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PSEUDO-ORBIT TRACING PROPERTY AND STRONG TRANSVERSALITY OF DIFFEOMORPHISMS ON CLOSED MANIFOLDS

KAZUHIRO SAKAI

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1. Introduction

We are interested in the dynamical property of a diffeomorphism f having the pseudo-orbit tracing property of a closed manifold M. Let d be a metric for M. A sequence of points $\{x_i\}_{i\in \boldsymbol{Z}}$ of M is called a *δ-pseudo-orbit* of f if $d(f(x_i), x_{i+1})<\delta$ for $i\in \boldsymbol{Z}$. A sequence $\{x_i\}_{i\in \boldsymbol{Z}}$ is said to be *f-ε-traced* by $y\in M$ if $d(f^i(y), x_i)<\varepsilon$ for $i\in \boldsymbol{Z}$.

We say that f has the *pseudo-orbit tracing property* (abbrev. **POTP**) if for every $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of f can be f-ε-traced by some point.

In [5] Robinson proved that every Axiom A diffeomorphism satisfying strong transversality has **POTP**. Thus it will be natural to ask whether **POTP** implies Axiom A and strong transversality. For this problem we have partial results that are answered in [4] for dim $M=2$ and in [7] for dim $M=3$. However we have no answer for higher dimensions.

Our aim is to prove the following

Theorem. *The C^1 interior of all diffeomorphisms having* **POTP** *of a closed manifold M, $\mathscr{P}(M)$, coincides with the set of all Axiom A diffeomorphisms satisfying strong transversality.*

We say that f has the *C^1 uniform pseudo-orbit tracing property* (abbrev. **C^1-UPOTP**) if there is a C^1 neighborhood $\mathscr{U}(f)$ of f with the property that for $\varepsilon>0$ there is $\delta>0$ such that every δ-pseudo-orbit of $g\in\mathscr{U}(f)$ is g-ε-traced by some point. Since every Axiom A diffeomorphism satisfying strong transversality has **C^1-UPOTP** (see [6, Theorem]), if we establish our theorem, then the following corollary is obtained.

Corollary. *The set of all diffeomorphism having* **C^1-UPOTP** *is characterized as the set of all Axiom A diffeomorphisms satisfying strong transversality.*

It was proved in [4] that all periodic points of $f \in \mathscr{P}(M)$ are hyperbolic. From this we can prove that each f belonging to $\mathscr{P}(M)$ satisfies Axiom A with no-cycle. Recently it was shown in general by Aoki [1]. Therefore, to conclude our theorem it remains only to prove the following proposition.

Proposition. *Every $f \in \mathscr{P}(M)$ satisfies strong transversality.*

Unfortunately this can not be proved by the techinques mentioned in [4] and [7]. Thus we need a new technique for the proof of the proposition.

2. Proof of Proposition

Let Diff(M) denote the set of all diffeomorphisms of M endowed with C^1 topology, and let $p=f^n(p)$ $(n>0)$ be a hyperbolic periodic point of $f \in$ Diff (M). Even if p is hyperbolic, when dim $M \geq 3$, it is not easy to construct an f^n-invariant foliation in a neighborhood of p that is compatible with the local stable manifold (i.e. the leaf passing through p is the local stable manifold of p). In this paper, by using Franks's lemma we make a new diffeomorphism g $(g^n(p)=p)$, arbitrarily near to f in C^1 topology, which has a g^n-invariant compatible foliation in a neighborhood of p (see lemmas 1 and 2). This foliation will play an essential role in the proof of the proposition.

Let $f \in$ Diff (M) satisfy Axiom A with no-cycle. The non-wandering set $\Omega(f)$ of f is expressed as a finite disjoint union of basic sets $\{\Lambda_i(f)\}$, and for a sufficiently small $\varepsilon_0>0$ and $x \in \Omega(f)$ there are a local stable manifold $W^s_{\varepsilon_0}(x, f)$ and a local unstable manifold $W^u_{\varepsilon_0}(x, f)$. Let $\Lambda(f)$ be a basic set of f. Since dim $W^s_{\varepsilon_0}(x, f)=$ dim $W^s_{\varepsilon_0}(y, f)$ $(x, y \in \Lambda(f))$, we denote by Ind $\Lambda(f)$ the dimension of $W^s_{\varepsilon_0}(x, f)$ for $x \in \Lambda(f)$. If $g \in$ Diff (M) is C^1 close to f, then the number of basic sets $\{\Lambda_i(g)\}$ of g coincides with that of basic sets $\{\Lambda_i(f)\}$ since f is Ω-stable.

Put $B_\varepsilon(x)=\{y \in M \mid d(x, y) \leq \varepsilon\}$ for $\varepsilon>0$ and let ρ be a usual C^1 metric of Diff (M). Then we have the following

Lemma 1. *Let $\varepsilon_0>0$ be as above and let $\Lambda(f)$ be a basic set such that $1 \leq$ Ind $\Lambda(f) \leq$ dim $M-1$. Then, for a periodic point $p \in \Lambda(f)$ $(f^n(p)=p$, $n>0)$, a neighborhood $\mathscr{U}(f) \subset$ Diff (M) and a number $\gamma>0$ there are $0<\varepsilon_1<\varepsilon_0/2$, $g \in \mathscr{U}(f)$ and a basic set $\Lambda(g)$ for g such that*

(i) $B_{4\varepsilon_1}(f^i(p)) \bigcap B_{4\varepsilon_1}(f^j(p))=\phi$ for $0 \leq i \neq j \leq n-1$,

$$\text{(ii)}\quad g(x)=\begin{cases} \exp_{f^{i+1}(p)}\circ D_{f^i(p)}f\circ \exp^{-1}_{f^i(p)}(x) \\ \qquad\qquad \text{if}\quad x\in B_{\varepsilon_1}(f^i(p)) \text{ for } 0\le i\le n-1, \\ f(x) \qquad \text{if } x\notin \bigcup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$$

(iii) *$g^n(p)=p\in\Lambda(g)$ and $\rho(W^\sigma_{\varepsilon 0}(p,f),\ W^\sigma_{\varepsilon 0}(p,g))<\gamma$ for $\sigma=s,\ u$ (i.e. there is a C^1 diffeomorphism ξ^σ: $W^\sigma_{\varepsilon 0}(p,f)\to W^\sigma_{\varepsilon 0}(p,g)$ such that $\rho(\xi^\sigma, id)<\gamma$ ($\sigma=s,u$)).*

