$  are unitarily equivalent, in fact

$$(23) \quad \begin{cases} \mathcal{D}(A_{-p}) = S^{SA} \mathcal{D}(A_p) = S^{AS} \mathcal{D}(A_p) \\ A_{-p} = S^{SA} A_p (S^{SA})^{-1} = S^{AS} A_p (S^{AS})^{-1}. \end{cases}$$

One can verify that the eigenvalues of  $A_p$ , in particular for  $p = 0$ , are at most double (for the details see [3], ch.5, 4.2). Denote  $\overline{H}^{SA}$  (resp.  $\overline{H}^{AS}$ ) the subspace of the functions  $\mathbf{u} = (u_1, u_2) \in \overline{H}$  such that  $u_2 = 0$  (resp.  $u_1 = 0$ ). These subspaces reduce  $A_0$  which is then the direct sum of the corresponding operators  $A_0^{SA}$  and  $A_0^{AS}$ . We check that  $A_0^{SA}$  and  $A_0^{AS}$  have simple eigenvalues and that an eigenvalue  $\lambda_n(0)$  of  $A_0$  is simple if and only if it is an eigenvalue of only one of these operators.

**Theorem 2.5.** *For all  $n$  there exists  $n'$  such that  $\lambda_n(-p) = \lambda_{n'}(p)$  for all  $p$ .*

*If  $\lambda_n(0)$  is a simple eigenvalue of  $A_0$  then the function  $\lambda_n$  is symmetric. If  $\lambda_n(0)$  is an eigenvalue of  $A_0^{SA}$  (resp.  $A_0^{AS}$ ) we can choose  $W_n$  such that  $W_n(-p) = S^{SA} W_n(p)$  (resp.  $W_n(-p) = S^{AS} W_n(p)$ ), for all  $p \in \mathbb{R}$ . In this case, for all  $\mathbf{u} \in H_s^{AS} := H_s \cap H^{AS}$  (resp.  $\mathbf{u} \in H_s^{SA} := H_s \cap H^{SA}$ ),  $s > 1/2$ ,  $F\mathbf{u} = (f^k)_{k \geq 1}$  is such that  $f^n$  is anti-symmetric.*

*Proof.* To prove the first assertion we suppose that  $\lambda_n(0)$  is a simple eigenvalue. The case where  $\lambda_n(0)$  is double is handled in a similar fashion.

According to Kato [9], ch.7, there exists  $\varepsilon, \eta > 0$  small enough such that  $\lambda_n(p)$  is the unique eigenvalue of  $A_p$  in  $(\lambda_n(0) - \varepsilon, \lambda_n(0) + \varepsilon)$  for  $|p| < \eta$ . Denote by  $\check{\lambda}_n$  the function  $p \rightarrow \lambda_n(-p)$ . Since  $A_{-p}$  is unitarily equivalent to  $A_p$ ,  $\check{\lambda}_n(p)$  is an eigenvalue of  $A_p$  and necessarily coincides with  $\lambda_n(p)$  for  $p$  sufficiently small, and thus everywhere because of the analyticity. Therefore  $\lambda_n(-p) = \lambda_{n'}(p)$  for all  $p$ , where  $n' = n$ , and  $\lambda_n$  is symmetric.

Denote  $Q_n(p)$  the orthogonal projection associated with the eigenvalue  $\lambda_n(p)$ . Since  $\lambda_n(0)$  is simple and according to [9], ch.7, the mapping  $p \rightarrow Q_n(p) \in \mathcal{B}(\overline{H})$  is analytic in a neighborhood of 0, and in view of (23) we have  $Q_n(-p) = S^{SA} Q_n(p) (S^{SA})^{-1} = S^{AS} Q_n(p) (S^{AS})^{-1}$ . Suppose up to the end of the proof that  $\lambda_n(0)$  is an eigenvalue of  $A_0^{SA}$ . Take an associated eigenvector  $\mathbf{v}_0 \in H^{SA}$  with  $\lambda_n(0)$  and define the function  $p \rightarrow \mathbf{v}(p) := Q_n(p) \mathbf{v}_0 \in \overline{H}$  which is analytic and does not vanish in a neighborhood of 0. Thus, in this neighborhood the function  $p \rightarrow \tilde{W}(p) := \mathbf{v}(p) / \|\mathbf{v}(p)\|_{\overline{H}}$  is well-defined, analytic and  $\|\tilde{W}(p)\|_{\overline{H}} = 1$ .

Since  $\mathbf{v}_0 \in H^{SA}$  we have  $(S^{SA})^{-1} \mathbf{v}_0 = \mathbf{v}_0$ , therefore

$$\tilde{W}(-p) = \frac{Q_n(-p) \mathbf{v}_0}{\|Q_n(-p) \mathbf{v}_0\|_{\overline{H}}} = \frac{S^{SA} Q_n(p) (S^{SA})^{-1} \mathbf{v}_0}{\|S^{SA} Q_n(p) (S^{SA})^{-1} \mathbf{v}_0\|_{\overline{H}}} = \frac{S^{SA} \mathbf{v}(p)}{\|S^{SA} \mathbf{v}(p)\|_{\overline{H}}} = S^{SA} \tilde{W}(p).$$

The function  $\tilde{W}$  can be continued analytically along the real axis (cf. [9], ch.7). This continuation, denoted  $W_n$ , obviously satisfies  $W_n(-p) = S^{SA} W_n(p)$  for all  $p$ .

If  $\mathbf{u} \in H_s$  with  $s > 1/2$  we verify (cf. (8)) that the integral

$$(24) \quad f^n(p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \langle \mathbf{u}(x_1, \cdot), W_n(p) \rangle_{\overline{H}} e^{-ipx_1} dx_1, \quad \forall p \in \mathbb{R},$$

with  $F\mathbf{u} = (f^k)_{k \geq 1}$ , is convergent. Using the variable change  $x_1 \rightarrow -x_1$  we obtain

$$(25) \quad f^n(-p) = (2\pi)^{-1/2} \int_{\mathbb{R}} \langle \mathbf{u}(-x_1, \cdot), W_n(-p) \rangle_{\overline{H}} e^{-ipx_1} dx_1.$$

Clearly, for  $\mathbf{u} \in H_s^{\text{AS}}$  we have  $S^{\text{AS}}\mathbf{u}(-x_1, \cdot) = \mathbf{u}(x_1, \cdot)$ . Since  $\lambda_n(0)$  is an eigenvalue of  $A_0^{\text{SA}}$  we have  $S^{\text{AS}}W_n(-p) = -S^{\text{SA}}W_n(-p) = -W_n(p)$ . Knowing that  $S^{\text{SA}}$  is unitary we thus have

$$\langle \mathbf{u}(-x_1, \cdot), W_n(-p) \rangle_{\overline{H}} = \langle S^{\text{AS}}\mathbf{u}(-x_1, \cdot), S^{\text{AS}}W_n(-p) \rangle_{\overline{H}} = -\langle \mathbf{u}(x_1, \cdot), W_n(p) \rangle_{\overline{H}}.$$

Therefore, in view of (24) and (25), the function  $f^n$  is anti-symmetric. □

### 3. Multiplication operator $M$ in $\bigoplus_{\infty} L^2(\mathbb{R})$ and unitarily equivalent operators to $M$

We suppose that  $(\mu_n)_{n \geq 1}$  (resp.  $Y^s$ ,  $s \geq 0$ ) is an arbitrary family of functions (resp. spaces) satisfying hypothesis **(H1)** (resp. **(H2)**). It is well-known that the multiplication operator  $M$  defined by (1) has an absolutely continuous spectrum (cf. [14]) and that its spectral measure  $E_M$  can be explicitly given. In this work we derive a LAP outside thresholds (cf. Theorem 3.3), and at thresholds in appropriate spaces (cf. Theorem 3.5), as well as a division theorem (cf. Theorem 3.7).

