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COMPACT EINSTEIN-WEYL MANIFOLDS AND THE ASSOCIATED CONSTANT

MITSUHIRO ITOH*

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1. Introduction

A manifold M is assumed in this paper always to be connected and smooth and have dimension $n \geq 3$.

A Weyl structure on a manifold M is a torsion free affine connection D preserving a conformal structure $[g]$. Namely, a torsion free affine connection D is called a Weyl structure if $Dg = \omega \otimes g$ for a 1-form ω.

The definition of Weyl structure goes back to the work of H. Weyl. In his famous book ([23]) he introduced Weyl structure to unify gravitational fields and electromagnetic fields.

The notion of Einstein-Weyl structure is originated in the paper of N.Hitchin ([11]) in which he developed the 3-dimensional minitwistor theory associated to the 3-dimensional monopole theory and observed that the minitwistor theory can be generalized over a 3-manifold endowed with a Weyl structure obeying a certain Ricci tensor condition, namely an Einstein-Weyl structure. Refer also to [12].

The exact definition of Einstein-Weyl structure is the following.

A Weyl structure $(D, [g])$ is Einstein-Weyl if the symmetrized Ricci tensor is proportional to a metric g representing $[g]$;

$$(1) \qquad Ric^D(X,Y) + Ric^D(Y,X) = \Lambda \; g(X,Y), \quad \Lambda \in C^\infty(M)$$

Thus an Einstein-Weyl structure is a generalization of Einstein metric in terms of affine connection.

The Levi-Civita connection ∇ of an Einstein metric g indeed gives an Einstein-Weyl structure $(\nabla, [g])$ with trivial ω.

Einstein-Weyl structures enjoy a conformal invariance as a significant feature. Gauduchon showed that after applying a suitable conformal factor every Einstein-Weyl structure on a compact manifold M is conformally equivalent to a standard structure, that is, one having coclosed 1-form ω; $d^\star\omega = 0$ ([7], [22]). As K.P. Tod claimed, this coclosed 1-form turns out to be the dual of a Killing field ([22]).

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We confirm ourselves in this paper instead of the full Einstein-Weyl equation to the Killing dual field equation together with the simplified Einstein-Weyl equation to investigate Einstein-Weyl geometry over compact manifolds.

For each compact Einstein-Weyl $n-$manifold ($n \geq 3$) with coclosed 1-form ω we can exhibit that the scalar $s_g - ((n+2)/4)|\omega|^2$ is constant (s_g is the scalar curvature of g) and observe that the ω satisfies a non-linear elliptic equation;

$$\nabla^\star \nabla \omega = \left(\frac{c}{n} - \frac{n-4}{4} |\omega|^2 \right) \omega, \tag{2}$$

where $c = s_g - ((n+2)/4)|\omega|^2$. This associated constant c behaves like the scalar curvature of an Einstein metric.

Notice that

$$c = s^D + \frac{n(n-4)}{4} |\omega|^2 \tag{3}$$

where $s^D = tr_g Ric^D$ is the scalar curvature of D with respect to g whose sign is conformal invariant.

The idea of this paper is to make crucial use of the associated constant c together with the strong maximum principle on the coclosed form ω.

The sign of the associated constant c causes difference in geometrical aspect of compact Einstein-Weyl manifolds. Actually, as will be shown in § 3 compact $n-$dimensional ($n \geq 4$) Einstein-Weyl structures of $c < 0$ and with coclosed 1-form are exhausted by Einstein manifolds of $s_g < 0$.

For Einstein-Weyl manifolds M of $c > 0$ the situation is quite similar to the Seiberg-Witten monopole equations in which the strong maximum principle was applied([14], [13], [5]). We obtain the sup-norm estimates as

Key Proposition (Theorem 3 in § 5). *Let M be a compact Einstein-Weyl n-manifold ($n \geq 5$) with coclosed form ω. If the associated constant $c > 0$, then*

$$\max_M |\omega|^2 \leq \frac{4}{n(n-4)} c \text{ and } \max_M |Ric_g|^2 \leq k_n c^2 \tag{4}$$

In addition, as shown in § 5, any compact Einstein-Weyl $n-$manifold M with coclosed ω and of $c > 0$ has positive (semi-)definite Ricci tensor; $Ric_g \geq 0$ and the first Betti number $b_1(M) \leq 1$. Furthermore for such an M having $b_1(M) = 1$ the universal covering $\tilde{M}$ splits into $\tilde{M} = N \times \mathbf{R}^1$ for an Einstein manifold N of positive scalar curvature.