Proof. Since $\Lambda(f)$ is hyperbolic, there is $e>0$ such that $d(f^n(x), f^n(y))\le e$ ($x, y\in\Lambda(f)$ and $n\in\boldsymbol{Z}$) implies $x=y$ (see [5]). By Ω-stability theorem, there exists a neighborhood $\mathscr{U}_0(f)\subset\mathscr{U}(f)$ of f such that for every $g\in\mathscr{U}_0(f)$ there is a homeomorphism h_g, which maps $\Omega(f)$ onto the non-wandering set $\Omega(g)$ of g, satisfying

$$\begin{cases} g\circ h_g=h_g\circ f, \\ d(h_g,\ id_{|\Omega(f)})<e, \\ \rho(W^\sigma_{\varepsilon 0}(p,\ f),\ W^\sigma_{\varepsilon 0}(h_g(p),\ g))<\gamma \text{ for } \sigma=s,\ u. \end{cases}$$

By Franks's lemma [2, lemma 1.1], we can find $g\in\mathscr{U}_0(f)$ and $0<\varepsilon_1<\varepsilon_0/2$ such that

$$B_{4\varepsilon_1}(f^i(p))\bigcap B_{4\varepsilon_1}(f^j(p))=\phi \ (0\le i\ne j\le n-1) \text{ and}$$

$$g(x)=\begin{cases} \exp_{f^{i+1}(p)}\circ D_{f^i(p)}f\circ \exp^{-1}_{f^i(p)}(x) \\ \qquad\qquad \text{if}\quad x\in B_{\varepsilon_1}(f^i(p)) \text{ for } 0\le i\le n-1, \\ f(x) \qquad \text{if } x\notin \bigcup_{i=0}^{n-1} B_{4\varepsilon_1}(f^i(p)), \end{cases}$$

We write $\Lambda(g)=h_g(\Lambda(f))$ for simplicity. Then $h_g(p)\in\Lambda(g)$ and Ind $\Lambda(f)=$ Ind $\Lambda(g)$. Clearly $g(f^i(p))=\exp_{f^{i+1}(p)}\circ D_{f^i(p)}f\circ\exp^{-1}_{f^i(p)}\ (f^i(p))=f^{i+1}(p)$ for $0\le i\le n-1$ and so $g(p)=f(p)$, $g^2(p)=f^2(p),\cdots,g^n(p)=f^n(p)=p$. Since

$$\begin{aligned} d(f^i(h_g^{-1}(p)),\ f^i(p)) &= d(h_g^{-1}(g^i(p)),\ f^i(p)) \\ &= d(h_g^{-1}(f^i(p)),\ f^i(p))<e \ (i\in\boldsymbol{Z}), \end{aligned}$$

we have $h_g(p)=p$. Therefore $\rho(W^\sigma_{\varepsilon 0}(p,\ f),\ W^\sigma_{\varepsilon 0}(p,\ g))<\gamma$ ($\sigma=s,\ u$) and $p\in\Lambda(g)$.

Since f satisfies Axiom A, by definition there is a Df-invariant continuous splitting $T_{\Omega(f)}M=E^s\oplus E^u$ and a constant $0<\lambda<1$ such that

$\|Df^m_{|E^s}\| \leq \lambda^m$ and $\|Df^{-m}_{|E^u}\| \leq \lambda^m$ for $m>0$. We denote by E^σ_x a fiber of E^σ at $x \in \Omega(f)$ ($\sigma = s, u$), and put $E^\sigma_x(\varepsilon) = \{v \in E^\sigma_x \mid \|v\| \leq \varepsilon\}$ for $\varepsilon>0$.

Let $g \in \mathrm{Diff}(M)$, $p = g^n(p) \in \Lambda(g)$ ($n>0$) and $\varepsilon_1>0$ be as in lemma 1. Then it is easily checked that for $0<\varepsilon\leq\varepsilon_1$, we have $\exp_p(E^\sigma_p(\varepsilon)) = W^\sigma_\varepsilon(p, g)$ and $\dim \exp_p(E^\sigma_p(\varepsilon)) = \dim W^\sigma_{\varepsilon 0}(p, g)$ ($\sigma = s, u$). Fix ε_2 with $0<\varepsilon_2=\varepsilon_2(g, n)<\varepsilon_1$ such that $x \in B_{\varepsilon_2}(p)$ implies $g^i(x) \in B_{\varepsilon_1}(g^i(p))$ for $0 \leq i \leq n-1$, and define

$$\tilde{W}^s_{\varepsilon_2}(x, g) = \exp_p\left(E^s_p(\varepsilon_2) + \exp_p^{-1}(x)\right)$$

for $x \in \exp_p(E^u_p(\varepsilon_2))$. Then, since $\bigcup_{v \in E^u_p(\varepsilon_2)} (E^s_p(\varepsilon_2)+v)$ is a foliation defined in a neighborhood of $O_p \in T_pM$ and since $\exp_p$ is a local diffeomorphism, we have that $\{\tilde{W}^s_{\varepsilon_2}(x, g) \colon x \in \exp_p(E^u_p(\varepsilon_2))\}$ is a foliation defined in a neighborhood of p in M such that $\tilde{W}^s_{\varepsilon_2}(p, g) = W^s_{\varepsilon_2}(p, g)$.

Lemma 2.