This enables to deduce similar results for any selfadjoint operator  $D$  in a Hilbert space  $Z^0$ , unitarily equivalent to such a  $M$ . More precisely, suppose there exists a unitary transformation  $U : Z^0 \rightarrow Y^0$  such that  $D = U^{-1}MU$ . Then  $D$  has an absolutely continuous spectrum and its spectral measure,  $E_D$ , is given by  $E_D = U^{-1}E_MU$ . Using the spaces  $Z^s := U^{-1}(Y^s)$  a LAP for  $D$  and a division theorem is directly derived.

We need the following lemmas.

**Lemma 3.1.** *Let  $k \geq 0$  be an integer and  $s > k + 1/2$ . Any  $f \in H^s(\mathbb{R})$  is Hölder continuous and belongs to  $C^k(\mathbb{R}) \cap L^\infty(\mathbb{R})$ . More precisely, there exist two constants  $c(s)$  and  $c(s, \delta)$ , with  $0 \leq \delta < \min(1, s - 1/2)$ , such that*

$$|f(p)| \leq c(s)\|f\|_{H^s(\mathbb{R})}, \quad |f(p') - f(p)| \leq c(s, \delta)|p' - p|^\delta \|f\|_{H^s(\mathbb{R})}, \quad \forall p, p' \in \mathbb{R}.$$

Let  $s_1, s_2 > 1/2$  and  $\mathbf{f} = (f^n)_{n \geq 1} \in Y^{s_1}$ ,  $\mathbf{g} = (g^n)_{n \geq 1} \in Y^{s_2}$ . According to **(H2)**,  $f^n$  and  $g^n$  belong locally to  $H^{s_1}(\mathbb{R})$  and  $H^{s_2}(\mathbb{R})$ , respectively. Then by the last lemma for  $\phi \in C_0^\infty(\mathbb{R})$  and  $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$  there exists a constant  $c > 0$  such that

$$(26) \quad |\phi(p')f^n(p')\overline{g^n(p')} - \phi(p)f^n(p)\overline{g^n(p)}| \leq c|p' - p|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

**Lemma 3.2.** *Let  $\bar{I}$  be a compact interval of  $\mathbb{R}$  and  $h$  a  $\delta$ -Hölder continuous function on  $\bar{I}$ ,  $\delta \in (0, 1)$ :*

$$\exists C(h), \quad |h(\mu') - h(\mu)| \leq C(h)|\mu' - \mu|^\delta, \quad \forall \mu', \mu \in \bar{I},$$

*vanishing at the end points of  $\bar{I}$ . Then, for all compact  $K \subset \mathbb{C}$ , the function*

$$z \longrightarrow b(z, h) := \int_I \frac{h(\mu)}{\mu - z} d\mu$$

*is  $\delta$ -Hölder continuous on  $K \cap \mathbb{C}^\pm$ : there exists a constant  $c$ , independent of  $h$ , such that*

$$|b(z', h) - b(z, h)| \leq cC(h)|z' - z|^\delta, \quad \forall z', z \in K \cap \mathbb{C}^\pm.$$

*For all  $\mu_0 \in I$  we have*

$$(27) \quad \lim_{\substack{z \rightarrow \mu_0 \\ \pm \operatorname{Im} z > 0}} b(z) = \text{p.v.} \int_I \frac{h(\mu)}{\mu - \mu_0} d\mu \pm i\pi h(\mu_0).$$

For the proof we refer to Gakhov ([8], ch.1, 5) or Muskhelishvili ([10], ch.2, 22).

To apply the last lemma we shall invert locally the functions  $\mu_n$ ,  $n \geq 1$ , thanks to variable changes, and use (26). To do so we fix an enumeration  $p_n^j$ ,  $j \in \mathcal{J}_n \subset \mathbb{Z}$ , of the roots of  $\mu'_n$  which form a discrete subset of  $\mathbb{R}$  by **(H1)**. We set  $p_n^0$  to be the root of  $\mu'_n$ , if it exists, with smallest modulus (if there are two roots with opposite signs and the same modulus we take the positive one). Then denote  $p_n^1$  (resp.  $p_n^{-1}$ ) the first root greater (resp. less) than  $p_n^0$ , if it exists, and so on. The roots are thus  $p_n^j$ ,  $j \in \mathcal{J}_n$ . If  $\mu'_n$  does not vanish, that is  $\mathcal{J}_n = \emptyset$ , we set  $p_n^0 = -\infty$  and  $p_n^1 = +\infty$ .

If  $j_{n,\max} := \max \mathcal{J}_n < \infty$  we set  $p_n^{j_{n,\max}+1} = +\infty$ . Similarly, if  $j_{n,\min} := \min \mathcal{J}_n > -\infty$  we set  $p_n^{j_{n,\min}-1} = -\infty$ . Let us define  $\overline{\mathcal{J}}_n$  as follows: (1)  $\overline{\mathcal{J}}_n = \{0\}$  if  $\mathcal{J}_n = \emptyset$ , in this case  $p_n^0 = -\infty$  and  $p_n^1 = +\infty$ . (2)  $\overline{\mathcal{J}}_n = \mathcal{J}_n \cup \{j_{n,\min} - 1\}$ , if  $j_{n,\min} > -\infty$ . (3)  $\overline{\mathcal{J}}_n = \mathcal{J}_n$ , otherwise. Thus, for all  $j \in \overline{\mathcal{J}}_n$ ,  $\mu_n : (p_n^j, p_n^{j+1}) \longrightarrow I_n^j := \mu_n((p_n^j, p_n^{j+1}))$  is  $C^\infty$  diffeomorphism, we denote  $P_n^j$  its inverse.

**3.1. Limiting absorption principle outside thresholds** For all  $s > 0$ , denote  $Y^{-s}$  the anti-dual of  $Y^s$ . Identifying  $Y^0$  to its anti-dual and using **(H2)** we verify that the embeddings  $Y^{s_1} \hookrightarrow Y^0 \hookrightarrow Y^{-s_2}$  are continuous and have dense ranges for all  $s_1, s_2 \geq 0$ . For all  $z \in \mathbb{C}^\pm$ ,  $R_M(z)$  can then be considered as an element of  $\mathcal{B}(Y^{s_1}, Y^{-s_2})$ . Likewise,  $R_D(z)$  can be considered as an element of  $\mathcal{B}(Z^{s_1}, Z^{-s_2})$ .

**Theorem 3.3.** *Let  $s_1, s_2 > 1/2$ . The mapping  $z \longrightarrow R_M(z) \in \mathcal{B}(Y^{s_1}, Y^{-s_2})$ , defined on  $\mathbb{C}^\pm$ , can be extended continuously to  $\overline{\mathbb{C}^\pm} \setminus \Gamma$ . This extension, denoted  $R_M^\pm(z)$ , is locally  $\delta$ -Hölder continuous,  $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$ .*

The precedent assertion holds when we replace  $M$  by  $D$  and  $Y$  by  $Z$ .  
 For all  $\mu_0 \in \mathbb{R} \setminus \Gamma$ ,  $\mathbf{f} = (f^n)_{n \geq 1} \in Y^{s_1}$  and  $\mathbf{g} = (g^n)_{n \geq 1} \in Y^{s_2}$ , we have

$$(28) \quad \langle R_D^\pm(\mu_0)U^{-1}\mathbf{f}, U^{-1}\mathbf{g} \rangle = \langle R_M^\pm(\mu_0)\mathbf{f}, \mathbf{g} \rangle = \sum_{n=1}^{\infty} \text{p.v.} \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - \mu_0} dp$$

$$(29) \quad \pm i\pi \sum_{((p_0, n); n \geq 1, \mu_n(p_0) = \mu_0)} \frac{f^n(p_0)\overline{g^n(p_0)}}{|\mu'_n(p_0)|}.$$

Proof. Let  $K \subset \mathbb{C}^\pm \setminus \Gamma$  be a compact set, we shall prove that there exists  $c = c_K$  such that

$$(30) \quad |\langle R_M(z')\mathbf{f} - R_M(z)\mathbf{f}, \mathbf{g} \rangle| \leq c|z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}$$

for all  $\mathbf{f} \in Y^{s_1}$ ,  $\mathbf{g} \in Y^{s_2}$  and  $z, z' \in K \cap \mathbb{C}^\pm$ .

