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TENSOR PRODUCT AND GENERALIZED OTT-SCHAEFFER PLANES

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1. Introduction

The Ott-Schaeffer planes (O-S) are translation planes of even order q^2 with kernel $K \cong GF(q)$ which admit a collineation group $\mathcal{G}$ isomorphic to $SL(2, q)$ where the involutions are Baer. Furthermore, if $\mathcal{S}_2$ is a Sylow 2-subgroup of $\mathcal{G}$ then no two nontrivial elements of $\mathcal{G}$ fix the same Baer subplane pointwise.

The O-S planes are also derivable and a plane may be defined for each automorphism α of $GF(q)$, $q=2^{2r+1}$ which has fixed field equal to $GF(2)$. Hence, the number of such translation planes of each order is $\Phi(2r+1)$ (the number of integers $\neq 1$ relatively prime to $2r+1$).

Further, the O-S planes may be defined by the tensor product of $SL(2, q)$ by a twisted version of the same (by an automorphism $\alpha \ni \text{Fix } \alpha = GF(2)$).

Note that $GL(2, q) = SL(2, q) \times \mathcal{Z}(GL(2, q))$ (center) when q is even so as the kernel is $GF(q)$, the O-S planes also admit $GL(2, q)$.

Conversely, in [7], for arbitrary kernel we have

Theorem (Johnson [7]). *Let π be a translation plane of even order $q^2 > 16$ that admits $GL(2, q)$ as a collineation group in the translation complement where the 2-groups are Baer and no two nontrivial elements fix the same Baer subplane pointwise. Then π is an Ott-Schaeffer plane.*

DEFINITION 1.1. Tensor Product Plane.

A translation plane π of order q^2, q even or odd, kernel $K \cong GF(q)$ that admits the collineation group

$$T = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^{\sigma} \\ 0 & 1 \end{bmatrix} \middle| \, a \in K \right\}, \ \sigma \in \text{Aut } K$$

where $\pi = \{(x_1, x_2, y_1, y_2) \mid x_i, y_i \in K,\ i=1, 2\}$, $x=(x_1, x_2)$, $y=(y_1, y_2)$ and $x=0$, $y=0$, $y=x$ are components in this representation is called a *tensor product plane*.

DEFINITION 1.2. Generalized Ott-Schaeffer Plane.

A translation plane π of order q^2, q even or odd, $q=p^r$ for p a prime, kernel

$K \simeq GF(q)$, that admits a p-group $\mathcal{B}$ of order q in the translation complement such that each element of $\mathcal{B}$ is Baer and no two nontrivial subgroups of $\mathcal{B}$ can fix the same Baer subplane pointwise is called a *generalized Ott-Schaeffer plane*.

In section 2, we consider the basic structure of tensor product planes and of generalized Ott-Schaeffer planes. In section 3, we consider translation planes (T-P and O-S) which admit groups of order $q(q-1)$ in the translation complement.

Our main results *completely* classify both tensor product planes of even order q^2 admitting a tensor group of order $q(q-1)$ (see (3.4)) *and* generalized Ott-Schaeffer planes admitting groups of order $q(q-1)$ in the translation complement with prescribed Sylow 2 subgroups (see (3.21)).

2. The fundamental structure

NOTES 2.1. For q odd $q=p^r$, $p \geq 3$, there is no tensor product plane. If $p \geq 5$ there is no generalized Ott-Schaeffer plane of order q^2.

Proof. Consider

$$\tau_a = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a^\sigma & a & a^{\sigma+1} \\ 0 & 1 & 0 & a \\ 0 & 0 & 1 & a^\sigma \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$\tau_a (a \neq 0)$ fixes $\pi_a = \{(0, x_2, -x_2 a^{1-\sigma}, y_2) \mid (x_2, y_2) \mid x_2, y_2 \in K\}$ pointwise. Furthermore, $\{\tau_a \mid a \in K \simeq GF(q)\} = \mathcal{S}_p$ is elementary abelian. By Foulser [4], $\mathcal{S}_p$ must fix some Baer subplane pointwise, which cannot be the case. Hence, there are no tensor product planes of characteristic ≥ 5.

Now assume $p=3$. The components $x=0$, $y=xM$ of π_a, for $M=\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, must satisfy $(0, x_2, x_2 m_3, x_2 m_4) = (0, x_2, -x_2 a^{1-\sigma}, y_2)$ so that $m_3 = -a^{1-\sigma}$.

On the other hand, in order that $y=xM$ is fixed by τ_a, we must have

$$\left(x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, x\left(\begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\right)\right) \in y = xM$$

$$\Leftrightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} m_1 & m_2 \\ -a^{1-\sigma} & m_4 \end{bmatrix} = \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + \begin{bmatrix} m_1 & m_2 \\ -a^{1-\sigma} & m_4 \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$$

$$\Leftrightarrow m_1 - a = a + m_1 \Leftrightarrow -a = +a .$$

Hence, $p \neq 3$. This argument, which is also valid for arbitrary odd order planes, was pointed out to the author by Rolando Pomareda. Now assume that π is a generalized Ott-Schaeffer plane of odd order q^2, $q=p^r$. Each element of $\mathcal{B}$ is Baer and for $p>3$, Foulser [4] has shown that the Baer subplanes involved must be disjoint. That is, since $|\mathcal{B}|=q$, $\mathcal{B}$ must fix a 1-dimensional subspace pointwise, which cannot be the case.

If $p=3$, it is possible that there are generalized Ott-Schaeffer planes of order 3^{2r}. However, hereafter, in this section, we shall consider only *even* order planes.

Theorem 2.2. *Any tensor product plane π of even order q^2 is derivable. The derivable net is a regulus and the derived plane $\bar{\pi}$ is a tensor product plane defined by the inverse σ^{-1} of the automorphism σ used in the definition of π. Moreover, Fix $\sigma=GF(2)$ and the spread may be represented by $x=0$,*

$$y=x\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad a\in K$$

$$y=x\begin{bmatrix} u & , m(u,a) \\ a^{1-\sigma} & , u+a \end{bmatrix}, m\colon K\times K\to K, \quad u, a\in K, a\neq 0\,.$$

Proof. Consider the notation of (2.1) with

$$\tau_a=\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}.$$

Then $y=0\xrightarrow{\tau_a} y=x\begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix}=\left[y=x\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}\right]$. But, $y=0$, $y=x\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$ for $a\in K\cong GF(q)$ is the vector form of a regulus in $PG(3, K)$.