(i) $\tilde{W}^s_{\varepsilon_2}(x, g)$ *is a* C^1 *manifold and* $\dim \tilde{W}^s_{\varepsilon_2}(x, g) = \dim \tilde{W}^s_{\varepsilon_2}(p, g)$,

(ii) $g^n(\tilde{W}^s_{\varepsilon_2}(x, g)) \subset \tilde{W}^s_{\varepsilon_2}(g^n(x), g)$ *for* $x \in \exp_p(E^u_p(\varepsilon_2)) \cap g^{-n}(\exp_p(E^u_p(\varepsilon_2)))$,

(iii) *there exists* $C>0$ *such that if* $\{x, g^n(x), \cdots, g^{nk}(x)\} \subset \exp_p(E^u_p(\varepsilon_2))$ *for some* $k>0$, *then* $d(g^{nk}(x), g^{nk}(y)) \leq C\lambda^{nk} d(x,y)$ *for* $y \in \tilde{W}^s_{\varepsilon_2}(x, g)$,

Proof. Assertion (i) is clear, and (ii) is easily obtained. To show (iii) put $T_p(\varepsilon_2) = \{v \in T_pM \mid \|v\| \leq \varepsilon_2\}$. Since $\exp_p \colon T_p(\varepsilon_2) \to M$ *and* $\exp_p^{-1}\colon B_{\varepsilon_2}(p) \to T_pM$ *are into diffeomorphisms there is* $K>0$ *such that*

$$d(\exp_p(v), \exp_p(w)) \leq K\|v-w\| \quad (v, w \in T_p(\varepsilon_2)),$$

$$\|\exp_p^{-1}(x) - \exp_p^{-1}(y)\| \leq K d(x, y) \quad (x, y \in B_{\varepsilon_2}(p)).$$

If $\{x, g^n(x), \cdots, g^{nk}(x)\} \subset \exp_p(E^u_p(\varepsilon_2))$ for some $k>0$, then for $y \in \tilde{W}^s_{\varepsilon_2}(x, g)$ there is $v_y \in E^s_p(\varepsilon_2)$ such that $y = \exp_p(v_y + \exp_p^{-1}(x))$. Thus we have

$$g^n(y) = \exp_p\left(D_pf^n(v_y) + \exp_p^{-1}(g^n(x))\right)$$

(since $D_pf^n(\exp_p^{-1}(x)) = \exp_p^{-1}(g^n(x))$), *and so*

$$\left(D_pf^n \circ \exp_p^{-1} \circ g^n\right)(y) = D_pf^{2n}(v_y) + D_pf^n(\exp_p^{-1}(g^n(x))),$$

from which

$$g^{2n}(y)=\operatorname{expp}\big(D_pf^{2n}(v_y)+D_pf(\exp_p^{-1}(g^{2n}(x)))\big).$$

Since $g^n(x)\in B_{\varepsilon_2}(p)$, we have $\big(\exp_p\circ D_pf^n\circ\exp_p^{-1}\big)$ $(g^n(x))=g^{2n}(x)$; i.e. $D_pf^n(\exp^{-1}(g^n(x)))=\exp_p^{-1}(g^{2n}(x))$. Thus $g^{2n}(y)=\exp_p\big(D_pf^{2n}(v_y)+\exp_p^{-1}(g^{2n}(x))\big)$. By repetition we have

$$g^{nk}(y)=\exp_p\big(D_pf^{nk}(v_y)+\exp_p^{-1}(g^{nk}(x))\big).$$

from which

$$\begin{aligned}d(g^{nk}(x),\ g^{nk}(y))&\le K\|\exp_p^{-1}(g^{nk}(x))-\exp_p^{-1}(g^{nk}(y))\|\\&=K\|D_pf^{nk}(v_y)\|\\&\le K\lambda^{nk}\|v_y\|.\end{aligned}$$

Clearly, $\|v_y\|=\|\exp_p^{-1}(x)-\exp_p^{-1}(y)\|\le Kd(x,\ y)$ since $\exp_p^{-1}(y)=v_y+\exp_p^{-1}(x)$. Therefore, $d(g^{nk}(x),\ g^{nk}(y))\le K^2\lambda^{nk}d(x,\ y)$. Assertion (iii) was proved.

Let f be as before, and denote by $W^s(x,\ f)$ the stable manifold and by $W^u(x,\ f)$ the unstable manifold for $x\in\Omega(f)$ respectively.

Lemma 3. *Let $\Lambda_1(f)$ and $\Lambda_2(f)$ be two distinct basic sets for f. Suppose that there are $p=f^n(p)\in\Lambda_1(f)$ $(n>0)$, $q\in\Lambda_2(f)$ and $x\in M\setminus\Omega(f)$ such that $x\in W^s(p,\ f)\bigcap W^u(q,\ f)$. Then, for neighborhood $\mathscr{U}(f)\subset$ Diff (M) there are $0<\varepsilon_1<\varepsilon_0/2$, $g\in\mathscr{U}(f)$ and two distinct basic sets $\Lambda_1(g)$ and $\Lambda_2(g)$ for g such that*

(I) $B_{4\varepsilon_2}(f^i(p))\bigcap B_{4\varepsilon_2}(f^j(p))=\phi$ for $0\le i\ne j\le n-1$,

$$\text{(II)}\quad g(z)=\begin{cases}\exp_{f^{i+1}(p)}\circ D_{f^i(p)}f\circ\exp_{f^i(p)}^{-1}(z)\\ \qquad \textit{if } z\in B_{\varepsilon_1}(f^i(p)) \textit{ for } 0\le i\le n-1,\\ f(z) \textit{ if } z\notin\bigcup_{i=0}^{n-1}B_{4\varepsilon_1}(f^i(p)),\end{cases}$$

$$\text{(III)}\quad\begin{cases}p=g^n(p)\in\Lambda_1(g),\ q\in\Lambda_2(g),\\ x\in W^s(p,\ g)\cap W^u(q,\ g),\\ T_xW^s(p,\ g)=T_xW^s(p,\ f) \textit{ and } T_xW^u(q,\ g)=T_xW^u(q,\ f).\end{cases}$$

Proof. Fix $\mathscr{U}(f)\subset$ Diff (M). By lemma 1, for any $\gamma>0$ there are

$0<\varepsilon_1<\varepsilon_0/2$, $g\in\mathscr{U}(f)$ and a basic set $\Lambda_1(g)$ satisfying properties (i), (ii) and (iii) of lemma 1. Put $\Lambda_2(g)=\Lambda_2(f)$. Then $q\in\Lambda_2(g)$. Since γ is arbitrarily small, by (iii) there are a new diffeomorphism $\tilde{g}\in\mathscr{U}(f)$ and a small neighborhood $U(x)$ of x such that $\tilde{g}(y)=g(y)$ for all $y\notin U(x)$ and such that

$$\begin{cases} x\in W^s(p,\ \tilde{g})\cap W^u(q,\ \tilde{g}), \\ T_xW^s(p,\ \tilde{g})=T_xW^s(p,\ f), \\ T_xW^s(q,\ \tilde{g})=T_xW^s(q,\ f), \end{cases}$$

For simplicity we identify $\tilde{g}$ with g. Thus (I), (II) and (III) are concluded.

