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AUTOMORPHISM GROUPS OF MULTILINEAR MAPS

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1. Introduction

In [2], R. Griess gave a beautiful construction of the Monster Simple Group. Namely, he constructed a 196883 dimensional (nonassociative) commutative algebra over the rational numbers and showed that a certain subgroup of the full automorphism group of this algebra is isomorphic to the Monster Simple Group.

Recently, Harada considered the automorphism group of a vector space possessing a (nonassociative) commutative algebra structure, which is closely related to a natural permutation representation of a multiply transitive group. ([3], also see Example 3.)

These two results suggest us to investigate the automorphism group of a general (nonassociative) algebra. It would be very interesting if such study would be useful in the theory of finite groups.

A (nonassociative) commutative algebra is nothing but a vector space having a binary operation which is bilinear and symmetric. Let V be a vector space over an algebraically closed field F and θ: $V\times V\rightarrow V$ a bilinear map (not necessarily symmetric). Set $\mathrm{Aut}\,\theta=\{g\in GL(V)\colon \theta(u^g, v^g)=\theta(u, v)^g$ for all $u, v\in V\}$. We are interested in the following three questions on $\mathrm{Aut}\,\theta$:

(1) When does $\mathrm{Aut}\,\theta$ become finite?

(2) Which finite group can be represented as $\mathrm{Aut}\,\theta$ for a suitable θ?

(3) What can we say about the order of $\mathrm{Aut}\,\theta$ for a given θ?

In this paper we shall give a result on (1). Namely, we shall show the finiteness of $\mathrm{Aut}\,\theta$ under the condition that θ satisfies a nonsingularity condition with a restriction on the characteristic of the field F. (See Theorem A.) However, sometimes it is not easy to show the nonsingularity of θ. For example, we could check the nonsingularity of the algebra Harada defined but we do not know whether the algebra defined by Griess is nonsingular or not.

The above criterion for the finiteness of $\mathrm{Aut}\,\theta$ will actually be proved for a more general θ, i.e., for a (not necessarily symmetric) multilinear map from $V\times \overset{r}{\cdots} \times V$ to V of degree $r\geqq 2$.

As a corollary, we shall also prove a result on a multilinear form θ: $V\times$

$\cdots \times V \to F$, where V is a vector space over an algebraically closed field F. (See Theorem B.)

There are several known results of this type when θ is a symmetric multilinear form [4], [5], [6] and [7]. The first one uses a lot of inequalities and is very difficult to read (at least to the author), the second and the third use theorems in algebraic geometry and our proof was motivated by the last paper which is very elementary. But we have to note that the proof in the last paper contains a serious error. (See Example 2.)

Finally we note that throughout this paper we shall frequently encounter the two kinds of maps:

(1) $\theta: V \times \cdots \times V \to V$.

(2) $\theta: V \times \cdots \times V \to F$.

The θ of type (1) will always be multilinear and called a "multilinear map", the θ of type (2) will be called a "Hermitian, (see Definition 1) or multilinear form".

2. Definition and theorem

We shall give some definitions, which we need to state our results.

Definition 1. Let V be a vector space over the complex number field C. A map $\theta: V \times \overset{r}{\cdots} \times V \to C$ is said to be a multilinear Hermitian form of degree r if it satisfies the following:

(1) θ is multilinear on the first $r-1$ terms.

(2) $\theta(v_1, \cdots, v_{r-1}, \lambda u + \mu w) = \bar{\lambda}\theta(v_1, \cdots, v_{r-1}, u) + \bar{\mu}\theta(v_1, \cdots, v_{r-1}, w)$,

where bars denote the complex conjugates.