Now derive π using this "regulus" partial spread. Recall, if

$$(x_1, x_2, y_1, y_2)\xrightarrow{\tau_a}(x_1, x_1a^\sigma+x_2, x_1a+y_1, x_1a^{\sigma+1}+x_2a+y_1a^\sigma+y_2)$$

represents τ_a in π by the standard representation of coordinates in $\bar{\pi}$ by (x_1, y_1, x_2, y_2), we obtain:

$$(x_1, y_1, x_2, y_2)\xrightarrow{\bar{\tau}_a}(x_1, x_1a+y_1, x_1a^\sigma+x_2, x_1a^{\sigma+1}+x_2a+y_1a^\sigma+y_2)\,,$$

$\bar{\tau}_a$ representing τ_a in $\bar{\pi}$ (see, e.g., Jha-Johnson [6]). Hence,

$$\bar{\tau}_a=\begin{bmatrix} 1 & a & a^\sigma & a^{\sigma+1} \\ 0 & 1 & 0 & a^\sigma \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{bmatrix}=\begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 1 & (a^\sigma)^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 1 & b^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix}$$

for $b=a^\sigma$. Hence, $\left\{\tau_a=\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}\right\}$ in π is $\left\{\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\otimes\begin{bmatrix} 1 & b^{\sigma^{-1}} \\ 0 & 1 \end{bmatrix}\right\}$ in $\bar{\pi}$.

We now consider the components $y=xM$, $M=\begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$ of $\pi_a=\{(0, x_2, x_2a^{1-\sigma},$

$y_2)\mid x_2, y_2 \in K\}$, $a \neq 0$ (Baer subplane of π). τ_a fixes

$$y = xM \Leftrightarrow \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} M = \begin{bmatrix} a & a^{\sigma+1} \\ 0 & a \end{bmatrix} + M \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}$$
$$\Leftrightarrow \begin{bmatrix} a+m_1, & a^{\sigma+1}+m_1 a^\sigma + m_2 \\ m_3 \quad , & a+m_3 a^\sigma + m_4 \end{bmatrix} = \begin{bmatrix} m_1+m_3 a^\sigma, & m_2+a^\sigma m_4 \\ m_3 \quad , & m_4 \end{bmatrix}$$
$$\Leftrightarrow m_3 = a^{1-\sigma},$$

and $a^{\sigma+1}+m_1 a^\sigma = a^\sigma m_4$ or rather that $a+m_1=m_4$.

Hence, $M=\begin{bmatrix} u \quad , & m(u, a) \\ a^{1-\sigma}, & u+a \end{bmatrix}$ for all u where m is a function from $K\times K$ to K. Note that if $[a^{1-\sigma}, u+a]$ does not take on all q^2 values then τ_a and τ_b for $a \neq b$ $(a, b \neq 0)$ fix the same component $y=xM \neq 0$ and hence $\langle \tau_1, \tau_2 \rangle$ must fix a 1-space pointwise on *both* $y=xM$ and $x=0$. Hence, $\tau_a=\tau_b$. Thus, $[a^{1-\sigma}, u]=[c^{1-\sigma}, u] \Leftrightarrow a=c \Leftrightarrow a^{1-\sigma}=c^{1-\sigma} \Leftrightarrow (ac^{-1})=(ac^{-1})^\sigma \Leftrightarrow \text{Fix } \sigma = GF(2)$.

Now we consider the general structure of a generalized Ott-Schaeffer plane π. Let $\mathcal{G}$ be a collineation group of π of order q in the linear translation complement and such that each involution in π is Baer. By Johnson and Ostrom [8], $\mathcal{G}$ is elementary abelian. Further, assume no two involutions in $\mathcal{G}$ fix the same Baer subplane pointwise. Then

Lemma 2.3. *The $q-1$ Baer subplanes corresponding to the involutions of $\mathcal{G}$ lie across $(q-1)\,q+1$ components. The remaining q components are in an orbit under $\mathcal{G}$.*

Proof. $\mathcal{G}$ fixes a component which we call $x=0$, $x=(x_1, x_2)$. If $g, h \in \mathcal{G}-\langle 1\rangle$ and Fix g and Fix h share a component $\mathcal{L} \neq (x=0)$ then $\langle g, h\rangle$ has fixed points on both $\mathcal{L}$ and $(x=0)$. Since $\mathcal{G}$ is linear, it follows that $\langle g, h\rangle$ is Baer—a contradiction. Hence, Fix g and Fix h cannot share a component $\neq (x=0)$. Thus, this accounts for $q(q-1)+1$ components. As the Baer subplanes corresponding to the involutions in $\mathcal{G}$ do not intersect the remaining set Γ of q components, Γ must be a $\mathcal{G}$-orbit.

Now choose $(y=0) \in \Gamma$. Then

Lemma 2.4. *We may choose a basis so that*

$$\mathcal{G} = \left\{ \tau_a = \left(\begin{array}{cc|cc} 1 & a & f(a) & g(a) \\ 0 & 1 & 0 & f(a) \\ \hline & & 1 & a \\ & & 0 & 1 \end{array} \right) \;\middle|\; a \in K \cong GF(q) \right\},$$

f, g functions $K \to K$, where f is a $1-1$ additive and $g(a+b)=g(a)+g(b)+af(b)+bf(a)$ for all $a, b \in K$.

Proof. Let $\mathcal{G}$ fix $x=0$. Choose a basis so one of the involutions fixes $\{(0, x_2, 0, y_2) \mid x_2, y_2 \in K\} = \pi_0$ pointwise and π_0 shares the components $x=0, y=0, y=x$. Then one of the involutions has the form $\tau = \left(\begin{array}{cc|cc} 1 & d & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{array}\right)$. Now $\rho \in \mathcal{G}$ fixes π_0 and $x=0$, so $\rho = \begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_4 & 0 & b_4 \\ 0 & 0 & c_1 & c_2 \\ 0 & 0 & 0 & c_4 \end{pmatrix}$. Since $|\rho| = 2$, we must have $a_1^2 = a_4^2 = c_1^2 = c_4^2 = 1$ so $a_1 = a_4 = c_1 = c_4 = 1$. Now $\rho\tau = \tau\rho$ so

$$\begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} = \begin{bmatrix} b_1 & b_2 \\ 0 & b_4 \end{bmatrix} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix}$$

and hence $b_1 = b_4$ $(d \neq 0)$. We assert that $a_2 = c_2$,

$$\rho^2 = \begin{bmatrix} I & \begin{bmatrix} 1 & a_2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix} + \begin{bmatrix} b_1 & b_2 \\ 0 & b_1 \end{bmatrix} \begin{bmatrix} 1 & c_2 \\ 0 & 1 \end{bmatrix} \\ 0 & I \end{bmatrix}$$

so $a_2 b_1 = b_1 c_2$.