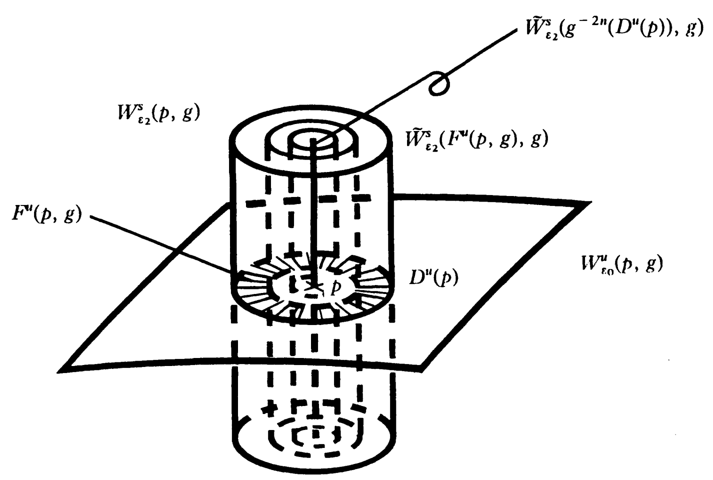
Let $g\in\mathscr{U}(f)$, $p=g^n(p)\in\Lambda_1(g)$ and $\varepsilon_1>0$ be as in lemma 3 and suppose that $\dim M-\operatorname{Ind}\Lambda_1(f)\geq 2$. Take $0<\varepsilon_2\leq\varepsilon_1$ be as in lemma 2, and fix $\alpha>0$ such that $D_pf^{2n}_{|E^u_p}(E^u_p(\alpha))\subset E^u_p(\varepsilon_2)$. Put $D^u(p)=\exp_p(E^u_p(\alpha))$. Then we have

$$d(\tilde{W}^s_{\varepsilon_2}(g^{2n}(F^u(p,\ g)),\ g),\ \tilde{W}^s_{\varepsilon_2}(F^u(p,\ g),\ g))>0,$$
$$d(\tilde{W}^s_{\varepsilon_2}(F^u(p,\ g),\ g),\ \tilde{W}^s_{\varepsilon_2}(g^{-2n}(D^u(p)),\ g))>0$$

where

$$F^u(p,\ g)=\overline{D^u(p)\setminus g^{-n}(D^u(p))} \tag{1}$$

is a fundamental domain of $W^u_{\varepsilon_2}(p,\ g)$ (recall that $\exp_p(E^u_p(\varepsilon))=W^u_\varepsilon(p,\ g)$ for $0<\varepsilon\leq\varepsilon_2$).



Let G be a linear subspace of E_p^u such that $1 \leq \dim G < \dim E_p^u$ and write $B_r^u(E) = B_r(E) \cap \exp_p(E_p^u(\varepsilon_2))$ for a subset E of M. Then we can find $0 < r_0 < \varepsilon_2$ such that

$$F^u(p, g) \setminus B_{r_0}^u\left(\exp_p(G \cap E_p^u(\varepsilon_2)) \cap F^u(p, g)\right) \neq \phi \tag{2}$$

for every G. Since

$$r_0' = d(\tilde{W}_{\varepsilon_2}^s(g^{2n}(F^u(p, g)), g), \tilde{W}_{\varepsilon_2}^s(F^u(p, g), g)) > 0,$$
$$r_0'' = d(\tilde{W}_{\varepsilon_2}^s(F^u(p, g), g), \tilde{W}_{\varepsilon_2}^s(g^{-2n}(D^u(p)), g)) > 0,$$

we define a positive number $r_1 = \frac{1}{4} \min\{r_0, r_0', r_0''\}$.

Put

$$\Gamma(p) = \bigcup_{y \in \exp_p(E_p^u(\varepsilon_2))} \tilde{W}_{\varepsilon_2}^s(y, g).$$

Then, for any $z \in \Gamma(p)$, we can find only one point $y \in \exp_p(E_p^u(\varepsilon_2))$ such that $z \in \tilde{W}_{\varepsilon_2}^s(y, g)$, and so we write

$$\pi(z) = y. \tag{3}$$

Then π: $\Gamma(p) \to \exp_p(E_p^u(\varepsilon_2))$ is differentiable and which plays an essential role in the proof of the proposition. For $z \in \Gamma(p) \setminus W_{\varepsilon_2}^s(p, g)$, there is an integer $N_z > 0$ such that $g^{ni}(\pi(z)) \in D^u(p)$ for $0 \leq i \leq N_z$ (especially $g^{nN_z}(\pi(z)) \in F^u(p, g)$) and $g^{n(N_z+1)}(\pi(z)) \notin D^u(p)$.

Lemma 4. *Under the above notations, there is* $0 < \varepsilon_3 < r_1$ *such that* $\operatorname{diam} \pi(B_{\varepsilon_3}(g^{nN_z}(z))) < r_1$ *for every* $z \in \left(\bigcup_{y \in W_{\varepsilon_3}^u(p,g)} \tilde{W}_{\varepsilon_3}^s(y, g) \right) \setminus W_{\varepsilon_3}^s(p, g)$.

Proof. If this is false, for $k > 0$ there are

$$z_k \in \left(\bigcup_{y \in W_{\frac{1}{k}}^u(p,g)} \tilde{W}_{\frac{1}{k}}^s(y, g) \right) \setminus W_{\frac{1}{k}}^s(p, g)$$

and $N_k = N_{z_k} > 0$ such that $\operatorname{diam} \pi(B_{\frac{1}{k}}(g^{nN_k}(z_k))) \geq r_1$. Since $z_k \in \tilde{W}_{\frac{1}{k}}^s(\pi(z_k), g)$, we have $N_k \to \infty$ as $k \to \infty$ (because of $\pi(z_k) \in W_{\frac{1}{k}}^u(p, g)$). From $g^{ni}(\pi(z_k)) \in D^u(p) \subset \exp_p(E_p^u(\varepsilon_2))$ for $0 \leq i \leq N_k$, we have

$$d(g^{nN_k}(\pi(z_k)), g^{nN_k}(z_k)) \leq C\lambda^{nN_k} d(\pi(z_k), z_k) \to 0 \text{ as } k \to \infty$$

by lemma 2 (iii).