Definition 2. Let $\theta: V \times \overset{r}{\cdots} \times V \to F$ be a form (i.e., a multilinear form or a multilinear Hermitian form) on a vector space over a field F. Then

$$\operatorname{Aut}\theta = \{g \in GL(V): \theta(v_1, \cdots, v_r) = \theta(v_1^g, \cdots, v_r^g) \quad \text{for all } v_1, \cdots, v_r \in V\}.$$

Definition 3. Let $\theta: V \times \overset{r}{\cdots} \times V \to V$ be a multilinear map on a vector space V. Then

$$\operatorname{Aut}\theta = \{g \in GL(V): \theta(v_1, \cdots, v_r)^g = \theta(v_1^g, \cdots, v_r^g) \quad \text{for all } v_1, \cdots, v_r \in V\}.$$

Now we state our main results.

Theorem A. *Let θ denote a multilinear map of degree $r \geq 2$ on a vector space V of dimension n over an algebraically closed field F of characteristic zero or greater than n. Then one of the following holds:*

(i) *there exists an element* $v \in V^{\sharp} = V - \{0\}$ *such that* $\theta(v, \cdots, v) = 0$; *or*
(ii) $\mathrm{Aut}\,\theta$ *is a finite group.*

Theorem B. *Let* θ *denote a multilinear form of degree* $r \geqq 3$ *on a vector space* V *of dimension* n *over an algebraically closed field* F *of characteristic zero or greater than* n. *Then one of the following holds*:

(i) *there exists an element* $v \in V^{\sharp}$ *such that* $\theta(v, \cdots, v, w) = 0$ *for all* $w \in V$; *or*

(ii) $\mathrm{Aut}\,\theta$ *is a finite group.*

Theorem C. *Let* θ *denote a multilinear Hermitian form of degree* $r \geqq 3$ *on a vector space* V *of dimension* n *over the complex number field. Then one of the following holds*:

(i) *there exists an element* $v \in V^{\sharp}$ *such that* $\theta(v, \cdots, v, w) = 0$ *for all* $w \in V$; *or*

(ii) $\mathrm{Aut}\,\theta$ *is a finite group.*

3. Theorem of Tate and Bott

In this section, we shall prove a key to our results which is related to a theorem of Tate and Bott. (See [7])

Proposition 1. *Let* θ *be a multilinear map of degree* $r \geqq 2$ *on a vector space* V *of dimension* n *over an algebraically closed field* F. *Suppose that* $\theta(v, \cdots, v) \neq 0$ *for all* $v \in V^{\sharp}$. *Then for every element* $\sigma \in \mathrm{Aut}\,\theta$, σ^m *is unipotent for some* m *at most* $(r^n - 1)^n$.

Proposition 2. *Let* θ *be a multilinear form of degree* $r \geqq 3$ *on a vector space of dimension* n *over an algebraically closed field* F. *Suppose that* $\theta(v, \cdots, v, w) = 0$ *for all* $w \in V$ *implies* $v = 0$. *Then for every element* $\sigma \in \mathrm{Aut}\,\theta$, σ^m *is unipotent for some* m *at most* $((r-1)^{2n} - 1)^n$.

Proposition 3. *Let* θ *be a multilinear Hermitian form of degree* $r \geqq 3$ *on a vector space* V *of dimension* n *over the complex number field. Suppose that* $\theta(v, \cdots, v, w) = 0$ *for all* $w \in V$ *implies* $v = 0$. *Then for every element* $\sigma \in \mathrm{Aut}\,\theta$, σ^m *is unipotent for some* m *at most* $((r-1)^{2n} - 1)^n$.

The proofs of these propositions are very similar and Tate and Bott first proved such kind of result in the form of Proposition 2. (See [7].) Hence in our paper we only give a proof of Proposition 1.