Remark that conversely any Einstein manifold N of positive scalar curvature yields an Einstein-Weyl structure on the product $N \times S^1$, which is locally conformal to Einstein manifold. Additionally we can characterize compact Einstein-Weyl manifolds which are locally conformal Einstein (see Theorem 5, § 5).

Another non-trivial example of Einstein-Weyl structure is constructed over the total space of circle bundle over a compact Einstein-Kähler manifold([19]). Recently it was shown by F. Narita([16]) that a Sasakian manifold of constant $\varphi-$sectional curvature k (≥ 1) carries an Einstein-Weyl structure. These manifolds are endowed with coclosed 1-form and have finite fundamental group. More nontrivial examples are constructed by using the connected sum argument in [20].

In the 4-dimensional case the square-norm of the associated constant c has the upper bound represented by $\chi(M) - (3/2)|\tau(M)|$, the 4-dim topological invariant, so that we can get the Thorpe-Hitchin inequality $\chi(M) \geq (3/2)|\tau(M)|$ for any compact Einstein-Weyl 4-manifold, which was already shown in [18].

Although not a few of conclusions of our theorems seem to have quite similar form to those given in [22], [20], [18] and [8], the method exploited in the present paper may have an advantage in formulating Einstein-Weyl geometry from the viewpoint of Riemannian geometry.

2. The Einstein-Weyl equation

Let $(D, [g])$ be an Einstein-Weyl structure on a manifold M.

By using the Levi-Civita connection ∇ of a metric g representing the conformal structure $[g]$ the affine connection D is then written as $D = \nabla + a$ for an $End(TM)-$valued 1-form a so that we can rewrite (1) as

$$Ric_g + \frac{n-2}{4}(\nabla^{sym}\omega + \omega \otimes \omega) = \Lambda g, \tag{5}$$

where $\nabla^{sym}\omega(X,Y) = (\nabla_X\omega)(Y) + (\nabla_Y\omega)(X)$ (see [19] for the details).

In the sequel we call a pair (g, ω) instead of an affine connection D an Einstein-Weyl structure when (g, ω) is a solution of (5).

Since from the equation (1) the affine connection D does not depend on conformal change of a metric, the Einstein-Weyl equation (5) is invariant under the conformal changes. More precisely, if (g, ω) is a solution of (5), so is $(\overline{g}, \overline{\omega})$, where $\overline{g} = e^{2f}g$, $\overline{\omega} = \omega + 2df$, $f \in C^\infty(M)$.

The equation (5) with trivial 1-form ω is just the Einstein metric equation. Moreover if a solution (g, ω) of (5) has closed 1-form ω, then ω is locally exact so that the metric g is locally conformal to an Einstein metric. Thus, Einstein-Weyl structure is considered as a generalization of Einstein metric from the viewpoint of conformal geometry on conformal structures together with the $\mathbf{R}^\star$ gauge action on 1-forms.

We assume now that M is compact.

From the results given by Gauduchon and Tod, as explained in § 1, by taking conformal change by a suitable positive function e^{2f} we can split the equation (5) into the Killing dual field equation and the simplified Einstein-Weyl equation([22]);

$$\nabla^{sym}\omega = 0 \quad \text{or} \quad \nabla_i\omega_j + \nabla_j\omega_i = 0 \tag{6}$$

$$Ric_g + \frac{n-2}{4}\omega \otimes \omega = \Lambda g \quad \text{or} \quad R_{ij} + \frac{n-2}{4}\omega_i\, \omega_j = \Lambda g_{ij} \tag{7}$$