If $b_1 = 0$ then ρ fixes π_0 pointwise. Hence, if $\rho \neq \tau$, $a_2 = c_2$. Now there exists a component $y = xT$ in the orbit of length q for $T = \begin{bmatrix} t_1 & t_2 \\ t_3 & t_4 \end{bmatrix}$. Change bases by $\begin{bmatrix} I & T \\ 0 & I \end{bmatrix}$. Then $x=0 \to x=0$,

$$y = 0 \leftrightarrow y = xT$$
$$y = x \to y = x(I+T) .$$

(That is, after the basis change, $y=x$ may not be an equation of a line.) Then for $\rho = \begin{pmatrix} 1 & a & b_1 & b_2 \\ 0 & 1 & 0 & b_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ we obtain

$$\begin{bmatrix} I & T \\ 0 & I \end{bmatrix} \rho \begin{bmatrix} I & T \\ 0 & I \end{bmatrix} = \begin{pmatrix} 1 & a & (b_1 + at_3), & b_2 + a(t_1 + t_4) \\ 0 & 1 & 0 & (b_1 + at_3) \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Letting

$$f(a) = b_1 + at_3$$
$$g(a) = b_2 + a(t_1 + t_4)$$

(note $\begin{pmatrix} 1 & a & c_1 & c_2 \\ 0 & 1 & 0 & c_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & a & \bar{c}_1 & \bar{c}_2 \\ 0 & 1 & 0 & \bar{c}_1 \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix} \in \mathcal{G}$ implies $c_1 = \bar{c}_1$, $c_2 = \bar{c}_2$ since there are no elations in $\mathcal{G}$) where $f, g\colon K \to K$, we have the proof to (2.4) since $\mathcal{G}$ is elementary abelian. Note

$$y = 0 \xrightarrow{\mathcal{G}} y = x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix}$$
$$\equiv y = x \begin{bmatrix} f(a), & g(a) + af(a) \\ 0\ , & f(a) \end{bmatrix}.$$

If $f(a) = 0$ for $a \neq 0$, then

$y = x \begin{bmatrix} 0 & g(a) \\ 0 & 0 \end{bmatrix} = \{(x_1, x_2, 0, x_1 g(a)) \mid x_1, x_2 \in K\} \cap \{(x_1, x_2, 0, 0) \mid x_1, x_2 \in K\}$ is $\{(0, x_2, 0, 0) \mid x_2 \in K\}$ and since both equations represent components, we have a contradiction. $\therefore$ f is 1–1. And, we have:

Lemma 2.5. *The $\mathcal{G}$-orbit of length q may be represented by $y = x \begin{bmatrix} f(a) & g(a) + af(a) \\ 0 & f(a) \end{bmatrix}$ where $a \in K$.*

At this point, let the components be $x = 0$, $y = 0$, $y = xM$, $M \in \mathcal{M}$ but I may not be in $\mathcal{M}$. Let $\tau_a = \begin{pmatrix} 1 & a & f(a) & g(a) \\ 0 & 1 & 0 & f(a) \\ & & 1 & a \\ & & 0 & 1 \end{pmatrix}$ for $a \neq 0$.

(2.6). τ_a fixes $y = xM \Leftrightarrow M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}$, $m_3 = a^{-1} f(a)$, $m_4 = m_1 + a^{-1} g(a)$.

Proof. $(x, xM) \xrightarrow{\tau_a} \left(x \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}, x \left(\begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \right) \right) \in y = xM \Leftrightarrow \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} M = \begin{bmatrix} f(a) & g(a) \\ 0 & f(a) \end{bmatrix} + M \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$.

Hence we have the following set of components

$$y = x \begin{bmatrix} u\ , & m(u, a) \\ a^{-1} f(a), & u + a^{-1} g(a) \end{bmatrix}$$

for all $a \neq 0$, $u \in K$, $m\colon K \times K \to K$,

$$x = 0$$
$$y = x \begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}$$

where $h(u)=g(f^{-1}(u))+uf^{-1}(u)$ for all $u\in K$. So, we obtain the following theorem,

Theorem 2.7. *Let π be a generalized Ott-Schaeffer plane of even order q^2, kernel $K\cong GF(q)$. Let $\mathcal{G}$ be a collineation group of order q in the linear translation complement such that each involution of $\mathcal{G}$ is Baer and no two involutions fix the same Baer subplane pointwise. Then π and $\mathcal{G}$ may be represented in the following form:*

$$\mathcal{G}=\left\{\tau_a=\begin{pmatrix}1 & a & f(a), & g(a)\\ 0 & 1 & 0 & f(a)\\ 0 & 0 & 1 & a\\ 0 & 0 & 0 & 1\end{pmatrix}\middle|\ a\in K, f\ 1-1 \text{ and additive}\right.$$
$$\left. g(a+b)=g(a)+g(b)+bf(a)+af(b)\right\}.$$

The components for π are $x=0, y=0, y=x\begin{bmatrix}u & h(u)\\ 0 & u\end{bmatrix}\ \forall\, u\in K, h(u)=uf^{-1}(u)+g(f^{-1}(u)), h(0)=0$, and $y=x\begin{bmatrix}u & , & m(u,a)\\ a^{-1}f(a), & & u+a^{-1}g(a)\end{bmatrix}$ for some function m: $K\times K\to K$.

Proof. Note $y=x$ may not represent a component.

3. Groups of order $q(q-1)$

We first assume that π is a tensor product plane of even order q^2 that admits a group $\mathcal{G}\mathcal{H}$ of order $q(q-1)$. Further, we assume $\mathcal{G}=\left\{\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix}\otimes\begin{bmatrix}1 & a^\sigma\\ 0 & 1\end{bmatrix}\middle|\ a\in K\equiv GF(q)\right\}$ as in section 2 and $\mathcal{H}=\left\{\begin{bmatrix}a & \\ & a^{-1}\end{bmatrix}\otimes\begin{bmatrix}a & \\ & a^{-\sigma}\end{bmatrix}\middle|\ a\in K\right\}$ for some $\sigma\in\operatorname{Aut} K$.

Recall from (2.2) that a spread for π may be represented in the form $x=0$, $y=x\begin{bmatrix}u & , & m(u,a)\\ a^{1-\sigma}, & & u+a\end{bmatrix}$ for all $u, a\in K$, m a function from $K\times K\to K$.

Let $\tau_a=\begin{pmatrix}1 & a^\sigma & a & a^{\sigma+1}\\ 0 & 1 & 0 & a\\ 0 & 0 & 1 & a^\sigma\\ 0 & 0 & 0 & 1\end{pmatrix}$ and $\rho_a=\begin{bmatrix}a & 0\\ 0 & a^{-1}\end{bmatrix}\otimes\begin{bmatrix}a^\sigma & 0\\ 0 & a^{-\sigma}\end{bmatrix}=\begin{pmatrix}a^{\sigma+1} & 0 & 0 & 0\\ 0 & a^{1-\sigma} & 0 & 0\\ 0 & 0 & a^{\sigma-1} & 0\\ 0 & 0 & 0 & a^{-\sigma-1}\end{pmatrix}$.

Consider the images of

$$y=x\begin{bmatrix}u & , & m(u,b)\\ b^{1-\sigma}, & & u+b\end{bmatrix}$$
$$\xrightarrow{\tau_a}\left[x\begin{bmatrix}1 & a^\sigma\\ 0 & 1\end{bmatrix}, x\left[\begin{bmatrix}a & a^{\sigma+1}\\ 0 & a\end{bmatrix}+\begin{bmatrix}u & , & m(u,b)\\ b^{1-\sigma}, & & u+b\end{bmatrix}\begin{bmatrix}1 & a^\sigma\\ 0 & 1\end{bmatrix}\right]\right]$$

$$\in y = x\begin{bmatrix} 1 & a^{\sigma} \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a+u, & a^{\sigma+1}+ua^{\sigma}+m(u,b) \\ b^{1-\sigma}, & a+b^{1-\sigma}a^{\sigma}+u+b \end{bmatrix}$$
$$= \left[y = x\begin{bmatrix} (a+u+a^{\sigma}b^{1-\sigma}), & m(u,b)+a^{\sigma}(b^{1-\sigma}a^{\sigma}+b) \\ b^{1-\sigma} & , \quad (a+u+a^{\sigma}b^{1-\sigma})+b \end{bmatrix} \right].$$

So,

Lemma 3.1.
$m(a+u+a^{\sigma}\, b^{1-\sigma}, b)=m(u, b)+a^{\sigma}(b^{1-\sigma}\, a^{\sigma}+b)$ for all $a, u, b, b \neq 0$ of K.