For $k>0$ there are w_k, $w'_k \in \exp_p(E^u_p(\varepsilon_2))$, $v_k \in \tilde{W}^s_{\varepsilon_2}(w_k, g) \cap B_{\frac{1}{k}}(g^{nN_k}(z_k))$ and $v'_k \in \tilde{W}^s_{\varepsilon_2}(w'_k, g) \cap B_{\frac{1}{k}}(g^{nN_k}(z_k))$ such that $d(w_k, w'_k) \geq r_1$. If $w_k \to w$ and $w'_k \to w'$ $(k \to \infty)$, then $w, w' \in \exp_p(E^u_p(\varepsilon_2))$ and $d(w, w') \geq r_1$. When $v_k \to v$ and $v'_k \to v'$ as $k \to \infty$, we have $v = v' \in \exp_p(E^u_p(\varepsilon_2))$ by the properties

$$\begin{cases} g^{nN_k}(\pi(z_k)) \in \exp_p(E^u_p(\varepsilon_2)), \\ d(g^{nN_k}(\pi(z_k)),\ g^{nN_k}(z_k)) \to 0 \text{ as } k \to \infty, \\ d(v_k,\ g^{nN_k}(z_k)) < \frac{1}{k} \text{ and } d(v'_k,\ g^{nN_k}(z_k)) < \frac{1}{k}. \end{cases}$$

Since $\tilde{W}^s_{\varepsilon_2}(y, g)$ $(y \in \exp_p(E^u_p(\varepsilon_2)))$ is continuous with respect to y, we have $v \in \tilde{W}^s_{\varepsilon_2}(w, g)$. Thus $v = w$ since $\tilde{W}^s_{\varepsilon_2}(w, g) \cap \exp_p(E^u_p(\varepsilon_2))$ is a single point and v, $w \in \exp_p(E^u_p(\varepsilon_2))$. In this way we get $w = v = v' = w'$, thus contradicting.

We are in a position to prove the proposition. Hereafter let $\dim M \geq 4$ and $f \in \mathscr{P}(M)$. Notice that f satisfies Axiom A with no-cycle.

Fix $x \in M \setminus \Omega(f)$. Then there are distinct basic sets $\Lambda_i(f)$ $(i=1, 2)$ such that $x \in W^s(\Lambda_1(f), f) \cap W^u(\Lambda_2(f), f)$. If $\operatorname{Ind} \Lambda_1(f) = \dim M$ or $\dim M - 1$, then by the proof of [4, Theorem 2] we have $T_x M = T_x W^s(x, f) + T_x W^u(x, f)$. Thus it is enough to prove the above equality for the case when $1 \leq \operatorname{Ind} \Lambda_1(f) \leq \dim M - 2$.

Since $\Omega(f) = \overline{P(f)}$, there is $f' \in \mathscr{P}(M)$ arbitrarily near to f in a C^1 topology satisfying

(a) $f(y) = f'(y)$ for all y outside of a small neighborhood of x,

(b) there are $p = f'^n(p) \in \Lambda_1(f)$ for some $n>0$ and $q \in \Lambda_2(f)$ such that $x \in W^s(p, f') \cap W^u(q, f')$, $T_x W^s(p, f') = T_x W^s(x, f)$ and $T_x W^u(q, f') = T_x W^u(x, f)$.

By (a) there are basic sets $\Lambda_i(f')$ $(i=1, 2)$ for f' such that $\Lambda_i(f') = \Lambda_i(f)$ $(i=1, 2)$ since f is Ω-stable. We shall prove that $T_x M = T_x W^s(p, f') + T_x W^u(q, f')$ for the case when $1 \leq \operatorname{Ind} \Lambda_1(f) \leq \dim M - 2$. For simplicity we identify f' with f.

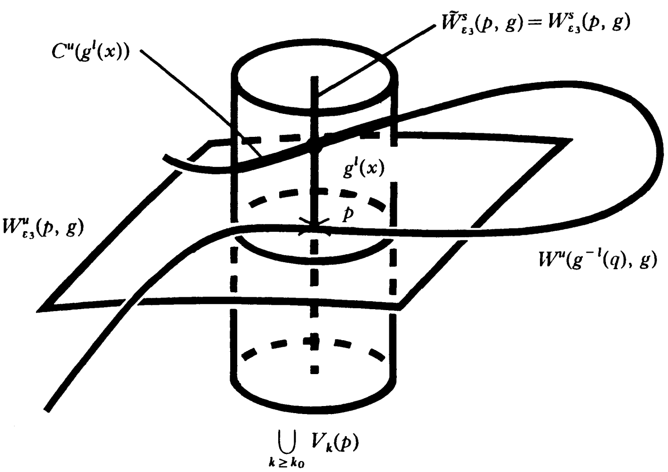
Let $\mathscr{U}(f)$ be a small neighborhood of f such that $\mathscr{U}(f) \subset \mathscr{P}(M)$. Then, by lemma 3 there are $g \in \mathscr{U}(f)$ and basic sets $\Lambda_i(g)$ $(i=1, 2)$ satisfying lemma 3 (I), (II) and (III). Thus $T_x W^s(p, g) = T_x W^s(x, f)$ and $T_x W^u(q, g) = T_x W^u(x, f)$. Let $\varepsilon_3 > 0$ be as in lemma 4 and define

$$V_k(p) = \bigcup_{y \in g^{-nk}(F^u(p, g))} \tilde{W}^s_{\varepsilon_3}(y, g) \text{ for } k \geq 0$$

where $F^u(p, g)$ is the fundamental domain of $W^u_{\varepsilon_2}(p, g)$ (see (1)). Notice that $V_k(p)\subset\Gamma(p)$ for $k\geq 0$ and that $V_k(p)\to\tilde{W}^s_{\varepsilon_3}(p, g)=W^s_{\varepsilon_3}(p, g)$ as $k\to\infty$ since $g^{-nk}(F^u(p, g))\to\{p\}$ as $k\to\infty$. Thus there is $k_0>0$ such that

$$V_{k_0}(p)\subset\bigcup_{y\in W^u_{\varepsilon_3}(p,g)}\tilde{W}^s_{\varepsilon_3}(y, g).$$

Obviously $\bigcup_{k\geq k_0}V_k(p)$ is a neighborhood of p in M.