From (7) we have $\Lambda = (1/n)(s_g + (n-2)/4|\omega|^2)$ where s_g is the scalar curvature of g, so (7) reads

$$\left(Ric_g - \frac{s_g}{n}\, g\right) + \frac{n-2}{4}\left(\omega \otimes \omega - \frac{1}{n}|\omega|^2 g\right) = 0 \tag{8}$$

EXAMPLE. Let (M, g) be the Riemannian product of an Einstein $(n-1)$−manifold (N, g_N) and the unit circle S^1. Since $Ric_g = Ric_{g_N} \oplus Ric_{S^1} = (s_N/(n-1))g_N \oplus 0$, the scalar curvature is $s_g = s_N$. So

$$Ric_g - \frac{s_g}{n} g = s_N \mathrm{diag}\left(\frac{1}{n(n-1)}, \cdots, \frac{1}{n(n-1)}, -\frac{1}{n}\right) \tag{9}$$

Let θ be the angular coordinate on S^1. Then $d\theta$ is a 1-form on M whose dual $\partial/\partial\theta$ is Killing on (M, g). We put $\omega = ad\theta$ so that

$$\frac{n-2}{4}\left(\frac{1}{n}|\omega|^2 g - \omega \otimes \omega\right) = \frac{a^2(n-2)}{4}\mathrm{diag}\left(\frac{1}{n}, \cdots, \frac{1}{n}, \frac{1-n}{n}\right) \tag{10}$$

and hence the equations (6) and (7) are fulfilled for the (g, ω), provided $a = \pm(2\sqrt{s_N}/\sqrt{(n-1)(n-2)})$. So the (M, g, ω) gives an Einstein-Weyl manifold with Killing dual 1-form ω. This is, however, locally conformal to an Einstein manifold.

Lemma 1. *Let (g, ω) be an Einstein-Weyl structure with Killing dual 1-form ω, namely (g, ω) be a solution of (6) and (7). Then,*
(i)

$$s_g - \frac{n+2}{4}|\omega|^2 \tag{11}$$

is constant which we denote by c and
(ii) *the form ω satisfies*

$$\nabla^\star \nabla \omega = \left(\frac{c}{n} - \frac{n-4}{4}|\omega|^2\right)\omega. \tag{12}$$

Proof. (i) is shown by taking the divergence of the both hand sides of (8).
In fact the first term of the left hand side reduces to $((1/2) - (1/n))\nabla_j s_g$ and the second term to $((n-2)/4)(-(1/2) - (1/n))\nabla_j(|\omega|^2)$ so that $\nabla_j(s_g - ((n+2)/4)|\omega|^2) = 0$.

To prove (ii) we have

$$\nabla^\star \nabla \omega = Ric(\omega), \text{ i.e., } -\nabla^i \nabla_i \, \omega_j = R^i_j \, \omega_i, \tag{13}$$

since the dual of ω is Killing.

On the other hand

$$Ric_g = \frac{1}{n}\left(c + \frac{n}{2}|\omega|^2\right) g - \frac{n-2}{4}\omega \otimes \omega. \tag{14}$$

So $Ric_g(\omega) = ((c/n) - ((n-4)/4)|\omega|^2)\omega$. □

REMARK 1. a. (i) is seen also in (31), [8], where the normalization is different from ours.

b. The conformal scalar curvature $s^D = tr_g Ric^D$ of a Weyl structure D is given in terms of s_g as $s^D = s_g - (n-1)d^\star\omega - ((n-1)(n-2)/4)|\omega|^2$ (see [19]) so that for an Einstein-Weyl structure with coclosed 1-form ω we have from Lemma 1

$$s^D = c - \frac{n(n-4)}{4}|\omega|^2 \tag{15}$$

so that the formula (12) is rewritten

$$\nabla^\star \nabla \omega = \frac{s^D}{n}\omega \tag{16}$$

which appears in [20].

c. When $n = 4$, $c = s^D$. If $n \geq 5$, then $c \leq 0$ implies $s^D \leq 0$.