Applying ρ_a we obtain

Lemma 3.2.

$$y = x\begin{bmatrix} u & , \ m(u,b) \\ b^{1-\sigma}, & u+b \end{bmatrix}$$
$$\xrightarrow{\rho_a} y = x\begin{bmatrix} a^{-(\sigma+1)} & \\ & a^{-(1-\sigma)} \end{bmatrix}\begin{bmatrix} u & , \ m(u,b) \\ b^{1-\sigma}, & u+b \end{bmatrix}\begin{bmatrix} a^{\sigma-1} & 0 \\ 0 & a^{-\sigma-1} \end{bmatrix}$$
$$= \left[y = x\begin{bmatrix} ua^{-2} & , \ (m(u,b)a^{-2(\sigma+1)} \\ (ba^{-2})^{1-\sigma}, & ua^{-2}+ba^{-2} \end{bmatrix} \right].$$

So, $m(ua^{-2}, ba^{-2})=m(u, b)a^{-2(\sigma+1)}$ for all u, a, b in K, $a \neq 0, b \neq 0$.

Hence,

Lemma 3.3.
$y=x\begin{bmatrix} b & , \ m(b,b) \\ b^{1-\sigma}, & 0 \end{bmatrix}$ has $\frac{q(q-1)}{2}$ images under $\mathcal{G}\mathcal{H}$ and this orbit includes all components with a zero in the $(2, 2)$-entry of the image matrix.

In particular, we have the orbits of length $q\left(\frac{q-1}{2}\right)$ defined by the images of $y=x\begin{bmatrix} 1, & m(1,1) \\ 1, & 0 \end{bmatrix}$ and $y=x\begin{bmatrix} 0, & m(0,1) \\ 1, & 1 \end{bmatrix}$ (by analogy). By (3.1) and (3.2), the orbit of $y=x\begin{bmatrix} 1, & m(1,1)=m_1 \\ 1, & 0 \end{bmatrix}$ is:

$$y = x\begin{bmatrix} 1 & m_1 \\ 1 & 0 \end{bmatrix} \xrightarrow{\tau_a} y = x\begin{bmatrix} a+1+a^{\sigma}, & m_1+a^{\sigma}(a^{\sigma}+1) \\ 1 & , \quad a+a^{\sigma} \end{bmatrix}$$
$$\xrightarrow{\rho_b} y = x\begin{bmatrix} (a+1+a^{\sigma})\, b^{-2}, & (m_1+a^{\sigma}(a^{\sigma}+1))\, b^{-2(\sigma+1)} \\ (b^{-2})^{1-\sigma} & , \quad (a+1+a^{\sigma})\, b^{-2}+b^{-2} \end{bmatrix}$$
$$\equiv \left[y = x\begin{bmatrix} (a+1+a^{\sigma})\, c, & (m_1+a^{\sigma}(a^{\sigma}+1))\, c^{1+\sigma} \\ c^{1-\sigma} & , \quad (a+a^{\sigma})\, c \end{bmatrix} \right]$$

for all $a, c \neq 0$ in K.

Similarly, the orbit of $\begin{bmatrix} 0, & m(0,1)=m_0 \\ 1, & 1 \end{bmatrix}$ is

$$\left\{ y = x \begin{bmatrix} (a+a^\sigma)\,c, & (m_0+a^\sigma(a^\sigma+1))\,c^{\sigma+1} \\ c^{1-\sigma} & , \quad (a+a^\sigma+1)\,c \end{bmatrix} \right\}.$$

Therefore, we obtain the following theorem:

Theorem 3.4. *A translation plane π of even order q^2 and kernel $k \cong GF(q)$ admits the group $H = \left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix} \middle| a \in K \right\}$. $\left\{ \begin{bmatrix} a & \\ & a^{-1} \end{bmatrix} \otimes \begin{bmatrix} a^\sigma & \\ & a^{-\sigma} \end{bmatrix} \middle| a \in K - \{0\} \right\}$ with components $x=0$, $y=0$, $y=x \Leftrightarrow$ there exists constants m_0, $m_1 \in K$ such that the spread for π may be represented by the matrix spread set:*

$$x=0,\ y = x \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$
$$y = x \begin{bmatrix} (a^\sigma+a+1)\,c, & (m_1+a^\sigma(a^\sigma+1))\,c^{\sigma+1} \\ c^{1-\sigma} & , \quad (a+a^\sigma)\,c \end{bmatrix}.$$
$$y = x \begin{bmatrix} (a^\sigma+a)\,c, & (m_0+a^\sigma(a^\sigma+1))\,c^{\sigma+1} \\ c^{1-\sigma} & , \quad (a+a^\sigma+1)\,c \end{bmatrix}$$

for all $u, a, c \in K$, $c \neq 0$. Also, the fixed field of $\sigma = GF(2)$, $q=2^r$, and r is odd.

Proof. It remains to prove that r is odd. We see that $\begin{bmatrix} 0 & m_0 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & m_1 \\ 1 & 1 \end{bmatrix}$ are in distinct H-orbits. Hence $\begin{bmatrix} 0 & m_1 \\ 1 & 1 \end{bmatrix} \xrightarrow{\tau_a} \begin{bmatrix} a+a^\sigma, & m_0+a^\sigma(a^\sigma+1) \\ 1 & , \quad a+a^\sigma+1 \end{bmatrix}$, so that $a+a^\sigma \neq 1$. Suppose $b^2=b+1$ for some $b \in K$. Then if $q=2^r$, assume r even, and $\sigma=2^s$ for s odd. Then $(b^2)^{\sigma/2} = b^{\sigma/2}+1 = b+1 \Leftrightarrow b^{\sigma/2}=b \Leftrightarrow b^\sigma = b^2 \Leftrightarrow b^{\sigma-2}=1 \Leftrightarrow$ (for $b \neq 1$), $\left[\frac{\sigma-2}{2}, 2^r-1\right] \neq 1$. Since $\frac{\sigma-2}{2} = 2^{s-1}-1$ and $(s-1, r) = 2 \cdot t$, we have that $GF(4)$ cannot be a subfield of $GF(q)$. That is, r is odd.

NOTES 3.5. In the Ott-Schaeffer planes $m_0=m_1=1$. Here, at least it is possible that there are other translation planes distinct from the O-S planes and admitting the same group of order $q(q-1)$ that the O-S planes admit.