Pick $l>0$ such that $g^l(x)\in\mathrm{int}\left(\bigcup_{k\geq k_0}V_k(p)\right)$ and $g^{-l}(x)\in W^u_{\varepsilon_0/2}(g^{-l}(q), g)$, and denote by $C^u(g^l(x))$ the connected component of $g^l(x)$ in $W^u(g^l(q), g)$ $\cap\left(\bigcup_{k\geq k_0}V_k(p)\right)$. Clearly, $\exp_p^{-1}(C^u(g^l(x)))\subset T_pM$.

For a linear subspace E of T_pM *and* $\nu>0$ we write

$$E_\nu(g^l(x))=\{v+\exp_p^{-1}(g^l(x))\mid v\in E\text{ with }\|v\|\leq\nu\}.$$

Then there are a linear subspace E' of T_pM and a number $0<\nu_0\leq\varepsilon_3$ such that

$$T_{g^l(x)}\exp_p(E'_\nu(g^l(x)))=T_{g^l(x)}C^u(g^l(x)) \tag{4}$$

and $\exp_p(E'_v(g^l(x)))\subset \bigcup_{k\geq k_0} V_k(p)$ for $0<v\leq v_0$.

Since $g^l(x)\notin\Omega(g)$, there exists $0<v_1\leq v_0$ such that $B_{v_1}(g^l(x))\cap g^i(B_{v_1}(g^l(x)))=\phi$ for $i\in \boldsymbol{Z}\setminus\{0\}$. Let $\mathscr{U}(g)$ be a neighborhood of g such that $\mathscr{U}(g)\subset\mathscr{U}(f)$. By (4) there are $0<v_2<v_1$ and $\varphi\in \mathrm{Diff}\,(M)$ such that

$$\begin{cases} \varphi_{|(B_{v_2}(g^l(x)))^c}=\mathrm{id}, \\ \varphi(g^l(x))=g^l(x), \\ \varphi(\exp_p(E'_{v_2}(g^l(x))))\subset C^u(g^l(x)), \\ \dim\ \varphi(\exp_p(E'_{v_2}(g^l(x))))=\dim\ C^u(g^l(x)), \\ g'\in\mathscr{U}(g) \text{ where } g'=\varphi^{-1}\circ g. \end{cases}$$

We denote $\exp_p(E'_{v_2}(g^l(x)))$ by $\exp_p(E'_{v_2}(g'^l(x)))$ because of $g'^l(x)=g^l(x)$.

It is clear that there are two distinct basic sets $\Lambda_i(g')$ $(i=1,\ 2)$ such that $\Lambda_i(g')=\Lambda_i(g)$ $(i=1,\ 2)$ since g is Ω-stable, and that

$$\begin{aligned} W^\sigma_{\varepsilon 0}(p,\ g')&=W^\sigma_{\varepsilon 0}(p,\ g), \\ W^\sigma_{\varepsilon 0}(q,\ g')&=W^\sigma_{\varepsilon 0}(q,\ g), \\ T_xW^\sigma(x,\ g')&=T_xW^\sigma(x,\ g)\ (\sigma=s,\ u), \\ \exp_p(E'_{v_2}(g'^l(x)))&\subset W^u(g'^l(q),\ g')\cap\Gamma(p), \\ \dim\ \exp_p(E'_{v_2}(g'^l(x)))&=\dim\ W^u(q,\ g')=\dim\ C^u(g^l(x)). \end{aligned}$$

Lemma 5. *Under the above notations,* $\exp_p(E'_{v_2}(g'^l(x)))$ *meets transversely* $W^s_{\varepsilon_3}(p,\ g')$ *at* $g'^l(x)$.

Proof. Let $\varepsilon_2>0$ be as in lemma 2. Since $W^s_{\varepsilon_3}(p,\ g')\subset\exp_p(E^s_p(\varepsilon_2))$ and $W^s_{\varepsilon_3}(p,g')\subset\exp_p(E^u_p(\varepsilon_2))$, to get the conclusion it is enough to prove

$$\dim\ \pi(\exp_p(E'_{v_2}(g'^l(x))))\geq\dim\ W^s_{\varepsilon_3}(p,\ g').$$

Here $\pi:\Gamma(p)\to\exp_p(E^u_p(\varepsilon_2))$ is the map defined as in (3).

Assume that $\dim\ \pi(\exp_p(E'_{v_2}(g'^l(x))))<\dim\ W^u_{\varepsilon_3}(p,\ g')$ and put $C^u_\varepsilon(g'^l(x))=B_\varepsilon(g'^l(x))\cap g'^{2l}(W^u_{\varepsilon 0}(g'^{-l}(q),\ g'))$ for $\varepsilon>0$. Take $0<\varepsilon<v_2$ such that $C^u_\varepsilon(g'^l(x))$ is the connected component of $g'^l(x)$ in $B_{\tilde\varepsilon}(g'^l(x))\cap g'^{2l}(W^u_{\varepsilon 0}(g'^{-l}(q),g'))$ for $0<\varepsilon\leq\tilde\varepsilon$, and such that $B_{\tilde\varepsilon}(g'^l(x))\cap g'^{2l}(W^u_{\varepsilon 0}(g'^{-l}(q),g'))\subset\exp(E'_{v_2}(g'^l(x)))$.

Claim 1. *Let* $0<\varepsilon\leq\tilde{\varepsilon}$. *If* $d(g'^{-i}(g'^{l}(x)), g'^{-i}(w))<\varepsilon$ *for* $i\geq 0$, *then* $w\in C^u_{\varepsilon}(g'^{l}(x))$.