The assertion of the theorem will follow since

$$(31) \quad \begin{aligned} \langle R_D^\pm(z)\mathbf{u}, \mathbf{v} \rangle_{Z^{-s_2} \times Z^{s_2}} &= \langle R_M^\pm(z)U\mathbf{u}, U\mathbf{v} \rangle_{Y^{-s_2} \times Y^{s_2}}, \\ \|R_D^\pm(z') - R_D^\pm(z)\|_{\mathcal{B}(Z^{s_1}, Z^{-s_2})} &= \|R_M^\pm(z') - R_M^\pm(z)\|_{\mathcal{B}(Y^{s_1}, Y^{-s_2})}. \end{aligned}$$

In view of **(H1)** and since  $K$  is compact, there exists  $n_0 \geq 1$  such that  $|\mu_n(p) - z|^{-1} \leq 1$  for all  $p \in \mathbb{R}$ ,  $z \in K$  and  $n \geq n_0$ . Set  $\mathbf{b}_0(z) = \sum_{n \geq n_0} \mathbf{b}_n(z)$  (cf. (13)). Using Schwartz inequality we have

$$|\mathbf{b}_0(z') - \mathbf{b}_0(z)| \leq |z' - z| \cdot \|\mathbf{f}\|_{Y^0} \|\mathbf{g}\|_{Y^0} \leq c_0|z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

To end the proof of (30) it remains to establish an analogous estimate for  $\mathbf{b}_n$ ,  $0 < n < n_0$ :

$$(32) \quad |\mathbf{b}_n(z') - \mathbf{b}_n(z)| \leq c_n|z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

Let  $\phi_1 \in C_0^\infty(\mathbb{R})$  be equal to 1 on a neighborhood of  $\mu_n^{-1}(K \cap \mathbb{R})$ . Put  $\phi_0(p) = 1 - \phi_1(p)$ ,

$$b_0(z) = \int_{\mathbb{R}} \frac{\phi_0(p)f^n(p)\overline{g^n(p)}}{\mu_n(p) - z} dp, \quad \text{and} \quad b_1(z) = \int_{\mathbb{R}} \frac{\phi_1(p)f^n(p)\overline{g^n(p)}}{\mu_n(p) - z} dp.$$

Due to the assumption on the support of  $\phi_1$  one has

$$\sup \left\{ \frac{|\phi_0(p)|}{|\mu_n(p) - z|^2}; p \in \mathbb{R}, z \in K \right\} := d < \infty,$$

so there exists  $\tilde{c}_0$  such that, for all  $z', z \in K \cap \mathbb{C}^\pm$ ,

$$(33) \quad |b_0(z') - b_0(z)| \leq d|z' - z| \cdot \|f^n\|_{L^2(\mathbb{R})} \|g^n\|_{L^2(\mathbb{R})} \leq \tilde{c}_0|z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}.$$

On the other hand, using the variable change  $p = P_n^j(\mu)$  we get

$$b_1(z) = \sum_{j \in \mathcal{J}_n} \int_{p_n^j}^{p_n^{j+1}} \frac{\phi_1(p) f^n(p) \overline{g^n(p)}}{\mu_n(p) - z} dp = \sum_{j \in \mathcal{J}_n} b_1^j(z),$$

where

$$(34) \quad b_1^j(z) := \int_{I_n^j} \frac{h_j(\mu)}{\mu - z} d\mu \quad \text{and} \quad h_j(\mu) := \frac{\phi_1(P_n^j(\mu)) f^n(P_n^j(\mu)) \overline{g^n(P_n^j(\mu))}}{|\mu'_n(P_n^j(\mu))|}.$$

Taking  $\phi_1$  with a sufficiently small support, one can assume that the support of its restriction to  $(p_n^j, p_n^{j+1})$  is compact. So, we verify, using (26), that  $\mu \rightarrow h_j(\mu)$ , which has a compact support in  $I_n^j$ , is  $\delta$ -Hölder continuous:  $|h_j(\mu') - h_j(\mu)| \leq c'_j \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}} |\mu' - \mu|^\delta$ , for all  $\mu', \mu \in I_n^j$ . Applying Lemma 3.2, it follows that  $|b_1^j(z') - b_1^j(z)| \leq c^j |z' - z|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}$ . Using this formula and (33) we deduce (32).  $\square$

**3.2. Limiting absorption principle at thresholds** In view of **(H1)**, for all  $n \geq 1$  and  $j \in \mathcal{J}_n$ , the order of the root  $p_n^j$  of  $p \rightarrow \mu_n(p) - \mu_n(p_n^j)$ , denoted  $N_n^j \geq 2$ , is finite. Thus there exist two functions  $G_{n,1}^j, G_{n,2}^j \in C^\infty(\mathbb{R})$ , such that  $G_{n,1}^j(p_n^j) \neq 0$ ,  $G_{n,2}^j(p_n^j) \neq 0$  and

$$(35) \quad \mu_n(p) - \mu_n(p_n^j) = (p - p_n^j)^{N_n^j} G_{n,1}^j(p), \quad \frac{d\mu_n}{dp}(p) = (p - p_n^j)^{N_n^j-1} G_{n,2}^j(p).$$

Since  $P_n^j$  (resp.  $P_n^{j-1}$ ) is the inverse of the restriction of  $\mu_n$  to  $[p_n^j, p_n^{j+1}]$  (resp.  $[p_n^{j-1}, p_n^j]$ ) we get

$$(36) \quad \frac{dP_n^j}{d\mu}(\mu) = \frac{1}{\mu'_n(P_n^j(\mu))} = \frac{1}{(P_n^j(\mu) - p_n^j)^{N_n^j-1} G_{n,2}^j(P_n^j(\mu))}$$

$$(37) \quad \frac{dP_n^{j-1}}{d\mu}(\mu) = \frac{1}{\mu'_n(P_n^{j-1}(\mu))} = \frac{1}{(P_n^{j-1}(\mu) - p_n^j)^{N_n^j-1} G_{n,2}^j(P_n^{j-1}(\mu))}.$$

The function  $P_n^j$  (resp.  $P_n^{j-1}$ ) is locally 1-Hölder continuous on  $I_n^j$  (resp.  $I_n^{j-1}$ ) but not on  $I_n^j \cup \{\mu_n^j\}$  (resp.  $I_n^{j-1} \cup \{\mu_n^j\}$ ), where it is locally  $(1/N_n^j)$ -Hölder continuous. In fact, using the left hand side equation in (35), one can verify that

$$(38) \quad |P_n^j(\mu') - P_n^j(\mu)| \leq c |\mu' - \mu|^{1/N_n^j}, \quad \forall \mu', \mu \in I_n^j \cup \{\mu_n^j\},$$

$$(39) \quad |P_n^{j-1}(\mu') - P_n^{j-1}(\mu)| \leq c |\mu' - \mu|^{1/N_n^j}, \quad \forall \mu', \mu \in I_n^{j-1} \cup \{\mu_n^j\}.$$

In view of **(H2)** and Lemma 3.1, for all  $n \geq 1, k \geq 1, j \in \mathcal{J}_n$  and  $s > k - 1/2$ ,

the subspace defined below is closed in  $Y^s$ . For  $k = 0$  we set  $NY_n^s(j, 0) := Y^s$ .

$$NY_n^s(j, k) := \left\{ \mathbf{f} = (f^m)_{m \geq 1} \in Y^s; \frac{d^l f^n}{dp^l}(p_n^j) = 0, l = 0, \dots, k - 1 \right\}.$$

**Lemma 3.4.** *Let  $k \geq 0$  be an integer and  $s > k + 1/2$ . For all  $f \in H^s(\mathbb{R})$  satisfying  $d^j f/dp^j(0) = 0, j = 0, \dots, k$ , we have  $v := f/p^{k+1} \in H^{s-(k+1)}(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$ , and there exists a constant  $c = c(k, s)$  such that  $\|v\|_{H^{s-(k+1)}(\mathbb{R})} \leq c\|f\|_{H^s(\mathbb{R})}$ .*

The proof for  $k = 0$  was given by Agmon [1], it can be generalized by recurrence to all  $k$  (for details see [3], annex C, Corollary C. 7). The case  $k = 1$  was proved in [5].