Proof of Proposition 1. It suffices to show that each eigenvalue of σ is a root of unity of order at most $(r^n - 1)$. Let λ be an eigenvalue of σ and v a corresponding eigenvector. Since σ is invertible, $\lambda \neq 0$. Define $u \in V$ by

$u=\theta(v, \cdots, v)$. By our hypothesis $u\neq 0$ and

$$\lambda^r u = \theta(\lambda v, \cdots, \lambda v) = \theta(v^\sigma, \cdots, v^\sigma) = u^\sigma .$$

So λ^r is also an eigenvalue of σ. Letting $\phi(\lambda)=\lambda^r$, we have $\phi^a(\lambda)=\phi^b(\lambda)$ for some $0\leqq a<b\leqq n$, as $\phi^b(\lambda)$ is an eigenvalue of σ. But now $|r^a-r^b|\leqq r^n-1$. Since $\lambda^{r^a-r^b}=1$, λ is a root of unity of order at most r^n-1.

4. Unipotent automorphisms

We study a multilinear map with a unipotent automorphism.

Proposition 4. *Let θ be a multilinear map of degree $r\geqq 2$ on a vector space V of dimension n over a field F. Let $\sigma\neq 1$ be a unipotent automorphism of θ. Assume that $(\sigma-1)^b\neq 0$ but $(\sigma-1)^{b+1}=0$. Then for any elements $v_1, v_2, \cdots, v_r$ in the image of $(\sigma-1)^b$, we have $\theta(v_1, v_2, \cdots, v_r)=0$, if the characteristic of F is zero or greater than $b+1$.*

Proof. Choose a basis $\{e(1), \cdots, e(n)\}$ for V so that the matrix $A=(a_{ij})$ of σ is in (lower triangular) Jordan normal form. Let $b_1, \cdots, b_1$ be the sizes of the blocks of A with $b_1= \cdots =b_d>b_{d+1}\geqq \cdots \geqq b_t\geqq 1$. Let $c_1=1$ and $c_{i+1}=c_i+b_i$ for $1\leqq i\leqq t-1$.

Note that $b+1=b_1$ and the image of $(\sigma-1)^b$ is spanned by $\{e(c_i)\colon 1\leqq i\leqq d\}$. Since θ is multilinear, it suffices to show that

$$\theta(e(h_1), e(h_2), \cdots, e(h_r)) = 0$$
$$\text{for } h_1, h_2, \cdots, h_r \text{ in } \{c_i\colon 1\leqq i\leqq d\} .$$

Now we fix $h_3, \cdots, h_r$ $\{c_i\colon 1\leqq i\leqq d\}$ and define

$$\theta^*(x, y) = \theta(x, y, e(h_3), \cdots, e(h_r)) .$$

Since $h_3, \cdots, h_r$ are fixed but arbitrary, it suffices to show that a bilinear map (i.e., a multilinear map of degree 2) θ^* has the property

$$\theta^*(e(h_1), e(h_2)) = 0 \quad \text{for } h_1, h_2\in \{c_i\colon 1\leqq i\leqq d\} .$$

Since A acts on $\{e(c_i)\colon 1\leqq i\leqq d\}$ trivially,

$$\theta^*(x^\sigma, y^\sigma) = \theta^*(x, y)^\sigma .$$

As A is in normal form, we have

$$\begin{aligned} a_{ij} &= 1 \quad \text{if } \ i=j \\ a_{ij} &= 1 \quad \text{if } \ i=j+1 \ \text{ unless } \ i\in \{c_k\colon 1\leqq k\leqq t\} \\ a_{ij} &= 0 \quad \text{otherwise.} \end{aligned}$$