3. The case of $c \leq 0$

We integrate over M the scalar product of $\nabla^\star\nabla\omega$ with ω. Then we have from (12)

$$\int_M |\nabla\omega|^2 dv_g = \frac{c}{n}\int_M |\omega|^2 dv_g - \frac{n-4}{4}\int_M |\omega|^4 dv_g \tag{17}$$

which gives (i), (ii) of the following theorem characterizing Einstein-Weyl structures of $c \leq 0$.

Theorem 2. *Let (g, ω) be an Einstein-Weyl structure on M with Killing dual 1-form ω. Then we have*

(i) *if $n \geq 5$ and $c \leq 0$, then $\omega = 0$, that is, g is an Einstein metric of $s_g \leq 0$.*

(ii) *if $n = 4$ and $c < 0$, then $\omega = 0$, that is, g is Einstein and $s_g < 0$, and*

(iii) *if $n = 4$ and $c = 0$, then the form ω is parallel so that either g is Ricci flat or M has $b_1(M) = 1$ and the universal covering of (M, g) is isometric to the Riemannian product $S^3 \times \mathbf{R}^1$, where S^3 is a round 3-sphere of constant curvature $(1/4)|\omega|^2$.*

In addition,

(iii) *if $n = 3$ and $4c \leq -3|\omega|^2$ but not identically equal, then $\omega = 0$, that is, g is an Einstein metric of $s_g < 0$ and*

(iv) *if $n = 3$ and $4c = -3|\omega|^2$, then ω is parallel so that either g is flat or $b_1(M) = 1$ and the universal covering of (M, g) is isometric to the Riemannian product $S^2 \times \mathbf{R}^1$, where S^2 is a round 2-sphere of constant curvature $(1/4)|\omega|^2$.*

REMARK 2.a. The statement (iii) characterizes almost completely compact Einstein-Weyl 4-manifolds with coclosed 1-form and of $c = s^D = 0$. See [8, Théorème 3] where we find a quite same statement.

b. From a, Remark 1 (i), (ii), (iv) in the theorem are easily shown from Proposition 2.3 in [20], proved originally in [21] and [8], since the hypotheses on c imply $s^D \leq 0$ or $s^D < 0$.

Proof. We will prove (iii), (iv) and (v). The proof of (iv) is similar to that of (i) and (ii), since the right hand in (12) is non-positive.

To prove (iii) and (v) let (g, ω) be an Einstein-Weyl structure with coclosed 1-form ω.

Suppose $n = 4$ and $c = 0$ or $n = 3$ and $4c = -3|\omega|^2$. Then from (12)

$$\nabla^\star \nabla \omega = 0 \tag{18}$$

from which on a compact M the form ω is parallel. For the case $\omega = 0$ g must be Einstein, and the scalar curvature $s_g = 0$ so that g is Ricci flat or flat according to $n = 4$ or $n = 3$.

If $\omega \neq 0$, then the Ricci tensor has eigenvalues 0 with multiplicity 1 and $(1/2)|\omega|^2$ with multiplicity 3 for $n = 4$ (resp. $(1/4)|\omega|^2 = -(1/3)c$ with 2 for $n = 3$). So by applying the splitting theorem on nonnegative Ricci curvature ([4]) we get the Riemannian product statements in (iii) and (v).

Next we will show $b_1(M) = 1$ for the both cases. Actually $\nabla^\star \nabla \omega = 0$ implies that ω is parallel and hence harmonic.

Let θ be any harmonic 1-form. Then $\nabla^\star \nabla \theta + Ric(\theta) = 0$. Since $Ric_g \geq 0$, θ is parallel. Decompose θ into $\theta = \phi + a\ \omega$, $a \in \mathbf{R}$, where ϕ is orthogonal to ω pointwise. Applying again the Weitzenböck formula to ϕ we conclude that ϕ must vanish, since Ric_g is positive in the direction to ϕ. □

4. The case of $c > 0$

Now we suppose that for a compact Einstein-Weyl $n-$manifold M with coclosed 1-form ω the associated constant is positive.

We can then make use of the strong maximum principle applied to the Seiberg-Witten monopole equation to get the sup-norm estimates on the 1-form ω and the Ricci tensor Ric_g.