Let  $\mu_0$  be a threshold. Set

$$\begin{aligned} \mathcal{J}^{\mu_0} &:= \{(n, j) \in \mathbb{N} \times \mathbb{Z}; \mu_n(p_n^j) = \mu_0\} \subset \mathbb{N} \times \mathbb{Z}, \\ n^{\mu_0} &:= \{n \geq 1; \exists j \in \mathbb{Z} \text{ such that } (n, j) \in \mathcal{J}^{\mu_0}\} \subset \mathbb{N}. \end{aligned}$$

A multi-index  $\alpha$  is defined by its values  $\alpha_{n,j} \in \mathbb{N}, (n, j) \in \mathcal{J}^{\mu_0}$ . Denote  $|\alpha| = \max\{\alpha_{n,j}; (n, j) \in \mathcal{J}^{\mu_0}\}$ . Recall that  $N_n^j$  is the order of the root  $p_n^j$  of  $p \rightarrow \mu_n(p) - \mu_n(p_n^j)$ , and denote  $\mathbf{N}_{\mu_0}$  the multi-index  $\{N_n^j; (n, j) \in \mathcal{J}^{\mu_0}\}$ . For all multi-index  $\alpha$  and  $s > |\alpha| - 1/2$ , let us define the following closed subspace of  $Y^s$ :

$$NY^s(\mu_0, \alpha) := \bigcap_{(n,j) \in \mathcal{J}^{\mu_0}} NY_n^s(j, \alpha_{n,j}).$$

Denote  $NY^{-s}(\mu_0, \alpha)$  its anti-dual. We also define  $NZ^s(\mu_0, \alpha)$  by  $U^{-1}(NY^s(\mu_0, \alpha))$  and denote its anti-dual  $NZ^{-s}(\mu_0, \alpha)$ .

We verify that  $NY^s(\mu_0, \alpha)$  is dense in  $Y^0$ , which enables to identify any  $\mathbf{f} \in Y^0$  to the anti-linear form  $\mathbf{g} \rightarrow \langle \mathbf{f}, \mathbf{g} \rangle_{Y^0}$  on  $NY^s(\mu_0, \alpha)$ . With this identification, if  $\alpha^1, \alpha^2$  are two multi-indexes and  $s_1 > |\alpha^1| + 1/2, s_2 > |\alpha^2| + 1/2$ , the embeddings  $NY^{s_1}(\mu_0, \alpha^1) \hookrightarrow Y^0 \hookrightarrow NY^{-s_2}(\mu_0, \alpha^2)$  are continuous. The resolvent  $R_M(z)$  can thus be considered as an element of  $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$ . Likewise  $R_D(z) \in \mathcal{B}(NZ^{s_1}(\mu_0, \alpha^1), NZ^{-s_2}(\mu_0, \alpha^2))$ . As in (31) we have

$$\begin{aligned} (40) \quad & \|R_D^\pm(z') - R_D^\pm(z)\|_{\mathcal{B}(NZ^{s_1}(\mu_0, \alpha^1), NZ^{-s_2}(\mu_0, \alpha^2))} \\ &= \|R_M^\pm(z') - R_M^\pm(z)\|_{\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))}. \end{aligned}$$

**Theorem 3.5.** *Assume that  $\alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j - 1$ , for all  $(n, j) \in \mathcal{J}^{\mu_0}$ . The mapping  $z \rightarrow R_M(z) \in \mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$ , defined on  $\mathbb{C}^\pm$ , can be extended continuously to  $\{\mu_0\} \cup (\mathbb{C}^\pm \setminus \Gamma)$ . This extension, denoted  $R_M^\pm(z)$ , is locally  $(\delta/|N_{\mu_0}|)$ -Hölder continuous,  $\delta < \min(1, s_1 - |\alpha^1| - 1/2, s_2 - |\alpha^2| - 1/2)$ .*

*In view of (40) the precedent assertion holds when we replace  $M$  by  $D$  and  $Y$  by  $Z$ .*

For all  $\mathbf{f} \in NY^{s_1}(\mu_0, \alpha^1)$  and  $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$ , we have

$$\begin{aligned}
 (41) \quad & \langle R_D^\pm(\mu_0)U^{-1}\mathbf{f}, U^{-1}\mathbf{g} \rangle = \langle R_M^\pm(\mu_0)\mathbf{f}, \mathbf{g} \rangle \\
 & = \sum_{n \geq 1} \text{p.v.} \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - \mu_0} dp \pm i\pi \sum_{((p_0, n); n \geq 1, \mu_n(p_0) = \mu_0 \text{ and } \mu'_n(p_0) \neq 0)} \frac{f^n(p_0)\overline{g^n(p_0)}}{|\mu'_n(p_0)|} \\
 (42) \quad & \pm i\pi \sum_{(n, j) \in \mathcal{J}^{\mu_0}} \frac{\epsilon(N_n^j)}{|\mu_n^{(N_n^j)}(p_n^j)|} \frac{(N_n^j - 1)! d^{\alpha_{n,j}^1} f^n}{\alpha_{n,j}^1 \alpha_{n,j}^2! dp^{\alpha_{n,j}^1}}(p_n^j) \frac{d^{\alpha_{n,j}^2} \overline{g^n}}{dp^{\alpha_{n,j}^2}}(p_n^j),
 \end{aligned}$$

where  $\mu_n^{(N_n^j)}$  is the derivative of  $\mu_n$  of order  $N_n^j$ , and  $\epsilon(\cdot)$  the mapping, defined on  $\mathbb{N}$ , which takes the value 0 if its argument is even and 1 otherwise.

Proof. As in the proof of Theorem 3.3, the same estimate holds for  $\mathbf{b}_0, b_0$  and  $b_1^j$  if  $\mu_n(p_n^{j'})$  and  $\mu_n(p_n^{j'+1})$  are not equal to  $\mu_0$ . It only suffices to prove a similar estimate for

$$(43) \quad \mathbf{b}_1^j(z) = b_1^{j-1}(z) + b_1^j(z) = \int_{p_n^{j-1}}^{p_n^{j+1}} \frac{\phi_1(p)f^n(p)\overline{g^n(p)}}{\mu_n(p) - z} dp,$$

where we suppose that  $\text{supp } \phi_1 \subset (p_n^{j-1}, p_n^{j+1})$ .