$$A=\left[\begin{array}{cccccc}
\left.\begin{array}{|cc|}\hline 1 & \\ 11 & 0\\ 1 & \\ 0 & 11\\\hline\end{array}\right\}b_1 & & & \mathbf{0} & & \\
& \left.\begin{array}{|cc|}\hline 1 & \\ 11 & 0\\ 1 & \\ 0 & 11\\\hline\end{array}\right\}b_2 & & & & \\
& & \ddots & & & \\
& & & \left.\begin{array}{|cc|}\hline 1 & \\ 11 & 0\\ 1 & \\ 0 & 11\\\hline\end{array}\right\}b_d=b_1 & & \\
\mathbf{0} & & & & \left.\begin{array}{|cc|}\hline 1 & 0\\ 1 & \\ 011 & \\\hline\end{array}\right\}b_{d+1}<b_1 & \\
& & & & & 1\\
& & & & & 1\}
\end{array}\right]
\begin{array}{l}
\longleftarrow\cdots\cdots c_1\\
\longleftarrow\cdots\cdots c_2\\
\\
\longleftarrow\cdots\cdots c_d\\
\longleftarrow\cdots\cdots c_{d+1}\\
\\
\longleftarrow\cdots\cdots c_t\\
\quad b_t
\end{array}$$

To calculate the value $\theta^*(e(h_1), e(h_2))$, we prove several lemmas. Let $C=\{c_k\colon 1\leqq k\leqq t\}$.

Lemma 1. *The following holds*:

$$\Big(\sum_{s=1}^{n}\mu_s e(s)\Big)^{\sigma}=\sum_{s=1}^{n}\mu_s e(s)+\sum_{\substack{s=1\\ s+1\notin C}}^{n-1}\mu_{s+1}e(s)\,.$$

Proof. As $e(s)^{\sigma}=\begin{cases} e(s) & s\in C\\ e(s)+e(s-1) & \text{otherwise}\end{cases}$

we have the formula above.

Lemma 2. *There exist constants* λ_{ij}^{k}, $1\leqq i, j, k\leqq t$ *such that*

$$\theta^*(e(c_i), e(c_j))=\sum_{k=1}^{t}\lambda_{ij}^{k}e(c_k)\,.$$

Proof. Since $e(c_i)^{\sigma}=e(c_i)$, we have

$$\theta^*(e(c_i), e(c_j))=\theta^*(e(c_i)^{\sigma}, e(c_j)^{\sigma})\,.$$

So the assertion follows from Lemma 1.

Lemma 3. *Let $1 \leqq i, j \leqq d$. There exist constants λ_{ij}^{uk}, $1 \leqq k \leqq t$, $0 \leqq u \leqq m$, $0 \leqq m \leqq b_i - 1$ such that*

$$\begin{aligned} &\theta^*(e(c_i+m), e(c_j)) \\ &= \sum_{k=1}^{t} (\lambda_{ij}^{0k} e(c_k+m) + \cdots + \lambda_{ij}^{uk} e(c_k+m-u) + \cdots + \lambda_{ij}^{mk} e(c_k)) \end{aligned}$$

and that $\lambda_{ij}^{0k} = \lambda_{ij}^{k}$, $\lambda_{ij}^{uk} = 0$ if $b_k + u \leqq m$.

Proof. We prove the assertion by induction on m. Suppose that $m=0$. Then this is the formula in Lemma 2 and $\lambda_{ij}^{0k} = \lambda_{ij}^{k}$. Suppose Lemma 3 holds for $m-1$, $m \geqq 1$. Let

$$\theta^*(e(c_i+m), e(c_j)) = \sum_{s=1}^{n} \mu_s e(s).$$

and apply σ on the both sides. Since $1 \leqq m \leqq b_i - 1$,

$$e(c_i+m)^{\sigma} = e(c_i+m) + e(c_i+m-1)$$

and $e(c_j)^{\sigma} = e(c_j)$. So

$$\begin{aligned} &\theta^*(e(c_i+m), e(c_j))^{\sigma} \\ &= \theta^*(\epsilon(c_i+m)^{\sigma}, e(c_j)^{\sigma}) \\ &= \theta^*(e(c_i+m), e(c_j)) + \theta^*(e(c_i+m-1), e(c_j)) . \end{aligned}$$

Hence by Lemma 1,

$$\begin{aligned} &\theta^*(e(c_i+m), e(c_j))^{(\sigma-1)} \\ &= \theta^*(e(c_i+m-1), e(c_j)) \\ &= \sum_{\substack{s=1 \\ q+1 \notin C}}^{n-1} \mu_{s+1} e(s) . \end{aligned}$$