It is clear that $d(g'^{-l-i}(x), g'^{-2l-i}(w))<\varepsilon\leq\varepsilon_0/2$ for all $i\geq 0$. On the other hand, since $d(g'^{-l-i}(x), g'^{-l-i}(q))<\varepsilon_0/2$ $(i\geq 0)$,

$$d(g'^{-2l-i}(w), g'^{-l-i}(q))\leq d(g'^{-2l-i}(w), g'^{-l-i}(x))+d(g'^{-l-i}(x), g'^{-l-i}(q))<\varepsilon_0$$

for all $i\geq 0$, and so $g'^{-2l}(w)\in W^u_{\varepsilon 0}(g'^{-l}(q), g')$. Thus $w\in C^u_{\varepsilon}(g'^{l}(x))=B_{\varepsilon}(g'^{l}(x))\cap g'^{2l}(W^u_{\varepsilon 0}(g'^{-l}(q), g'))$ since $d(g'^{l}(x), w)<\varepsilon$.

We divide the proof of this lemma into two cases:

Case 1. $C^u_{\tilde{\varepsilon}}(g'^{l}(x))\subset W^s_{\varepsilon_3}(p, g')$,

Case 2. $C^u_{\tilde{\varepsilon}}(g'^{l}(x))\not\subset W^s_{\varepsilon_3}(p, g')$,

For case 1, put $\varepsilon=\tilde{\varepsilon}/2$ and let $0<\delta=\delta(\varepsilon, g')<\varepsilon$ be the number in the definition of **POTP** of g'. Recall that $F^u(p, g')=F^u(p, g)$ and fix $y\in\bigcup_{k\geq k_0} g'^{-nk}(F^u(p, g))\setminus\{p\}$ such that $\tilde{W}^s_{\varepsilon_3}(y, g')\cap B_{\delta}(g'^{l}(x))\neq\phi$. For $z\in\tilde{W}^s_{\varepsilon_3}(y, g')\cap B_{\delta}(g'^{l}(x))$,

$$\{\cdots, g'^{-1}(x), x, g'(x),\cdots, g'^{l-1}(x), z, g'(z), g'^{2}(z),\cdots\}$$

is a δ-pseudo-orbit of g'. Thus there exists $w\in M$ such that $d(g'^{i}(w), g'^{i}(z))<\varepsilon$ $(i\geq 0)$ and $d(g'^{-i}(w), g'^{-i}(g'^{l}(x)))<\varepsilon$ $(i\geq 1)$. Since $d(w, z)<\varepsilon$ and $d(z, g'^{l}(x))<\delta<\tilde{\varepsilon}/2$, we have $d(g'^{l}(x), w)<\tilde{\varepsilon}$. Therefore $d(g'^{-i}(w), g'^{-i}(g'^{l}(x)))<\tilde{\varepsilon}$ for all $i\geq 0$, and so $w\in C^u_{\tilde{\varepsilon}}(g'^{l}(x))$ by claim 1.

Obviously, there is $\tilde{k}=\tilde{k}(z)>0$ such that $g'^{n\tilde{k}}(z)\in V_0(p)=\bigcup_{y\in F^u(p,g')}\tilde{W}^s_{\varepsilon_3}(y, g')$. By the choice of ε and by the definition of $F^u(p, g)$ we have $B_{\varepsilon}(g'^{n\tilde{k}}(z))\cap W^s_{\varepsilon_3}(p, g')=\phi$. However, $w\in C^u_{\tilde{\varepsilon}}(g'^{l}(x))\subset W^s_{\varepsilon_3}(p, g')$ implies $(g'^{n\tilde{k}}(w)\in W^s_{\varepsilon_3}(p, g')$. Thus $g'^{n\tilde{k}}(w)\in B_{\varepsilon}(g'^{n\tilde{k}}(z))\cap W^s_{\varepsilon_3}(p, g')\neq\phi$ (since $d(g'^{n\tilde{k}}(z), g'^{n\tilde{k}}(w))<\varepsilon$). This is a contradiction and so the lemma is proved for case 1.

For case 2, take $k_1 k_0$ such that $k\geq k_1$ implies $C^u_{\tilde{\varepsilon}}(g'^{l}(x))\cap V_k(p)\neq\phi$. By the choice of $\tilde{\varepsilon}$,

$$\pi\left(B_{\tilde{\varepsilon}}(g'^{n\tilde{k}}(C^u_{\tilde{\varepsilon}}(g'^l(x))\cap V_k(p)))\right)\not\supset F^u(p,\ g')$$

for all $k\geq k_1$ since dim $\pi(C^u_{\tilde{\varepsilon}}(g'^l(x)))<$ dim $W^u_{\varepsilon_3}(p, g')$ (see (2)). To simplify we write

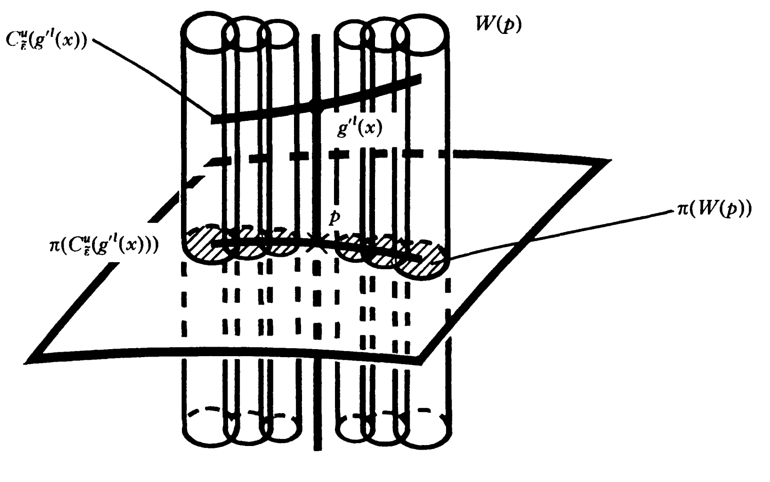
$$W_k(p)=\bigcup_{y\in g'^{-nk}(\pi(B_{\tilde{\varepsilon}}(g'^{nk}(C^u_{\tilde{\varepsilon}}(g'^l(x))\cap V_k(p)))))}\tilde{W}^s_{\varepsilon_3}(y,\ g'),$$

$$W(p)=\Big(\bigcup_{k\geq k_1} W_k(p)\Big)\cup W^s_{\varepsilon_3}(p,\ g').$$

Then $W(p)\subset\Gamma(p)$ and

$$\begin{aligned}\pi(W(p))&=\Big(\bigcup_{k\geq k_1}\pi(W_k(p))\Big)\cup\{p\}\\ &=\Big(\bigcup_{k\geq k_1} g'^{-nk}\big(\pi(B_{\tilde{\varepsilon}}(g'^{nk}(C^u_{\tilde{\varepsilon}}(g'^l(x))\cap V_k(p))))\big)\Big)\cup\{p\}\end{aligned}$$

is not a neighborhood of p in $W^u_{\varepsilon_3}$ $(p,\ g')$.