Since  $\mathbf{f} \in NY^{s_1}(\mu_0, \alpha^1)$  and  $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$  we have  $f^n(p) = (p - p_n^j)^{\alpha_{n,j}^1} h_1(p)$  and  $g^n(p) = (p - p_n^j)^{\alpha_{n,j}^2} h_2(p)$  where, according to Lemma 3.4,  $h_l, l = 1, 2$ , is  $\delta^l$ -Hölder continuous, with  $\delta^l < \min(1, s_l - |\alpha^l| - 1/2)$ . Setting  $h = h_1 \overline{h_2}$ , we then verify that

$$(44) \quad |h(p') - h(p)| \leq c|p' - p|^\delta \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}}, \quad \text{with } \delta = \min(\delta^1, \delta^2).$$

Since  $\alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j - 1$  we have

$$(45) \quad f^n(p)\overline{g^n(p)} = (p - p_n^j)^{N_n^j - 1} h(p).$$

Using the variable changes  $p = P_n^{j-1}(\mu)$  and  $p = P_n^j(\mu)$ , and taking (36) and (37) into account, we get

$$\mathbf{b}_1^j(z) = \int_{I_n^j} \frac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{(\mu - z)|G_{n,2}^j(P_n^j(\mu))|} d\mu - (-1)^{N_n^j} \int_{I_n^{j-1}} \frac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{(\mu - z)|G_{n,2}^j(P_n^{j-1}(\mu))|} d\mu.$$

Assume that  $\mu_n^{j+1} > \min(\mu_n^j, \mu_n^{j-1})$  and define the function  $\mathcal{H}(\cdot)$  as follows:

- if  $N_n^j$  is even (here we suppose in addition that  $\mu_n^j \leq \mu_n^{j-1}$ ):

$$\mathcal{H}(\mu) = \begin{cases} \frac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|} - \frac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{|G_{n,2}^j(P_n^{j-1}(\mu))|}, & \text{if } \mu \in [\mu_n^j, \mu_n^{j-1}], \\ \frac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|}, & \text{if } \mu \in [\mu_n^{j-1}, \mu_n^{j+1}], \\ 0 & \text{otherwise,} \end{cases}$$

- if  $N_n^j$  is odd:

$$\mathcal{H}(\mu) = \begin{cases} \frac{\phi_1(P_n^{j-1}(\mu))h(P_n^{j-1}(\mu))}{|G_{n,2}^j(P_n^{j-1}(\mu))|}, & \text{if } \mu \in [\mu_n^{j-1}, \mu_n^j], \\ \frac{\phi_1(P_n^j(\mu))h(P_n^j(\mu))}{|G_{n,2}^j(P_n^j(\mu))|}, & \text{if } \mu \in [\mu_n^j, \mu_n^{j+1}], \\ 0 & \text{otherwise.} \end{cases}$$

Knowing that  $\mathcal{H}(\cdot)$  vanishes at  $\mu_n^j$  if  $N_n^j$  is even and that it is continuous at the same point if  $N_n^j$  is odd, we verify, using (44) and (38)–(39) that:

$$\exists c > 0, \quad |\mathcal{H}(\mu') - \mathcal{H}(\mu)| \leq c \|\mathbf{f}\|_{Y^{s_1}} \|\mathbf{g}\|_{Y^{s_2}} |\mu' - \mu|^{\delta/N_n^j}, \quad \forall \mu', \mu \in \mathbb{R}.$$

The desired estimate for  $\mathbf{b}_1^j(z)$  follows by invoking Lemma 3.2 since  $\mathbf{b}_1^j(z) = \int_{\mathbb{R}} \mathcal{H}(\mu)/(\mu - z) d\mu$ . Formula (41)–(42) is a consequence of (27) since  $\mathcal{H}(\mu_n^j) = 0$  if  $N_n^j$  is even. □

Other variant of the LAP at thresholds: let  $\alpha^1, \alpha^2$  such that

$$(46) \quad \alpha_{n,j}^1, \alpha_{n,j}^2 \geq 1, \quad \alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j, \quad \forall (n, j) \in \mathcal{J}^{\mu_0}.$$

For  $s_1 > |\alpha^1| + 1/2$  and  $s_2 > |\alpha^2| + 1/2$  the previous theorem gives a LAP with  $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$ . One can in fact prove a similar result only with:

$$(47) \quad s_1 > |\alpha^1| - 1/2, \quad s_2 > |\alpha^2| - 1/2 \quad \text{and} \quad s_1 + s_2 > |\alpha^1| + |\alpha^2|.$$

This will be used below in the case where the function  $\mu_n$  is symmetric, and applied later to obtain a LAP at some thresholds for the operators associated with a half-strip (see Theorem 4.1). The conditions on  $s_1$  and  $s_2$  are thus improved compared to applying Theorem 3.5.

The difference from the proof of Theorem 3.5 lies in the treatment of  $\mathbf{b}_1^j(z)$  (cf. (43)). Indeed, for  $\mathbf{f} \in NY^{s_1}(\mu_0, \alpha^1)$  and  $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$  we have, according to Lemma 3.4,  $f^n(p) = (p - p_n^j)^{\alpha_{n,j}^1 - 1} h_1(p)$  and  $\overline{g^n(p)} = (p - p_n^j)^{\alpha_{n,j}^2 - 1} h_2(p)$  where,  $h_l$ ,  $l = 1, 2$ , are locally Hölder continuous. Given (46) and (47) there exist  $r_1$  and  $r_2$  such that  $0 < r_1 < s_1 - |\alpha^1| + 1/2$ ,  $0 < r_2 < s_2 - |\alpha^2| + 1/2$  and  $r_1 + r_2 = 1$ . Verifying that

$h_1(p_n^j) = 0$ , we check that  $h_1(p)/|p - p^j|^{r_1}$  and  $h_2(p)/|p - p_n^j|^{r_2}$  are locally  $\delta_1$  and  $\delta_2$  Hölder continuous, respectively, for  $\delta_1 < s_1 - |\alpha^1| + 1/2 - r_1$  and  $\delta_2 < s_2 - |\alpha^2| + 1/2 - r_2$ . The function  $h$  defined below is then  $\delta$ -Hölder continuous with  $\delta = \min(\delta_1, \delta_2)$ .

$$h(p) := \pm \frac{h_1(p)}{|p - p_n^j|^{r_1}} \frac{h_2(p)}{|p - p_n^j|^{r_2}}, \quad \text{if } \pm(p - p_n^j) > 0.$$

Since  $\alpha_{n,j}^1 + \alpha_{n,j}^2 = N_n^j$  and  $r_1 + r_2 = 1$  we have, as in (45),  $f^n(p)\overline{g^n(p)} = (p - p_n^j)^{N_n^j - 1} h(p)$ . We then proceed as above by using the variable changes  $p = P_n^{j-1}(\mu)$  and  $p = P_n^j(\mu)$ , and applying Lemma 3.2. Hence  $z \rightarrow R_M^\pm(z)$  is locally  $(\delta/|\mathbf{N}_{\mu_0}|)$ -Hölder on  $\{\mu_0\} \cup (\mathbb{C}^\pm \setminus \Gamma)$ .

Considering the upper-bound over the set of  $(r_1, r_2)$  one verifies that  $\delta > 0$  could be any real such that

$$\delta < \min(1, s_1 - |\alpha^1| + 1/2, s_2 - |\alpha^2| + 1/2, (s_1 + s_2 - |\alpha^1| - |\alpha^2|)/2).$$

*Optimality:* let  $s_1$  and  $s_2$  be sufficiently large. The previous results concerning the LAP at thresholds require the condition  $\alpha_j^1 + \alpha_j^2 \geq N^j - 1$  for all  $(j, n) \in \mathcal{J}^{\mu_0}$ . One can prove its optimality (in some cases). More precisely, if  $\alpha_{n,j}^1 + \alpha_{n,j}^2 < N_n^j - 1$ , and if  $\alpha_{n,j}^1 + \alpha_{n,j}^2$  is even or  $\alpha_{n,j}^1 + \alpha_{n,j}^2$  and  $N_n^j$  are odd then we do not have a LAP with  $\mathcal{B}(NY^{s_1}(\mu_0, \alpha^1), NY^{-s_2}(\mu_0, \alpha^2))$ . Indeed, for all  $s_1 > N_n^j - 3/2$  and  $\mathbf{f} \in Y^{s_1}$  such that  $f^n \in NY_n^{s_1}(j, \alpha_{n,j}^1) \setminus NY_n^{s_1}(j, \alpha_{n,j}^1 + 1)$  we prove (cf. [3]) that there exists  $\mathbf{g} \in NY^{s_2}(\mu_0, \alpha^2)$ , with  $s_2 > \alpha_{n,j}^2 + 1/2$ , such that

$$\lim_{\varepsilon \rightarrow 0} |\langle R_D(\mu_0 \pm i\varepsilon)U^{-1}\mathbf{f}, U^{-1}\mathbf{g} \rangle| = \lim_{\varepsilon \rightarrow 0} |\langle R_M(\mu_0 \pm i\varepsilon)\mathbf{f}, \mathbf{g} \rangle| = +\infty.$$