By induction's hypothesis, we have

$$\begin{aligned} &\sum_{\substack{s=1 \\ q+1 \notin C}}^{n-1} \mu_{s+1} e(s) \\ &= \sum_{k=1}^{t} (\lambda_{ij}^{0k} e(c_k+m-1) + \cdots + \lambda_{ij}^{uk} e(c_k+m-1-u) + \cdots + {}_{ij}^{m-1k} e(c_k)) \\ &\quad \lambda_{ij}^{0k} = \lambda_{ij}^{k}, \ \lambda_{ij}^{uk} = 0 \ \text{ if } \ b_k + u \leqq m-1 . \end{aligned}$$

Now it follows from the equation above that $\lambda_{ij}^{uk} = 0$ if $b_k + u \leqq m$, as we may assume that $b_k + u = m$ and that $\lambda_{ij}^{vk} = 0$ for $0 \leqq v < u$. Moreover, $\mu_{s+1} = 0$ if $c_k + m \leqq s \leqq c_k + b_k - 1$. Let $\mu_{c_k} = \lambda_{ij}^{mk}$. Then

$$\begin{aligned} &\theta^*(e(c_i+m), e(c_j)) \\ &= \sum_{k=1}^{t} (\lambda_{ij}^{0k} e(c_k+m) + \cdots + \lambda_{ij}^{uk} e(c_k+m-u) + \cdots + \lambda_{ij}^{mk} e(c_k)) \end{aligned}$$

holds for all $0\leqq m\leqq b_i-1$.

Lemma 3′. *Let* $1\leqq i, j\leqq d$. *There exist constants* λ'^{uk}_{ij}, $1\leqq k\leqq t$, $0\leqq u\leqq m$, $0\leqq m\leqq b_i-1$ *such that*

$$\begin{aligned}&\theta^*(e(c_i), e(c_j+m))\\&=\sum_{k=1}^{t}(\lambda'^{0k}_{ij}e(c_k+m)+\cdots+\lambda'^{uk}_{ij}e(c_k+m-u)+\cdots+\lambda^{mk}_{ij}e(c_k))\end{aligned}$$

and that $\lambda'^{0k}_{ij}=\lambda^k_{ij}$, $\lambda'^{uk}_{ij}=0$ *if* $b_k+u\leqq m$.

Proof. Let $\theta'(x, y)=\theta^*(y, x)$. Then θ' is a bilinear map invariant under the action of σ and

$$\begin{aligned}\theta'(e(c_j), e(c_i)) &= \theta^*(e(c_i), e(c_j))\\&=\sum_{k=1}^{t}\lambda^k_{ij}e(c_k)\,.\end{aligned}$$

Applying the last lemma to θ', we have the desired assertion.

Corollary 4. *For* $1\leqq i, j\leqq d$,

$$\theta^*(e(c_i), e(c_j))=\sum_{k=1}^{d}\lambda^k_{ij}e(c_k)\,.$$

Proof. The corollary follows from Lemma 2 and Lemma 3 with $m=b_k<b_i=b_1$ for $k>d$.

Lemma 5. *For* $1\leqq i, j, k\leqq d$, $0\leqq m\leqq b_i-2$, *let*

$$\theta^*(e(c_i+m), e(c_j+1))=\sum_{s=1}^{n}\mu^{ms}_{ij}e(s)\cdots(*)$$

Then the following hold:

(1) $\mu^{ms}_{ij}=0$ *if* $c_k+b_k-1\geqq s\geqq c_k+m+2$.

(2) $\mu^{mc_k+m+1}_{ij}=(m+1)\lambda^k_{ij}$.