We now further consider generalized Ott-Schaeffer planes π of order q^2 and kernel $K \cong GF(q)$. We may use the representation given in (2.7). Assume there is a linear collineation group H such that $HK^*/K^* \cong H(K^* = K - \{0\})$ and $|H| = q(q-1)$. Note that we use the notation HK^* to refer to the product of H by the kernel homology group of order $q-1$.

Lemma 3.6. *A Sylow 2-subgroup $S_2 \trianglelefteq H$ or π is Ott-Schaeffer.*

Proof. Let $\mathcal{S}_2$ fix the component $\mathcal{L}$. Since the involutions of H (see (2.7)) are all Baer, it follows from Johnson-Ostrom [8] (3.27) that if π is not Ott-Schaeffer then the group R generated by the Sylow 2-subgroups of H is reducible and solvable and by the argument to [8] (3.27), R must be a 2-group. That is, $\mathcal{S}_2 \trianglelefteq H$.

Lemma 3.7. *$H=\mathcal{S}_2 \cdot C$ where C is a 2-complement of $\mathcal{S}_2$. Then C fixes two components.*

Proof. Clearly, $\mathcal{S}_2$ is a Hall normal subgroup so let C be a 2-complement of order $q-1$.

C fixes $\mathcal{L}$ and by Maschke's Theorem, decompose $\pi=\mathcal{L}\oplus\mathcal{W}$ where $\mathcal{W}$ is a C-invariant 2-space. Either $\mathcal{W}$ is a component and (3.7) is finished or $\mathcal{W}$ is a C-invariant Baer subplane. Further, $\mathcal{W}$ is Desarguesian and $C\,|\,\mathcal{W} \leq GL(2, q)$ acting on $\mathcal{W}$. Hence, C must fix two components of $\mathcal{W}$ which are 1-spaces of π. Hence, C fixes the components of π which contain the C-invariant components of $\mathcal{W}$.

Lemma 3.8. *H acts faithfully on $\mathcal{L}$.*

Proof. If $h \in H$ fixes $\mathcal{L}$ pointwise then h is a homology and by the orbit structure of π (see section 2), it must be that the coaxis of h is moved by $\mathcal{S}_2$. That is, there must be elations in H by André [1]. Hence, we have the proof to (3.8).

Lemma 3.9. *C acts regularly on $\mathcal{S}_2-\langle 1\rangle$ by conjugation.*

Proof. Represent $\mathcal{S}_2$ as in (2.7), then let

$$h=\left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]\in C.$$

So $A, C \in N_{GL(2,q)}\left[\left\{\begin{bmatrix}1 & a\\ 0 & 1\end{bmatrix} \,\middle|\, a\in K\right\}\right]$ so that

$$A=\begin{bmatrix}a_1 & a_2\\ 0 & a_4\end{bmatrix}, \quad C=\begin{bmatrix}c_1 & c_2\\ 0 & c_4\end{bmatrix}.$$

Let $B=\begin{bmatrix}b_1 & b_2\\ b_3 & b_4\end{bmatrix}$. If $\tau_a^h=\tau_a$ for some $a\neq 0$ (see (2.7)) then clearly $a_1=a_4$, $c_1=c_4$ and so $\tau_b^h=\tau_b\ \forall\ b\neq 0$. This implies

$$[A+C]\begin{bmatrix}f(b) & g(b)\\ 0 & f(b)\end{bmatrix}=B\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}+\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}B=\begin{bmatrix}bb_3, & b(b_1+b_4)\\ 0, & bb_3\end{bmatrix}$$

for all $b\neq 0$.

Hence

$$\begin{bmatrix} a_1+c_1, & a_2+c_2 \\ 0 \;, & a_4+c_4 \end{bmatrix} \begin{bmatrix} f(b) & g(b) \\ 0 & f(b) \end{bmatrix} = \begin{bmatrix} bb_3 & b(b_1+b_4) \\ 0 & bb_3 \end{bmatrix}.$$

So $(a_2+c_2)f(b)+(a_1+c_1)g(b)=b(b_1+b_4)$. That is, if $a_1 \neq c_1$ then $g(b)=bK_1+f(b)K_2$ for

$$K_1 = \left[\frac{b_1+b_4}{a_1+c_1}\right] \quad \text{and} \quad K_2 = \left[\frac{a_2+c_2}{a_1+c_1}\right].$$

Thus, if $a_1 \neq c_1$ then g is additive.

However, $g(b+t)=g(b)+g(t)+bf(t)+tf(b)$ so that $bf(t)=tf(b)$ for all $b, t \in K$. Hence $f(t)=tf(1)$.

But the components include the $q(q-1)$ elements

$$y = x\begin{bmatrix} u \quad , & m(u, a) \\ a^{-1}f(a), & u+a^{-1}g(a) \end{bmatrix} \quad \text{for} \quad u, a \in K,\ a \neq 0$$

(see (2.7)) so that the (2, 1)-entries are always $a^{-1}af(1)=f(1)$. That is,

$$y = x\begin{bmatrix} u \quad , & m(u, a) \\ f(1), & u+a^{-1}g(a) \end{bmatrix}$$

for $f(1)$ a constant, represents $q(q-1)$ components which clearly is a contradiction

Hence, $a_1+c_1=0$ so that $a_1=a_4=c_1=c_4$ and we have

$$(a_2+c_2f(b)) = b(b_1+b_4)\,.$$

If $a_2+c_2 \neq 0$ then $f(b)=b \cdot \alpha$ for α some constant. Then we still have $y=x\begin{bmatrix} u \quad , & m(u, a) \\ f(1), & u+a^{-1}g(a) \end{bmatrix}$ to represent $q(q-1)$ components—a contradiction.

Thus, $a_2=c_2$ and $b_1=b_4$. So the element in C has the form $\begin{pmatrix} a_1 & a_2 & b_1 & b_2 \\ 0 & a_1 & 0 & b_1 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & 0 & a_1 \end{pmatrix}$. Multiplying by $a_1^{-1} I_4$ we obtain $\rho=\begin{pmatrix} 1 & a_2a_1^{-1} & b_1a_1^{-1} & b_2a_1^{-1} \\ 0 & 1 & 0 & b_1a_1^{-1} \\ 0 & 0 & 1 & a_2a_1^{-1} \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Let $a_2a_1^{-1}=e$. We have $\tau_e \in \mathcal{S}_2$, $\tau_e=\begin{pmatrix} 1 & e & f(e) & g(e) \\ 0 & 1 & 0 & f(e) \\ 0 & 0 & 1 & e \\ 0 & 0 & 0 & 1 \end{pmatrix}$. Since $\mathcal{S}_2 \trianglelefteq HK^*$ we must have that $\rho=\tau_e$. In other words, the element is in $\mathcal{S}_2K^*$. However, we assumed that $HK^*/K^* \cong H$. This proves (3.9).

Thus, C must fix one of the components in the orbit of $\mathcal{S}_2$ of length q. That is, C must fix a component of the form $y=x\begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}$ for some $u \in K$.

So, we may choose a 2-complement C for S_2 so that C fixes $y=0$.