**3.3. Symmetry property and the limiting absorption principle at thresholds**

Let  $\mu_0 = \mu_{n_0}(0)$  be a threshold such that: (1)  $\mu_{n_0}$  is a symmetric function, (2)  $\mu_0 = \mu_n(p_0)$  with  $\mu_n'(p_0) = 0$  implies  $n = n_0$  and  $p_0 = 0$ , (3)  $\mu_{n_0}''(0) \neq 0$ , that is to say  $N_n^0 = 2$ . Denote

$$(48) \quad Y_{\mu_0, AS}^s := \{\mathbf{f} \in Y^s; f^{n_0}(-p) = -f^{n_0}(p)\} \quad \text{and} \quad Z_{\mu_0, AS}^s = U^{-1}Y_{\mu_0, AS}^s.$$

According to Theorem 3.5 there is a LAP at  $\mu_0$  with  $\mathcal{B}(NY_{n_0}^{s'}(0, 1), Y^{-s})$  and  $\mathcal{B}(Y^s, NY_{n_0}^{-s'}(0, 1))$ , where  $s > 1/2$ ,  $s' > 3/2$ . We also remarked that a LAP holds with:

$$(49) \quad \mathcal{B}(NY_{n_0}^{s_1}(0, 1), NY_{n_0}^{-s_2}(0, 1)) \quad \text{where } s_1, s_2 > 1/2 \text{ and } s_1 + s_2 > 2.$$

Taking the symmetry property into account the following theorem improves this result.

**Theorem 3.6.** *Let  $s_1, s_2 > 1/2$  with  $s_1 + s_2 > 2$ . The mapping  $z \rightarrow R_M(z) \in \mathcal{B}(Y_{\mu_0, AS}^{s_1}, Y^{-s_2})$ , defined on  $\mathbb{C}^\pm$ , can be extended continuously to  $\{\mu_0\} \cup (\mathbb{C}^\pm \setminus \Gamma)$ . This*

extension is locally  $(\delta/2)$ -Hölder continuous,  $\delta < \min(1, s_1 - 1/2, s_2 - 1/2, (s_1 + s_2 - 2)/2)$ .

The precedent assertion holds when we replace, respectively,  $M$  by  $D$  and  $Y$  by  $Z$ .

Proof. The key idea is the construction of a linear continuous mapping  $Q : Y^s \rightarrow NY_{n_0}^s(0, 1)$  such that  $\text{Ran}(I_d - Q)$  is a subset of  $(Y_{\mu_0, AS}^0)^\perp$  which is the orthogonal subspace to  $Y_{\mu_0, AS}^0$  in the Hilbert space  $Y^0$ .

Take a symmetric function  $\phi \in C_0^\infty(\mathbb{R})$  such that  $\phi(0) = 1$ . Thanks to **(H2)**,  $(\phi\delta_{n_0}^k)_{k \geq 1}$  belongs to  $Y^s$  for all  $s \geq 0$ , and a suitable mapping  $Q$  is given by

$$Q\mathbf{f} = \mathbf{f} - f^{n_0}(0)(\phi\delta_{n_0}^k)_{k \geq 1}, \quad \forall \mathbf{f} \in Y^s.$$

For all  $\mathbf{f} \in Y_{\mu_0, AS}^{s_1}$ ,  $\mathbf{g} \in Y^{s_2}$  and  $z \in \mathbb{C}^\pm$ , we have  $R_M(z)\mathbf{f} \in Y_{\mu_0, AS}^0$  and  $\mathbf{g} - Q\mathbf{g} \in (Y_{\mu_0, AS}^0)^\perp$ , therefore

$$\langle R_M(z)\mathbf{f}, \mathbf{g} \rangle = \langle R_M(z)\mathbf{f}, Q\mathbf{g} \rangle.$$

Since  $\mathbf{f}$  and  $Q\mathbf{g}$  belong to  $NY_{n_0}^{s_1}(0, 1)$  and  $NY_{n_0}^{s_2}(0, 1)$ , respectively, it suffices to use the LAP with (49). □

**3.4. Division theorem** We have the following division theorem outside thresholds. One can prove a similar result at thresholds.

**Theorem 3.7.** *Let  $\mu_0 \in \mathbb{R} \setminus \Gamma$  and  $s > 1/2$ . Set  $\tilde{s} = \max(0, 1-s)$ . If  $\mathbf{f} \in Y^s$  verifies one of the following three equivalent conditions: (1) for all  $n \geq 1$  and  $p_0$  such that  $\mu_n(p_0) = \mu_0$ , we have  $f^n(p_0) = 0$ , (2)  $R_M^+(\mu_0)\mathbf{f} = R_M^-(\mu_0)\mathbf{f}$ , (3)  $\text{Im}\langle R_M^\pm(\mu_0)\mathbf{f}, \mathbf{f} \rangle = 0$ , then  $R_M^\pm(\mu_0)\mathbf{f} \in Y^{-\tilde{s}}$ . In particular,  $R_M^\pm(\mu_0)\mathbf{f} \in Y^0$  if  $s \geq 1$ .*

*Set  $\mathbf{u} = U^{-1}\mathbf{f}$ . We can replace the last two conditions by  $R_D^+(\mu_0)\mathbf{u} = R_D^-(\mu_0)\mathbf{u}$  and  $\text{Im}\langle R_D^\pm(\mu_0)\mathbf{u}, \mathbf{u} \rangle = 0$ , respectively. In this case  $R_D^\pm(\mu_0)\mathbf{u} \in Z^{-\tilde{s}}$ .*

Proof. We verify, using (28)–(29), that the three conditions of the theorem are equivalent and that in such a case

$$\langle R_D^\pm(\mu_0)\mathbf{u}, \mathbf{v} \rangle_{Z^{-s} \times Z^s} = \langle R_M^\pm(\mu_0)\mathbf{f}, \mathbf{g} \rangle_{Y^{-s} \times Y^s} = \sum_{n \geq 1} \int_{\mathbb{R}} \frac{f^n(p)\overline{g^n(p)}}{\mu_n(p) - \mu_0} dp,$$

where  $\mathbf{v} = U^{-1}\mathbf{g}$ . To prove the theorem it suffices to show that

$$|\langle R_M^\pm(\mu_0)\mathbf{f}, \mathbf{g} \rangle_{Y^{-s} \times Y^s}| \leq c\|\mathbf{g}\|_{Y^s}, \quad \forall \mathbf{g} \in Y^s \subset Y^{\tilde{s}}.$$

Let us go back to the proof of Theorem 3.3 (where  $K$  is now  $\{\mu_0\}$ , and  $\phi_1 = \psi^2$

with  $\psi \in C_0^\infty((p_n^j, p_n^{j+1}))$  and remark that it suffices to prove

$$(50) \quad |b_1^j(\mu_0)| \leq c'_0 \|\mathbf{g}\|_{Y^s}.$$

In fact, similar results obviously hold for  $b_0(\mu_0)$  and  $\mathbf{b}_0(\mu_0)$ . Recall (cf. (34)) that

$$(51) \quad b_1^j(\mu_0) = \int_{p_n^j}^{p_n^{j+1}} \frac{\phi_1(p) f^n(p) \overline{g^n(p)}}{\mu_n(p) - \mu_0} dp = \int_{\mathbb{R}} \frac{\psi(p) f^n(p) (p - p_0) \overline{\psi(p) g^n(p)}}{p - p_0 \mu_n(p) - \mu_0} dp.$$