Proof. We prove this lemma by induction on m. Let $m=0$. Then it corresponds to the case $m=1$ in Lemma 3′. So $\mu^{0s}_{ij}=0$ if $c_k+2\leqq s\leqq c_k+b_k-1$. Hence (1) holds. Since $\mu^{0c_k+1}_{ij}=\lambda'^{0k}_{ij}=\lambda^k_{ij}$ by Lemma 3′, we have (2). Suppose Lemma 5 holds for $m-1$, $1\leqq m\leqq b_i-1$. We apply σ on the equation (*). Since $m\geqq 1$,

$$\begin{aligned}&\theta^*(e(c_i+m), e(c_j+1))^{(\sigma-1)}\\&=\theta^*(e(c_i+m)^\sigma, e(c_j+1)^\sigma)-\theta^*(e(c_i+m), e(c_j+1))\\&=\theta^*(e(c_i+m), e(c_j))+\theta^*(e(c_i+m-1), e(c_j+1))\\&\quad+\theta^*(e(c_i+m-1), e(c_j))\,.\end{aligned}$$

We have already had the formulas for the first and the third term. So by Lemma 1, we have

$$\sum_{\substack{s=1\\ s+1\notin C}}^{n-1} \mu_{ij}^{m\,s+1} e(s)$$
$$= \sum_{k=1}^{t} \sum_{u=0}^{m} \lambda_{ij}^{uk} e(c_k+m-u) + \sum_{k=1}^{t} \sum_{u=0}^{m-1} \lambda_{ij}^{uk} e(c_k+m-1-u)$$
$$+\theta^*(e(c_i+m-1),\ e(c_j+1))\,.$$

Let $c_k+b_k-1 \geqq s \geqq c_k+m+2$. Computing the coefficients of $e(s-1)$, we can obtain

$$\mu_{is}^{mj} = 0+0+\mu_{ij}^{m-1\,s-1}\,.$$

Since $c_k+b_k-2 \geqq s-1 \geqq c_k+(m-1)+2$,

$$\mu_{ij}^{m-1\,s-1} = 0$$

by induction's hypothesis. Hence (1) is established. Comparing the coefficients of $e(c_k+m)$, we have

$$\mu_{ij}^{m\,c_k+m+1} = \lambda_{ij}^{0k}+0+\mu_{jj}^{m-1\,c_k+m}\,.$$

Since

$$\mu_{ij}^{m-1\,c_k+m} = \mu_{ij}^{m-1\,c_k+(m-1)+1} = ((m-1)+1)\lambda_{ij}^{k}$$

and $\lambda_{ij}^{0k}=\lambda_{ij}^{k}$,

$$\mu_{ij}^{m\,c_k-m-1} = (m+1)\lambda_{ij}^{k}\,,$$

as desired.

Now we finish our proof of Proposition 4. Let

$$\theta^*(e(c_i+b_i-1),\ e(c_j+1)) = \sum_{s=1}^{n} \mu_{ij}^{s} e(s)\,.$$

Then as in the proof of Lemma 5, we have

$$\sum_{\substack{s=1\\ s+1\notin C}}^{n-1} \mu_{ij}^{s+1} e(s)$$
$$= \theta^*(e(c_i+b_i-1),\ e(c_j)) + \theta^*(e(c_i+b_i-2),\ e(c_j))$$
$$+\theta^*(e(c_i+b_i-2),\ e(c_j-1))\,.$$

So the coefficients of $e(c_k+b_k-1)$ on the left hand side of the equality are zero for $1 \leqq i, j, k \leqq d$. On the other hand, they are

$$\lambda_{ij}^{k}+0+(b_i-1)\lambda_{ij}^{k} = b_i\lambda_{ij}^{k} = (b+1)\lambda_{ij}^{k}\,.$$

Since the characteristic of the field F is zero or greater than $b+1$, $\lambda_{ij}^{k}=0$ for all $1 \leqq i, j, k \leqq d$. As

$$\theta^*(e(c_i), e(c_j)) = \sum_{i=1}^{d} \lambda_{ij}^k e(c_k)$$

by Corollary 4, we finally obtained zero as the value of each $\theta^*(e(c_i), e(c_j))$ for all $1 \leqq i, j \leqq d$. This completes the proof of Proposition 4.