(3.10). Thus, the elements of C have the form

$$\left(\begin{array}{c|c} \begin{bmatrix} a_1 & a_3 \\ 0 & a_4 \end{bmatrix} & 0 \\ \hline 0 & \begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \end{array}\right).$$

$$\left[x, x\begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}\right] \to \left[x\begin{bmatrix} a_1 & a_2 \\ 0 & a_4 \end{bmatrix}, x\begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix}\right]$$

so that

$$\begin{aligned} &\begin{bmatrix} a_1^{-1}, & a_2a_1^{-1}a_4^{-1} \\ 0\,, & a_4^{-1} \end{bmatrix}\begin{bmatrix} u & h(u) \\ 0 & u \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \\ &= \begin{bmatrix} ua_1^{-1}, & a_1^{-1}h(u)+a_2a_1^{-1}a_4^{-1}u \\ 0\,, & ua_4^{-1} \end{bmatrix}\begin{bmatrix} c_1 & c_2 \\ 0 & c_4 \end{bmatrix} \\ &= \begin{bmatrix} ua_1^{-1}c_1, & ua_1^{-1}c_2+(a_1^{-1}h(u)+a_2a_1^{-1}a_4^{-1}u)\,c_4 \\ 0\,, & ua_4^{-1}c_4 \end{bmatrix} \in \left\{\begin{bmatrix} u & f(u) \\ 0 & u \end{bmatrix} \,\middle|\, u\in K\right\}. \end{aligned}$$

(3.11). So $a_1^{-1}c_1=a_4^{-1}c_4$.

Since $HK^*/K^*\cong H$, then $CK^*/K^*\cong C \Leftrightarrow C\cap K^*=\{1\}$. So multiplying a typical element by $\begin{bmatrix} a_1^{-1} & & & \\ & a_1^{-1} & & \\ & & a_1^{-1} & \\ & & & a_1^{-1} \end{bmatrix}$ then in CK^*, there are $q-1$ elements of the general form $\begin{bmatrix} 1 & a_2 & & \\ 0 & a_4 & & \\ & & c & c_2 \\ & & 0 & a_4c \end{bmatrix}$. Assume two such elements have equal (2,2)-entries. Consider

$$\rho = \begin{bmatrix} 1 & \bar{a}_2 & & \\ 0 & \bar{a}_4 & & \\ & & \bar{c} & \bar{c}_2 \\ & & 0 & \bar{a}_4\bar{c} \end{bmatrix} \quad\text{and}\quad \chi = \begin{bmatrix} 1 & a_2 & & \\ 0 & a_4 & & \\ & & c & c_2 \\ & & 0 & a_4c \end{bmatrix}$$

when $\bar{a}_4=a_4$. Then the product

$$\rho\chi^{-1} = \left(\begin{array}{c|c} \begin{bmatrix} 1 & d \\ 0 & 1 \end{bmatrix} & \\ \hline & \end{array}\right)$$

where $d=\bar{a}_2a_4^{-1}+a_2a_4^{-1}$ and $(\rho\chi^{-1})^2$ must fix $x=0$ pointwise. By (3.8), $(\rho\chi^{-1})^2=$

1. However, $|CK^*|\,|(q-1)^2$, so $\rho\chi^{-1}=1\Leftrightarrow\rho=\chi$. Thus, there must exist a collineation $\rho=\begin{bmatrix}1 & a_2 & & \\ 0 & a & & \\ & & c & c_2 \\ & & 0 & ac\end{bmatrix}$ where $|a|=q-1$.

Hence, since $\mathcal{S}_2\trianglelefteq HK^*$, we obtain

$$\begin{bmatrix}A^{-1} & \\ & C^{-1}\end{bmatrix}\begin{bmatrix}1 & b & f(b) & g(b) \\ 0 & 1 & 0 & f(b) \\ & & 1 & b \\ & & 0 & 1\end{bmatrix}\begin{bmatrix}A & \\ & C\end{bmatrix}$$
$$=\begin{bmatrix}A^{-1}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}A & A^{-1}\begin{bmatrix}f(b) & g(b)\\ 0 & f(b)\end{bmatrix}C \\ 0 & C^{-1}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}C\end{bmatrix}\in\mathcal{S}_2$$

where $A=\begin{bmatrix}1 & a_2\\ 0 & a\end{bmatrix}$, $C=\begin{bmatrix}c & c_2\\ 0 & ac\end{bmatrix}$ so that

$$A^{-1}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}A=\begin{bmatrix}1 & a_2a^{-1}\\ 0 & a^{-1}\end{bmatrix}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & a_2\\ 0 & a\end{bmatrix}=\begin{bmatrix}1, & ba\\ 0, & 1\end{bmatrix}=C^{-1}\begin{bmatrix}1 & b\\ 0 & 1\end{bmatrix}C$$

and

$$A^{-1}\begin{bmatrix}f(b) & g(b)\\ 0 & f(b)\end{bmatrix}C=\begin{bmatrix}1 & a_2a^{-1}\\ 0 & a^{-1}\end{bmatrix}\begin{bmatrix}f(b) & g(b)\\ 0 & f(b)\end{bmatrix}\begin{bmatrix}c & c_2\\ 0 & ac\end{bmatrix}$$
$$=\begin{bmatrix}cf(b), & f(b)\,(c_2+a_2c)+g(b)\,ac\\ 0\ , & cf(b)\end{bmatrix}.$$

Since $|a|=q-1$, $\exists\, j\ni a^j=c$, so that we obtain the elements

$$\begin{bmatrix}1, & ab & a^jf(b), & f(b)\,(c_2+a_2a^j)+g(b)\,a^{j+1} \\ 0, & 1 & 0\ , & a^jf(b) \\ & & 1 & ab \\ & & 0 & 1\end{bmatrix}$$

in $\mathcal{S}_2$. Hence,

(3.12) $f(ab)=a^jf(b)$ for all b in K^* and

(3.13) $g(ab)=f(b)\,(c_2+a_2a^j)+g(b)\,a^{j+1}$ for all b in K^*.

Also,

$$\rho^i=\left(\begin{array}{c|c}\begin{matrix}1 & a_2\left[\dfrac{a^i-1}{a-1}\right]\\ 0 & a^i\end{matrix} & \\ \hline & \begin{matrix}a^{ij} & c_2a^{(i-1)j}\left[\dfrac{a^i-1}{a-1}\right]\\ & a^{i(j+1)}\end{matrix}\end{array}\right).$$

The previous argument applied to ρ^i implies

(3.14) $f(a^i b) = a^{ij} f(b)$ for all b in K^* and

(3.15) $g(a^i b) = f(b)\,(c_2 a^{(i-1)j}\left[\dfrac{a^i-1}{a-1}\right]+a_2 a^{ij}\left[\dfrac{a^i-1}{a-1}\right]\Big]+g(b)\,a^{i(j+1)}$

for all $i\geq 1$, b in K.

Let $b=1$ in (3.14) and $a^i=c$ to obtain $f(c)=c^j f(1)$ and further

(3.16) $f(c)=c^\tau f$ where $c^j=c^\tau$ and $\tau\in$ Aut K, $f=f(1)$.