Denote  $p_0$  the unique root of  $p \rightarrow \mu_n(p) - \mu_0$  in  $(p_n^j, p_n^{j+1})$ . The function  $\psi f^n$  vanishes at  $p_0$  and belongs to  $H^s(\mathbb{R})$  by **(H2)**. According to Lemma 3.4 (in which we consider the case  $k = 0$ ) we have  $\psi(p) f^n(p) = (p - p_0) \psi(p) v(p)$ , where  $\psi v$  belongs to  $H^{s-1}(\mathbb{R}) \cap L^1_{loc}(\mathbb{R})$  and there exists a constant  $c_1$  such that

$$(52) \quad \left| \int_{\mathbb{R}} \psi(p) v(p) \overline{\phi(p)} dp \right| \leq c'_1 \|\phi\|_{H^{1-s}(\mathbb{R})} \leq c_1 \|\phi\|_{H^s(\mathbb{R})}, \quad \forall \phi \in C_0^\infty(\mathbb{R}).$$

This inequality is still valid, using the density of  $C_0^\infty(\mathbb{R})$  in  $H^s(\mathbb{R})$ , for all  $\phi \in H^s(\mathbb{R})$ . On the other hand, since  $p_0$  is a simple root of  $p \rightarrow \mu_n(p) - \mu_0$  (because  $\mu_0 \notin \Gamma$ ) we have

$$\frac{(p - p_0) \psi(p)}{\mu_n(p) - \mu_0} \in C_0^\infty(\mathbb{R}),$$

thus

$$\phi(p) := \frac{(p - p_0) \psi(p)}{\mu_n(p) - \mu_0} \overline{g^n(p)} \in H^s(\mathbb{R}) \subset H^{\bar{s}}(\mathbb{R}),$$

and, in view of **(H2)**,  $\|\phi\|_{H^s(\mathbb{R})} \leq c''_1 \|\mathbf{g}\|_{Y^s(\mathbb{R})}$ . Combining this inequality with (52) we obtain (50), with  $c'_0 = c_1 c''_1$ . □

#### 4. Application to the elastic strip and generalization to other differential operators

**4.1. The elastic strip case** Let us go back to the elastic strip. Now  $(\mu_n) = (\lambda_n)$  and  $Y^s := F(H_s)$ ,  $s \geq 0$ . In view of Theorems 2.2 and 2.3 hypotheses **(H1)** and **(H2)** are satisfied and we are then within the framework of Section 3. More precisely, the conditions of application of this section are satisfied with  $D = A$ ,  $U = F$  and  $Z^s = H_s$ .

Thus  $A$  has an absolutely continuous spectrum (in particular  $A$  has no eigenvalues), and there is a LAP outside thresholds (cf. Theorem 3.3), and at thresholds in appropriate spaces (cf. Theorem 3.5), as well as a division theorem (cf. Theorem 3.7).

**4.2. Validity of a LAP for  $A_{\Omega_k}^{SA}$  and  $A_{\Omega_k}^{AS}$  at some thresholds** The simple eigenvalues  $\lambda_n(0)$  belong to  $\Gamma$  (cf. (11)) since, in view of Theorem 2.5,  $\lambda_n$  is symmetric so that  $\lambda'_n(0) = 0$ . According to  $(\mathcal{P})$  the order  $N_n^0 \geq 2$  of the root  $p = 0$  of

$p \rightarrow \lambda_n(p) - \lambda_n(0)$  may be such that  $N_n^0 > 2$ . In the following definitions of the disjoint subsets  $\Gamma^{\text{SA}}$  and  $\Gamma^{\text{AS}}$  of  $\Gamma$  we only consider the case  $N_n^0 = 2$ .

$$\Gamma^{\text{SA}}(\text{resp. } \Gamma^{\text{AS}}) = \{\lambda_0 = \lambda_n(0); \lambda_0 \text{ is an eigenvalue of } A_0^{\text{AS}}(\text{resp. } A_0^{\text{SA}}), N_n^0 = 2, \\ \text{and there are no } p \neq 0 \text{ and } n' \geq 1 \text{ such that } \lambda_0 = \lambda_{n'}(p) \text{ and } \lambda'_{n'}(p) = 0\}.$$

According to Theorem 2.5, if  $\lambda_0 \in \Gamma^{\text{SA}}$  (resp.  $\Gamma^{\text{AS}}$ ) we have  $H_s^{\text{SA}} \subset F^{-1}Y_{\lambda_0, \text{AS}}^s$  (cf. (48)) (resp.  $H_s^{\text{AS}} \subset F^{-1}Y_{\lambda_0, \text{AS}}^s$ ). Applying Theorem 3.6 we deduce that  $z \rightarrow R_A(z) \in \mathcal{B}(H_{s_1}^{\text{SA}}, H_{-s_2})$  (resp.  $\mathcal{B}(H_{s_1}^{\text{AS}}, H_{-s_2})$ ) can be extended continuously to  $\Gamma^{\text{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$  (resp.  $\Gamma^{\text{AS}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$ ). This extension is locally  $(\delta/2)$ -Hölder continuous where

$$(53) \quad \delta < \min(1, s_1 - 1/2, s_2 - 1/2, (s_1 + s_2 - 2)/2).$$

The following theorem gives a LAP for  $A_{\Omega_k}^{\text{SA}}$  and  $A_{\Omega_k}^{\text{AS}}$  (cf. (20) and (21)). Denote  $H_{\Omega_k, s}$  the weighted space equipped with the norm  $\|\mathbf{u}\|_{H_{\Omega_k, s}} := \|(1 + x_1^2)^{s/2} \mathbf{u}\|_{H_{\Omega_k}}$ . One can verify that the transformation  $\mathcal{U}_k^{\text{SA}}$  (resp.  $\mathcal{U}_k^{\text{AS}}$ ), used in (22), is also unitary from  $H_{\Omega_k, s}$  onto  $H_s^{\text{SA}}$  (resp.  $H_s^{\text{AS}}$ ).

**Theorem 4.1.** *The operator  $A_{\Omega_k}^{\text{SA}}$  has an absolutely continuous spectrum (in particular it has no eigenvalues). Let  $s_1, s_2 > 1/2$ . The mapping  $z \rightarrow R_{A_{\Omega_k}^{\text{SA}}}(z) \in \mathcal{B}(H_{\Omega_k, s_1}, H_{\Omega_k, -s_2})$  can be continuously extended to  $\overline{\mathbb{C}^\pm} \setminus \Gamma$  (resp.  $\Gamma^{\text{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$  if in addition  $s_1 + s_2 > 2$ ). This extension is locally  $\delta$ -Hölder continuous,  $\delta < \min(1, s_1 - 1/2, s_2 - 1/2)$ , (resp.  $(\delta/2)$ -Hölder continuous with (53)).*

*The precedent assertion holds when we replace  $A_{\Omega_k}^{\text{SA}}$  by  $A_{\Omega_k}^{\text{AS}}$  and  $\Gamma^{\text{SA}}$  by  $\Gamma^{\text{AS}}$ .*

Proof. Since  $A$  is the direct sum of  $A^{\text{SA}}$  and  $A^{\text{AS}}$  the latter have, as  $A$ , an absolutely continuous spectrum. Therefore, this also holds for  $A_{\Omega_k}^{\text{SA}}$  and  $A_{\Omega_k}^{\text{AS}}$  since they are, respectively, unitarily equivalent to the latter (cf. (22)).