Though we may follow the similar lines of the proof of Proposition 4 to obtain the multilinear form (or multilinear Hermitian form) version of Proposition 4, we shall take a different approach which would make clear the relations between multilinear maps and mulitlinear forms.

Proposition 5. *Let θ be a multilinear form of degree $r+1 \geqq 3$ on a vector space V and $\{e(i)\}$ be a basis of V. Suppose θ admits a unipotent automorphism σ. Let σ^* denote the transpose inverse of σ according to the basis above. Then the following hold:*

(1) *There is an element τ in $GL(V)$ such that*

$$\tau^{-1}\sigma^*\tau = \sigma, \quad or \quad \tau\sigma^{-1} = \sigma^t\tau\,,$$

where σ^t denotes an element of $GL(V)$ corresponding to the transpose of σ according to the basis.

(2) *Let*

$$\theta^*(v_1, \cdots, v_r) = \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i))e(i)^\tau\,.$$

Then θ^* is a multilinear map of degree r admitting σ as a unipotent automorphism.

(3) *If $\theta(v, \cdots, v, w)=0$ for all $w \in V$ implies $v=0$, $\theta^*(v, \cdots, v)=0$ implies $v=0$.*

Proof. (1) is well-known, and it can be checked easily by transforming the matrix of σ into Jordan's canonical form. By the definition of θ^*, it is easy to see that θ^* is a multilinear map of degree r, and that (3) holds. To see (2) let (λ_{ij}) be the matrix of σ according to the basis above. Since σ is an automorphism of θ, we have

$$\begin{aligned}
&\theta^*(v_1^{\sigma^{-1}}, \cdots, v_r^{\sigma^{-1}}) \\
&= \sum_{i=1}^{n} \theta(v_1^{\sigma^{-1}}, \cdots, v_r^{\sigma^{-1}}, e(i))e(i)^\tau \\
&= \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i)^\sigma)e(i)^\tau \\
&= \sum_{i=1}^{n} \theta(v_1, \cdots\ldots, v_r, \sum_{j=1}^{n} \lambda_{ij}e(j))e(i)^\tau \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \theta(v_1, \cdots, v_r, e(j))\lambda_{ij}e(i)^\tau\,.
\end{aligned}$$

On the other hand we have

$$\begin{aligned}
&(\theta^*(v_1, \cdots, v_r))^{\sigma^{-1}} \\
&= \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i))e(i)^{\tau\sigma^{-1}} \\
&= \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i))e(i)^{\tau t_\tau} \\
&= \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i))\,(\sum_{j=1}^{n} \lambda_{ji}e(j))^{\tau} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{n} \theta(v_1, \cdots, v_r, e(i))\lambda_{ji}e(j)^{\tau}\,.
\end{aligned}$$

Thus $\theta^*(v_1^{\sigma^{-1}}, \cdots, v_r^{\sigma^{-1}})=\theta^*(v_1, \cdots, v_r)^{\sigma^{-1}}$.
Therefore σ is an automorphism of θ^* as well as θ. This is a proof of (2).

5. Theorems of Burnside on linear groups

In this section we collect theorems on linear groups which we need in the following section. We note that they are essentially proved by Burnside.

Lemma 6. *Let G be a group (not necessarily finite), and let F be an algebraically closed field. Suppose that $T_1, \cdots, T_k$ are inequivalent irreducible matrix representations of G in F, and let*

$$T_r(g) = (f^r_{ij}(g))_{1\leqq i,j\leqq n_r},\ g\in G\,.$$

Then the coordinate functions

$$f^r_{ij}\colon 1\leqq i, j\leqq n_r,\ 1\leqq r\leqq k \quad \textit{for all}\ \ g\in G$$

are linearly independent over F; that is

$$\sum_{i,j,r} a^r_{ij}f^r_{ij}(g) = 0 \quad \textit{for all}\ \ g\in G$$

implies that each $a^r_{ij}=0$.