Pf. f is additive and i is arbitrary as $|a|=q-1$.
From (3.10),

$$y = x\begin{bmatrix} f(b), & bf(b)+g(b) \\ 0\ , & f(b) \end{bmatrix}$$

$$\xrightarrow{\rho} y = x\begin{bmatrix} f(b)a^\tau, & \{f(b)\,(c_2+a_2a^\tau)+f(b)\,ba^{\tau+1}+g(b)\,a^{\tau+1}\} \\ 0\ , & f(b)\,a^\tau \end{bmatrix}.$$

Using (3.16), $f(b)a^\tau=f(ab)$ so that the (2,2) entry of the preceding matrix is $df(d)+g(d)$ for $d=ab$ since $(ab)f(ab)=f(b)ba^{\tau+1}$, we obtain

(3.17) $g(ab) = g(b)\,a^{\tau+1}+b^\tau f(1)\,(c_2+a_2a^\tau)$ for all b in K^*.

Let $b=1$ in (3.15) to obtain:

(3.18) $g(a^i) = g(1)\,a^{i(\tau+1)}+f(1)\,(c_2+a_2a^\tau)\left[a^{(i-1)\tau}\left[\dfrac{a^i-1}{a-1}\right]\right]$ all $i\geq 1$ in Z.

Since $|a|=q-1$, we obtain

$$g(c)=g(1)\,c^{\tau+1}+f(1)\,\frac{(c_2+a_2a^\tau)}{(a-1)}\,a^{-\tau}(c^\tau(c-1))$$

for all $c\neq 0$. Let $s=f(1)\,\dfrac{(c_2+a_2a^\tau)}{(a-1)}\,a^{-\tau}$, $g(1)=g$, $f(1)=f$ to obtain

(3.19) $g(c) = c^{\tau+1}(s+g)+c^\tau s$

for all $c\neq 0$ in K.

Recall $f(d)=d^\tau f$ for all d and $g(e+d)=g(e)+g(d)+ef(d)+df(e)$ for all e, d in K. Thus,

$$\begin{aligned} g(e+d) &= (e+d)^{\tau+1}(s+g)+(e+d)^\tau s \\ &= (e^{\tau+1}(s+g)+e^\tau s)+(d^{\tau+1}(s+g)+d^\tau s)+ed^\tau f+de^\tau f \end{aligned}$$

so that

(3.20) $(ed^{\tau}+ed^{\tau})(s+g+f)=0$

so that either $\tau=1$ or $s+g+f=0$. If $\tau=1$ then from (2.7), the components have the form

$$\left[y=x\begin{bmatrix}u & h(u)\\ 0 & u\end{bmatrix}\right],\ \left[y=x\begin{bmatrix}u\ , & m(u,d)\\ d^{-1}f(d), & u+d^{-1}g(d)\end{bmatrix}\right]$$
$$\equiv\left[y=x\begin{bmatrix}u, & m(u,d)\\ f, & u+d^{-1}g(d)\end{bmatrix}\right].$$

But, the latter set for all $u, d\neq 0$ represents $q(q-1)$ components. For $d_1\neq d_2$

$$\begin{bmatrix}u, & m(u,d_1)\\ f, & u+d^{-1}g(d_1)\end{bmatrix}-\begin{bmatrix}u, & m(u,d_2)\\ f, & u+d^{-1}g(d_2)\end{bmatrix}$$

is singular so $\tau\neq 1$.

Change bases by

$$\chi=\left(\begin{array}{cc|cc}1 & t & & \\ 0 & 1 & \multicolumn{2}{c}{0}\\ \hline & & 1 & t\\ \multicolumn{2}{c|}{0} & 0 & 1\end{array}\right)$$

where $t=\dfrac{a_2}{1+a}$.

Recall

$$\tau_d=\begin{pmatrix}1 & d & f(d) & g(d)\\ 0 & 1 & 0 & f(d)\\ & & 1 & d\\ \multicolumn{2}{c}{0} & 0 & 1\end{pmatrix}$$

(from section 2) and χ commutes with τ_d. Recall

$$\rho=\left(\begin{array}{cc|cc}1 & a_2 & & \\ 0 & 1 & & \\ \hline & & a^{\tau} & c_2\\ & & 0 & a^{\tau+1}\end{array}\right)$$

so that

$$\bar{\rho}=\chi\rho\chi^{-1}=\left(\begin{array}{cc|cc}1 & 0 & & \\ 0 & a & & \\ \hline & & a^{\tau} & \dfrac{a^{\tau}(a+1)s}{f}\\ \multicolumn{2}{c|}{0} & 0 & a^{\tau+1}\end{array}\right)$$

as

$$\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}1 & a_2\\ 0 & a\end{bmatrix}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}=\begin{bmatrix}1 & t+a_2+ta\\ 0 & a\end{bmatrix}$$

when $t=\dfrac{a_2}{1+a}$ so that $t+a_2+ta=\dfrac{a_2}{1+a}+a_2+\dfrac{a_2}{1+a}\,a=0$ and

$$\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}\begin{bmatrix}a^\tau & c_2\\ 0 & a^{\tau+1}\end{bmatrix}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}=\begin{bmatrix}a^\tau & a^\tau t+c_2+ta^{\tau+1}\\ 0 & a^{\tau+1}\end{bmatrix}$$

where

$$a^\tau t+c_2+ta^{\tau+1}=a^\tau\frac{a_2}{1+a}+\frac{a^\tau(a+1)}{f}\,s+a_2a^\tau+\frac{a_2}{1+a}\,a^{\tau+1}=\frac{a^\tau(a+1)\,s}{f}$$

as

$$s=f\,\frac{(c_2+a_2a^\tau)\,a^{-\tau}}{(1+a)}\,.$$

We originally had the components in the form

$$y=x\begin{bmatrix}f(b) & g(b)+bf(b)\\ 0 & f(b)\end{bmatrix}$$

and

$$y=x\begin{bmatrix}u\ , & m(u,c)\\ c^{-1}f(c), & u+c^{-1}g(c)\end{bmatrix}.$$

Apply χ, the forms become:

$$y=x\begin{bmatrix}f(b) & g(b)+bf(b)\\ 0 & f(b)\end{bmatrix}$$

and

$$\left[y=x\begin{bmatrix}u+c^{\tau-1}f\cdot t, & \overline{}\\ c^{\tau-1}f\ , & u+c^{\tau-1}f\cdot t+c^{-1}g(c)\end{bmatrix}\right]$$
$$\equiv\left[y=x\begin{bmatrix}v\ , & \overline{}\\ c^{\tau-1}f, & v+c^{-1}g(c)\end{bmatrix}\right]$$

for $v=u+c^{\tau-1}f\cdot t$. In other words the form is invariant under χ.

Now $g(b)+bf(b)=(b^{\tau+1}(s+g)+b^\tau s+b^{\tau+1}f)$ by (3.19). By (3.20) $s+g+f=0$ so that $g(b)+bf(b)=b^\tau s$.