Since $\nabla^\star\nabla(|\omega|^2) \leq 2(\nabla^\star\nabla\omega, \omega)$, at a point where $|\omega|^2$ attains the maximum one has from (12)

$$0 \leq \frac{1}{2}\nabla^\star\nabla(|\omega|^2) \leq \frac{c}{n}|\omega|^2 - \frac{n-4}{4}|\omega|^4. \tag{19}$$

So, if $|\omega|^2(p) > 0$, then $((n-4)/4)|\omega|^2(p) \leq (c/n)$. Thus we have the sup-norm estimate.

Theorem 3. *Let (M, g, ω) be a compact Einstein-Weyl $n-$manifold with coclosed 1-form ω. If $n \geq 5$ and $c > 0$, then*

$$\max_M |\omega|^2 \leq \frac{4}{n(n-4)}\, c \tag{20}$$

$$\max_M |Ric_g|^2 \leq k_n\, c^2 \tag{21}$$

where k_n is a universal positive constant depending only on n.

The sup-norm estimate (21) on Ric_g is easily derived from (14) and (20).

Similar estimates on ω and Ric_g valid for all dimension $n \geq 3$ are available in terms of the scalar curvature s_g.

In fact, let (g, ω) be an Einstein-Weyl structure with coclosed 1-form ω. Then, since $\nabla^\star\nabla\omega = Ric(\omega)$, we have from (8)

$$\nabla^\star\nabla\omega = \left(\frac{s_g}{n} - \frac{(n-1)(n-2)}{4n}|\omega|^2\right)\omega \tag{22}$$

So, suppose $\max_M s_g \geq 0$. Then

$$\max_M |\omega|^2 \leq \frac{4}{(n-1)(n-2)} \max_M s_g \tag{23}$$

$$\max_M |Ric_g|^2 \leq \ell_n(\max_M s_g)^2 \tag{24}$$

where ℓ_n is a universal constant depending only on n.

From the uniform bound on the Ricci tensor in Theorem 3 we can investigate the space of compact Einstein-Weyl $n-$manifolds satisfying certain geometric inequalities(see for instance, [2], [15] and [1]).

5. The Ricci positivity

That the Ricci tensor Ric_g is positive definite for any Einstein-Weyl structure of $c > 0$ follows from (8) and Theorem 3. Actually this will be stated in the following way.

Theorem 4. *Let (g, ω) be an Einstein-Weyl structure with coclosed 1-form ω defined on a compact n-manifold M. If the constant $c > 0$, then Ric_g is positive semi-definite.*

In particular (i) *if $n = 3, 4$, then Ric_g is strictly positive definite, so that $\pi_1(M) < \infty$,*

(ii) *if $n \geq 5$ and ω satisfies $|\omega|^2 < (4/(n(n-4)))c$, then Ric_g is strictly positive definite so $\pi_1(M) < \infty$, and*

(iii) *if $n \geq 5$ and $|\omega|^2 = (4/(n(n-4)))c$, then $b_1(M) = 1$ and the universal covering of (M, g) is isometric to the Riemannian product of (N, h) and the straight line $(\mathbf{R}^1, g_1)$, where (N, h) is a simply connected Ricci positive Einstein manifold.*

REMARK 3.a. In the case where $n \geq 5$ and $|\omega|^2 \leq (4/(n(n-4)))c$, but not identically equal, $b_1(M) = 0$ is concluded.

b. H.K. Pak obtained in [17] $b_1 = 1$ for certain Einstein-Weyl manifolds.

Proof. We make use of the formula (14);

$$Ric_g = \frac{1}{n}\left(c + \frac{n}{2}|\omega|^2\right) g - \frac{n-2}{4}\omega \otimes \omega \tag{25}$$

It is seen that Ric_g is positive definite where ω vanishes.

So, suppose $\omega \neq 0$ at a point p.