We will restrict ourselves to proving a LAP for  $A_{\Omega_k}^{\text{SA}}$  on  $\Gamma^{\text{SA}} \cup (\overline{\mathbb{C}^\pm} \setminus \Gamma)$ . From (22) follows that  $R_{A_{\Omega_k}^{\text{SA}}}(z) = (\mathcal{U}_k^{\text{SA}})^{-1} R_{A^{\text{SA}}}(z) \mathcal{U}_k^{\text{SA}}$ . Therefore, and this will end the proof, for any  $\mathbf{u} \in H_{\Omega_k, s_1}$

$$\|R_{A_{\Omega_k}^{\text{SA}}}(z')\mathbf{u} - R_{A_{\Omega_k}^{\text{SA}}}(z)\mathbf{u}\|_{H_{\Omega_k, -s_2}} = \|R_{A^{\text{SA}}}(z')\mathcal{U}_k^{\text{SA}}\mathbf{u} - R_{A^{\text{SA}}}(z)\mathcal{U}_k^{\text{SA}}\mathbf{u}\|_{H_{-s_2}} \\ \leq \|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}^{\text{SA}}, H_{-s_2})} \|\mathcal{U}_k^{\text{SA}}\mathbf{u}\|_{H_{s_1}} = \|R_A(z') - R_A(z)\|_{\mathcal{B}(H_{s_1}^{\text{SA}}, H_{-s_2})} \|\mathbf{u}\|_{H_{\Omega_k, s_1}}. \quad \square$$

**4.3. Non validity of a LAP for  $A_{\Omega_k}^{\text{SA}}$  and  $A_{\Omega_k}^{\text{AS}}$  at all thresholds** In the acoustic case the set of thresholds is  $\{\lambda_{n, B}(0); n \geq 1\}$ , and for each  $n$  the function  $\lambda_{n, B}$  is symmetric and  $\lambda'_{n, B} \neq 0$ . Then a similar result to the previous theorem gives a LAP at all thresholds for the operators associated with a half-strip. The difference with the elastic case is that Theorem 4.1 does not provide such a result. The reason is that, due to  $(P)$ ,  $\Gamma^{\text{SA}} \cup \Gamma^{\text{AS}}$  does not necessarily coincide with  $\Gamma$ .

**Theorem 4.2.** *Assume that the eigenvalue  $\lambda_0 = \lambda_n(0)$  is simple and that  $\lambda'_n(0) = \lambda''_n(0) = \lambda'''_n(0) = 0$ . The limits of  $R_{A_{\Omega_k}^{SA}}(z)$  and  $R_{A_{\Omega_k}^{AS}}(z)$ , as  $z \rightarrow \lambda_0$  with  $\pm \text{Im } z > 0$ , do not exist for the topology of  $\mathcal{B}(H_{\Omega_k, s_1}, H_{\Omega_k, -s_2})$ , where  $s_1$  and  $s_2$  are sufficiently large.*

*Proof.* We will prove the non-existence of the first limit and suppose that  $\lambda_0$  is an eigenvalue of  $A_0^{AS}$ . The other cases can be treated in a similar fashion.

Let  $\phi \in C_0^\infty(\mathbb{R})$  be an anti-symmetric function with a small enough support such that  $\phi'(0) \neq 0$ . We claim that the following function  $\mathbf{u}$  belongs to  $H_s^{SA}$  with  $s$  sufficiently large.

$$\mathbf{u}(x_1, x_2) = \int_{\mathbb{R}} \phi(p) W_n(p, x_2) e^{ipx_1} dp.$$

Indeed, in view of Theorem 2.5 and since  $\lambda_0$  is an eigenvalue of  $A_0^{AS}$  we have  $S^{SA} W_n(-p, \cdot) = -W_n(p, \cdot)$  for all  $p$ . Thus, using variable change  $p \rightarrow -p$  we obtain

$$\begin{aligned} S^{SA} \mathbf{u}(x_1, x_2) &= \int_{\mathbb{R}} \phi(-p) S^{SA} W_n(-p, x_2) e^{-ipx_1} dp \\ &= \int_{\mathbb{R}} \phi(p) W_n(p, x_2) e^{ip(-x_1)} dp = \mathbf{u}(-x_1, x_2). \end{aligned}$$

We also verify that  $\mathbf{u} \in H_s$ , for all  $s$ , and, using (8), that  $F\mathbf{u} = (\phi \delta_n^k)_{k \geq 1}$ .

Denote  $\mathbf{u}' = \sqrt{2} \mathbf{u}_{|\Omega_k}$ , so

$$\langle R_{A_{\Omega_k}^{SA}}(z) \mathbf{u}', \mathbf{u}' \rangle_{H_{\Omega_k}} = \langle R_A(z) \mathbf{u}, \mathbf{u} \rangle_H = \int_{\mathbb{R}} \frac{|\phi(p)|^2}{\lambda_n(p) - z} dp,$$

and put  $I^\varepsilon := \langle R_{A_{\Omega_k}^{SA}}(\lambda_0 + i\varepsilon) \mathbf{u}', \mathbf{u}' \rangle_{H_{\Omega_k}}$ . Let us prove that  $|I^\varepsilon| \rightarrow \infty$ , as  $\varepsilon \rightarrow 0$ , which will end the proof. Since  $\lambda_n(p) - \lambda_0 = p^N G(p)$ , where  $N = N_n^0$ , and  $G(p) = G_{n,1}^j(p)$  (see (35)) we have

$$I^\varepsilon = \int_{\mathbb{R}} \frac{p^N G(p) |\phi(p)|^2}{p^{2N} G^2(p) + \varepsilon^2} dp + i \int_{\mathbb{R}} \frac{\varepsilon |\phi(p)|^2}{p^{2N} G^2(p) + \varepsilon^2} dp.$$

It then suffices to prove that the real part  $I_1^\varepsilon$  of  $I^\varepsilon$  goes to infinity. Since  $\phi'(0) \neq 0$  and  $N$  is even and greater or equal to 4, there exist  $r > 0$  and  $c_r$  such that

$$|I_1^\varepsilon| \geq c_r \int_{-r}^r \frac{|p|^{N+2}}{p^{2N} + \varepsilon^2} dp \rightarrow c_r \int_{-r}^r \frac{1}{|p|^{N-2}} dp = +\infty.$$

The convergence above is a consequence of the monotonic convergence theorem. □

**4.4. Generalization to other differential operators** Our objective here is not to state a general result concerning a precise class of differential operators but rather to

propose a path to follow in which we use the general results of Section 3. It remains that in any particular case some steps have to be verified.

For a given differential selfadjoint operator  $D$  on a stratified strip  $\Omega$ , or more generally on  $\Omega' \times \mathbb{R}$  where  $\Omega'$  is a regular and bounded open set in  $\mathbb{R}^n$ ,  $n \geq 1$ , we propose to follow the following steps. Using the partial Fourier transform defined in (4) it is expected to obtain a family of reduced operators whose eigenvalues and eigenvectors are represented by two families of analytic real and vectorial functions denoted, respectively,  $\lambda_{n,D}$  and  $W_{n,D}$ ,  $n \geq 1$ . We have to verify that  $D$  is unitarily equivalent to the multiplication operator in  $\bigoplus_{n \geq 1}^{\infty} L^2(\mathbb{R})$  by  $(\lambda_{n,D})_{n \geq 1}$ , via the unitary transformation defined by (8) where  $W_n$  have been replaced by  $W_{n,D}$ . The proof of Theorem 2.3 is still valid if we replace  $W_n$  by  $W_{n,D}$  so that **(H2)** is satisfied. It only remains to verify **(H1)**. The main difficulty is to prove  $\inf_{p \in \mathbb{R}} |\lambda_{n,D}(p)| \rightarrow +\infty$ , as  $n \rightarrow \infty$ . One can follow the proof of Theorem 2.2 and the idea of utilizing (10).

It is easy to verify this procedure when  $D = B$  and recover the well-known results of the acoustic case obtained in [4] and [5].

For some differential operators allowing separation of variables, such as  $B$ , we have  $\lambda_{n,D}(p) = \lambda(p) + c_n$ , where  $c_n \rightarrow +\infty$ , as  $n \rightarrow +\infty$  and  $\lambda(p) \rightarrow +\infty$ , as  $p \rightarrow \infty$ ; or  $\lambda_{n,D}(p) \geq \lambda_{n,D}(0)$  with  $\lambda_{n,D}(0) \rightarrow +\infty$ , as  $n \rightarrow +\infty$ . In both cases **(H1)** is satisfied.

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