Change bases by

$$\gamma=\begin{bmatrix}I & 0\\ 0 & \begin{bmatrix}f^{-1} & sf^{-2}\\ 0 & f^{-1}\end{bmatrix}\end{bmatrix}$$

so that

$$y = x\begin{bmatrix} b^\tau f, & b^\tau s \\ 0\ , & b^\tau f \end{bmatrix} \xrightarrow{\gamma} y = x\begin{bmatrix} b^\tau & 0 \\ 0 & b^\tau \end{bmatrix},$$

$$\left[y = x\begin{bmatrix} v\ , & \overline{\qquad} \\ c^{\tau-1}f, & v+c^{-1}g(c) \end{bmatrix}\right] = \left[y = x\begin{bmatrix} v\ , & \overline{\qquad} \\ c^{\tau-1}f, & v+c^\tau(s+g)+c^{\tau-1}s \end{bmatrix}\right]$$

$$\xrightarrow{\gamma} y = x\begin{bmatrix} vf^{-1}, & \overline{\qquad} \\ c^{\tau-1}f, & vf^{-1}+c^\tau \end{bmatrix}$$

since $s+g+f=0$.

Furthermore,

$$\gamma^{-1}\bar{p}\gamma = \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & a \end{matrix} & \\ \hline & \begin{bmatrix} 1 & sf^{-1} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a^\tau, & a^\tau(a+1)\dfrac{s}{f} \\ 0\ , & a^{\tau+1} \end{bmatrix}\begin{bmatrix} 1 & sf^{-1} \\ 0 & 1 \end{bmatrix} \end{array}\right)$$

$$= \left(\begin{array}{c|c} \begin{matrix} 1 & 0 \\ 0 & a \end{matrix} & \\ \hline & \begin{matrix} a^\tau & 0 \\ 0 & a^{\tau+1} \end{matrix} \end{array}\right)$$

and

$$\gamma^{-1}\tau_d\gamma = \left(\begin{array}{c|c} \begin{matrix} 1 & d \\ 0 & 1 \end{matrix} & \begin{bmatrix} f(d) & g(d) \\ 0 & f(d) \end{bmatrix}\begin{bmatrix} f^{-1}, & sf^{-2} \\ 0\ , & f^{-1} \end{bmatrix} \\ \hline & \begin{matrix} 1 & d \\ 0 & 1 \end{matrix} \end{array}\right)$$

and

$$\begin{bmatrix} f(d) & g(d) \\ 0 & f(d) \end{bmatrix}\begin{bmatrix} f^{-1} & sf^{-2} \\ 0 & f^{-1} \end{bmatrix} = \begin{bmatrix} d^\tau f & d^{\tau+1}(s+g)+d^\tau s \\ 0 & d^\tau f \end{bmatrix}\begin{bmatrix} f^{-1} & sf^{-2} \\ 0 & f^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} d^\tau & d^{\tau+1}\dfrac{(s+g)}{f} \\ 0 & d^\tau \end{bmatrix} = \begin{bmatrix} d^\tau & d^{\tau+1} \\ 0 & d^\tau \end{bmatrix}.$$

Thus

$$\gamma^{-1}\tau_d\gamma = \begin{pmatrix} 1 & d & d^\tau & d^{\tau+1} \\ 0 & 1 & 0 & d^\tau \\ & & 1 & d \\ \multicolumn{2}{c}{0} & 0 & 1 \end{pmatrix}.$$

Now let $\tau=\sigma^{-1}$. Then

$$\begin{pmatrix} 1 & b & b^\tau & b^{\tau+1} \\ 0 & 1 & 0 & b^\tau \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a^\sigma & a & a^{\sigma+1} \\ 0 & 1 & 0 & a \\ & & 1 & a^\sigma \\ & & 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & a^\sigma \\ 0 & 1 \end{bmatrix}$$

for $b=a^{\sigma}$ and

$$\begin{pmatrix} 1 & 0 & & \\ 0 & a & & \\ & & a^{\tau} & 0 \\ & & 0 & a^{\tau+1} \end{pmatrix} = \begin{pmatrix} 1 & 0 & & \\ 0 & d^{-2\sigma} & & \\ & & d^{-2} & 0 \\ & & 0 & d^{-2-2\sigma} \end{pmatrix}$$

for $a=d^{-2\sigma}$. Multiplying by

$$\begin{bmatrix} d^{\sigma+1} & & & \\ & d^{\sigma+1} & & \\ & & d^{\sigma+1} & \\ & & & d^{\sigma+1} \end{bmatrix}$$

of the kernel homology group, we obtain

$$\begin{bmatrix} d^{\sigma+1} & & & \\ & d^{1-\sigma} & & \\ & & d^{\sigma-1} & \\ & & & d^{-\sigma-1} \end{bmatrix} = \begin{bmatrix} d & \\ & d^{-1} \end{bmatrix} \otimes \begin{bmatrix} d^{\sigma} & \\ & d^{-\sigma} \end{bmatrix}.$$

Hence, we may apply the results on tensor product planes with groups of order $q(q-1)$.

Thus we obtain:

Theorem 3.21. *Let π be a translation plane of order q^2 and kernel $K\cong GF(q)$, q even. Let H be a group in the linear translation complement of order $q(q-1)$ and $HK^*/K^*\cong H$. Further, if S_2 is a Sylow 2-subgroup of H, assume the involutions are Baer and no two involutions fix the same subplane pointwise.*

Then π is a tensor product plane and the spread is completely determined. A matrix spread set may be represented as follows:

$$x=0,\ y=x\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix},$$

$$y=x\begin{bmatrix} (a^{\sigma}+a+1)\,c, & (m_1+a^{\sigma}(a^{\sigma}+1))\,c^{\sigma+1} \\ c^{1-\sigma} \quad, & (a+a^{\sigma})\,c \end{bmatrix},$$

$$y=x\begin{bmatrix} (a^{\sigma}+a)\,c, & (m_0+a^{\sigma}(a^{\sigma}+1))\,c^{\sigma+1} \\ c^{1-\sigma} \quad, & (a+a^{\sigma}+1)\,c \end{bmatrix}$$

for all $u, a, c\neq 0$ of K, m_0, m_1 constants in K and $\sigma\in Aut\ K$ such that the fixed field of σ is $GF(2)$ and $q=2^{2r+1}$.

NOTES 3.22. Several authors, [2], [3], [5], have recently studied translation planes of order q^2 that admit H groups of order $q(q-1)$ where H is an autotopism group. In this situation, there are many different classes of nonisomorphic translation planes. So, we see that the assumption on the nature of the Sylow p-subgroups for $p^r=q$ is crucial in (3.21).

(3.23) Open Problems and Related Questions.

1) (a) What are the Tensor Product Planes?
   (b) Are there nontrivial generalized Ott-Schaeffer Planes?
2) Study translation planes of order q^2, $p^r=q$ kernel $GF(q)$ that admit linear collineation groups of order $q(q-1)$: The Sylow p-subgroups are
   (a) planar.
   (b) quartic.
   (a) 1) if planar the group is an autotropism group.
   (a) 2) no two p-elements fix the same Baer subplane pointwise.

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