Let ξ be the tangent vector at p dual of ω. Since $\omega(\xi) = |\omega|^2$,

$$Ric_g(\xi, \xi) = \left(\frac{c}{n} - \frac{n-4}{4}|\omega|^2\right)|\omega|^2. \tag{26}$$

For any tangent vector X orthogonal to ξ

$$Ric_g(X, \xi) = 0 \text{ and } Ric_g(X, X) = \frac{1}{n}\left(c + \frac{n}{2}|\omega|^2\right) g(X, X) \tag{27}$$

from which it follows that when $n = 3$ or 4 Ric_g is positive definite at p.

When $n \geq 5$ we make use of the estimate on $|\omega|^2$ obtained in Theorem 3 so that from (26) $Ric_g(\xi, \xi) \geq 0$, that is, Ric_g is positive semidefinite.

(ii) is easily derived from (26). To see (iii) suppose $|\omega|^2 = (4/(n(n-4)))c$. Then from (26) the Ricci tensor is degenerate in the direction to ξ. The Ricci curvature splitting theorem ([4]) can be again applied so that the universal covering space of (M, g) is isometric to the Riemannian product of (N, h) and the straight line $\mathbf{R}^1$. Since the zero eigenspace is one-dimension, (N, h) must be Einstein.

The proof of $b_1(M) = 1$ may be given, same as in the proof of Theorem 2. $\square$

Finally we will remark on locally conformal Einstein, Einstein-Weyl manifolds. By applying Theorems 2, 3 and 4 we get

Theorem 5. *Let (M, g, ω) be a compact Einstein-Weyl $n-$manifold $(n \geq 4)$. If M is locally conformal Einstein, but not globally conformal, then M has $b_1(M) = 1$ and the universal covering space of (M, g) is globally conformal to $N \times \mathbf{R}^1$, where N is an Einstein manifold of positive scalar curvature.*

Proof. By a conformal change we assume that the closed 1-form ω is coclosed. So ω is non-trivial and harmonic, because M is not globally conformal.

In addition, we have from Theorem 2 the associated constant $c > 0$, if $n \geq 5$ (resp. $c \geq 0$ if $n = 4$). So from Theorem 2 together with (iii), Theorem 4 we get $b_1(M) = 1$ and the proof is completed. $\square$

6. Four-dimensional case

We now restrict ourselves to Einstein-Weyl 4-manifolds.

The following theorem tells us that 4-dimensional Einstein-Weyl structures are closely related to the topological invariants, the Euler characteristic $\chi(M)$, the signature $\tau(M)$, same as Einstein 4-manifolds ([9], [3]).

Theorem 6. *Let (M, g, ω) be a compact, oriented Einstein-Weyl 4-manifold. Then the inequality holds;*

$$\frac{1}{4\pi^2}\int_M |W^{\pm}|^2 + \frac{1}{192\pi^2}c^2 vol(M) \leq \chi(M) \pm \frac{3}{2}\tau(M) \tag{28}$$

from which the following holds;

$$\chi(M) \geq \frac{3}{2}|\tau(M)|, \tag{29}$$

The equality holds here if and only if either (M, g, ω) is conformally equivalent to a Ricci flat, half conformally flat (i.e., (anti-)self-dual) 4-manifold with $\omega = 0$ or $b_1(M) = 1$ and the universal covering space $(\tilde{M}, \tilde{g}, \tilde{\omega})$ is conformally equivalent to $S^3 \times \mathbf{R}^1$ with a parallel 1-form $\omega = 2\sqrt{k}\, dt$, where S^3 is a 3-sphere of constant curvature k.

We remark that Pedersen, Poon and Swann obtained in [18] a quite similar integral inequality from which they asserted (29).

Proof. For each oriented Riemannian 4-manifold the following holds ([3],[6]);

$$\chi(M) \pm \frac{3}{2}\tau(M) = \frac{1}{4\pi^2}\int_M |W^{\pm}|^2 + \frac{1}{48\pi^2}\int_M (s_g^2 - 3|Ric_g|^2) \tag{30}$$

where $W^{\pm}$ denotes the (anti-)self-dual Weyl conformal curvature.