Claim 2. *Put* $\varepsilon=\tilde{\varepsilon}/2$ *and let* $\delta=\delta(\varepsilon,\ g')<\varepsilon$ *be the number in the definition of* **POTP** *of* g'. *Then we have* $B_\delta(g'^l(x))\subset W(p)$.

For every $z \in B_\delta(g'^l(x)) \setminus W^s_{\varepsilon_0}(p, g)$, there exists $w \in M$ such that $d(g'^i(w), g'^i(z)) < \varepsilon$ and $d(g'^{-i-1}(w), g'^{-i-1}(g'^l(x))) < \varepsilon$ for all $i \geq 0$ since

$$\{\cdots, g'^{-1}(x), x, g'(x), \cdots, \\ g'^{l-1}(x), z, g'(z), g'^2(z), \cdots\}$$

is a δ-pseudo-orbit of g'. Thus $d(g'^{-i}(w), g'^{-i}(g'^l(x))) < \tilde{\varepsilon}$ for all $i \geq 0$ (since $d(g'^l(x), w) \leq d(g'^l(x), z) + d(z, w) < \varepsilon + \delta < \tilde{\varepsilon}$), and so $w \in C^u_{\tilde{\varepsilon}}(g'^l(x))$ by claim 1. Fix $\tilde{k} = \tilde{k}(w) \geq k_1$ such that $w \in V_{\tilde{k}}(p) \cap C^u_{\tilde{\varepsilon}}(g'^l(x))$. Then $g'^{n\tilde{k}}(z) \in B_\varepsilon(V_0(p) \cap g'^{n\tilde{k}}(C^u_{\tilde{\varepsilon}}(g'^l(x))))$ since $d(g'^{n\tilde{k}}(w), g'^{n\tilde{k}}(z)) < \varepsilon$. Thus we have $z \in W_{\tilde{k}}(p) \subset W(p)$.

By claim 2 we have $\pi(B_\delta(g'^l(x))) \subset \pi(W(p))$. If we establish that $\pi(B_\delta(g'^l(x)))$ is a neighborhood of p in $W^u_{\varepsilon_3}(p, g')$, then we get a contradiction and therefore the proof of this lemma is completed.

If $\pi(B_\delta(g'^l(x)))$ is not a neighborhood of p in $W^u_{\varepsilon_3}(p, g)$, then for every $i > 0$ there is $y_i \in W^u_{\varepsilon_3}(p, g')$ such that $y_i \notin \pi(B_\delta(g'^l(x)))$ and $d(y_i, p) < \frac{1}{i}$. Since $\tilde{W}^s_{\varepsilon_3}(y_i, g') \to W^s_{\varepsilon_3}(p, g')$ as $i \to \infty$,

$$\tilde{W}^s_{\varepsilon_3}(y_i, g') \cap B_\delta(g'^l(x)) \neq \phi$$

for sufficiently large i and thus $y_i \in \pi(B_\delta(g'^l(x)))$. This is a contradiction and so $\pi(W(p))$ is a neighborhood of p in $W^u_{\varepsilon_3}(p, g')$. For any case lemma 5 was proved.

The proof of the transversality at x for case $1 \leq \operatorname{Ind} \Lambda_1(f) \leq \dim M - 2$ follows from lemma 5. Indeed, since $\exp_p(E'_{v_2}(g'_l(x)))$ meets transversely $W^s_{\varepsilon_3}(p, g')$ at $g'^l(x)$, we have

$$\begin{aligned} T_{g'^l(x)}M &= T_{g'^l(x)}\exp_p(E'_{v_2}(g'^l(x))) + T_{g'^l(x)}W^s_{\varepsilon_3}(p, g') \\ &= T_{g'^l(x)}W^u(g'^l(q), g') + T_{g'^l(x)}W^s_{\varepsilon_3}(p, g') \end{aligned}$$

by (4). Thus

$$\begin{aligned} T_xM = T_xW^s(p, g') + T_xW^u(q, g') \\ &= T_xW^s(x, g) + T_xW^u(x, g) \\ &= T_xW^s(x, f) + T_xW^u(x, f). \end{aligned}$$

Therefore the proof of the proposition is completed.

References

[1] N. Aoki: *The set of Axiom A diffeomorphisms with no cycle*, Bol. Soc. Bras. Mat. 23-New Series (1992), 21–65.

[2] J. Franks: *Necessary conditions for stability of diffeomorphisms*, Trans. A.M.S. **158** (1971), 301–308.

[3] M. Hirsch and C. Pugh: *Stable manifold and hyperbolic sets*, in Global Analysis, Proc. Sympos. Pure Math., A.M.S. **14** (1970), 133–163.

[4] K. Moriyasu: *The topological stability of diffeomorphisms*, Nagoya Math. J. **123** (1991), 91–102.

[5] C. Robinson: *Stability theorems and hyperbolicity in dynamical systems*, Rocky Mountain J. Math. **7** (1977), 425–437.

[6] K. Sakai: *The C^1 uniform pseudo-orbit tracing property*, Tokyo J. Math. **15** (1992), 99–109.

[7] K. Sakai: *Diffeomorphisms with pseudo orbit tracing property*, Nagoya Math. J. **126** (1992), 125–140.

[8] S. Smale: *The Ω-stability theorem*, *in* Global Analysis Proc. A.M.S. **14** (1970), 289–297.

Department of Mathematics
Kanagawa University
Rokkakubashi, Kanagawa-ku
Yokohama 221, Japan