Let (g, ω) be an Einstein-Weyl structure on a 4-manifold M. Without loss of generality we may assume that (g, ω) satisfies the Killing dual field equation and the simplified Einstein-Weyl equation so that for the (g, ω) $s_g = c + (3/2)|\omega|^2$. Then

$$s_g^2 = c^2 + 3c|\omega|^2 + \frac{9}{4}|\omega|^4 \tag{31}$$

and from (14)

$$|Ric_g|^2 = \frac{c^2}{4} + \frac{3}{4}c|\omega|^2 + \frac{3}{4}|\omega|^4, \tag{32}$$

so, $s_g^2 - 3|Ric_g|^2 = (1/4)c^2 + (3/4)c|\omega|^2$. Thus, (30) reads as

$$\chi(M) \pm \frac{3}{2}\tau(M) = \frac{1}{4\pi^2}\int_M |W^{\pm}|^2 + \frac{1}{48\pi^2}\int_M \left(\frac{1}{4}c^2 + \frac{3}{4}c|\omega|^2\right) \tag{33}$$

It is easily seen that $c\int_M |\omega|^2 \geq 0$ for any case of $c \geq 0$ and $c < 0$. Therefore

$$\chi(M) \pm \frac{3}{2}\tau(M) \geq \frac{1}{4\pi^2}\int |W^{\pm}|^2 + \frac{1}{192\pi^2}c^2 \mathrm{vol}(M) \tag{34}$$

and hence we obtain the Thorpe-Hitchin inequality (29).

Suppose $\chi(M) = (3/2)|\tau(M)|$. Then from the above inequality either W^+ or W^- vanishes and c must be zero.

So, from (iii), Theorem 2 (M, g, ω) must be either Ricci flat, half conformally flat and with $\omega = 0$, or the universal covering of (M, g, ω) is isometric to the Riemannian product $S^3 \times \mathbf{R}^1$. □

REMARK 4.a. From the Thorpe-Hitchin inequality we can claim like the Einstein 4-manifold case (see [18])that a connected sum of certain compact 4-manifolds carries no Einstein-Weyl structures. For instance a connected sum of ℓ copies of the complex projective plane $P^2(\mathbf{C})$ can admit no Einstein-Weyl structures, if $\ell \geq 4$.

b. The inequality (28) imples that the constant $|c|$ has the uniform upper bound, just given by the topological invariants, provided the volume of g is unit;

$$c^2 \leq 192\pi^2 \left(\chi(M) - \frac{3}{2}|\tau(M)|\right) \tag{35}$$

Finally, we consider an Einstein-Weyl 4-manifold M whose metric is half-conformally flat (i.e., self-dual; $W^- = 0$). We have actually

Theorem 7. *Let M be a compact, oriented Einstein-Weyl 4-manifold of $c > 0$. If M is half-conformally flat, then M is conformal to S^4 or $P^2(\mathbf{C})$ with the canonical conformal structure.*

REMARK 5. From (ii), (iii) of Theorem 2, a compact half-conformally flat, Einstein-Weyl 4-manifold of $c \leq 0$ is either conformal to a compact half-conformally flat, Einstein 4-manifold of non-positive scalar curvature or has the universal covering space which is conformal to $S^3 \times \mathbf{R}^1$.

Proof. Since M is Einstein-Weyl, M carries a half-conformally flat metric g with a coclosed 1-form ω. For this Einstein-Weyl structure (g, ω) one has from (11) $s_g = c + (3/2)|\omega|^2 > 0$.

Because of $c > 0$ we have from Theorem 4 $\pi_1(M) < \infty$ so that the first cohomology group $H^1(M) = 0$. It follows then from [20, Cor. 3.3] that M has an Einstein metric g_1 of positive scalar curvature in the conformal structure $[g]$. One can apply Hitchin's theorem (see [10] or [3, Theorem 13.30]). So, (M, g_1) is isometric to S^4 or $P^2(\mathbf{C})$ with their canonical metrics. $\square$

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Institute of mathematics
University of Tsukuba
305-8571, Tsukuba, JAPAN
e-mail: itohm@sakura.cc.tsukuba.ac.jp