Proof. See Corollary 27.13 in [1].

Proposition 6. *Let V be a vector space of dimension n over a field F, and G be a subgroup of $GL(V)$. Suppose that the exponent m of G is finite. Moreover, assume one of the following:*

(i) *the characteristic of F is zero;*
(ii) *G is absolutely irreducible; or*
(iii) *the order of any unipotent normal subgroup of G is finite.*

Then G is a finite group.

Proof. We may assume that F is algebraically closed. Consider the case where G is an (absolutely) irreducible group of linear transformations. Letting $\chi(g)$ denote the trace of the linear transformation g, we see that for each

$g \in G$, the number $\chi(g)$ is a sum of n m-th roots of unity. Therefore the set $\{\chi(g): g \in G\}$ has only a finite number of distinct elements. Relative to some fixed F-basis of V, let the matrix of $g \in G$ be

$$(f_{ij}(g))_{1 \leqq i, j \leqq n}, f_{ij}(g) \in F .$$

By Lemma 6, there exist n^2 elements $g_1, \cdots, g_{n^2} \in G$ such that n^2 n^2-tuples

$$\{f_{ij}(g_k): 1 \leqq i, j \leqq n, 1 \leqq k \leqq n^2\}$$

are linearly independent over F. For $g \in G$, we have

$$\chi(g_k g) = \sum_{i=1}^{n} f_{ii}(g_k g) = \sum_{i,j=1}^{n} f_{ij}(g_k) f_{ji}(g) .$$

Regarding this as a set of n^2 linear equations in the n^2 unknowns $\{f_{ji}(g)\}$, we see that the rows of the matrix of coefficients are linearly independent over F, and hence there is a unique solution for the $\{f_{ji}(g)\}$, this solution of course depending on the values $\{\chi(g_k g)\}$. But χ takes on only a finite number of possible values and hence so does each $f_{ji}(g)$. Therefore the group G is finite in this case.

We now use induction on n, and note that we have established the result when G is irreducible (the case (ii) above). Now let G be a reducible set of linear transformations. Relative to some F-basis of V, the matrices corresponding to the elements g of G take the form

$$\begin{bmatrix} T(g) & 0 \\ U(g) & V(g) \end{bmatrix}.$$

The m-th power of such matrix has $T(g)^m$ and $V(g)^m$ as diagonal blocks, and the groups

$$\{T(g): g \in G\}, \{V(g): g \in G\}$$

are groups with finite exponents. We may assume that the first group acts irreducibly on the corresponding subspace. By the induction hypothesis, both of these groups are finite.

Set

$$H_1 = \{g \in G: T(g) = I\}, \ H_2 = \{g \in G: V(g) = I\} .$$

Then H_1 and H_2 are normal in G and are of finite index. Hence also $|G: H_1 \cap H_2|$ is finite. Since $H_1 \cap H_2$ is normal in G and if $g \in H_1 \cap H_2$, the matrix of g is just

$$\begin{bmatrix} I & 0 \\ U(g) & I \end{bmatrix},$$

$H_1 \cap H_2$ is a unipotent normal subgroup of G, so the case (iii) is done. Now we may assume that the characteristic of the field F is zero. Since the m-th power of the matrix above is

$$\begin{bmatrix} I & 0 \\ mU(g) & I \end{bmatrix},$$

we must have $U(g)=0$, as m is the exponent of G. Thus $g=1$ in this case. This completes the proof of Proposition 6.

REMARK. The proof of Proposition 6 is essentially taken from [1] Theorem 36.1, which states only the case (i) explicitly.

6. Proofs of theorems

Now we give the proofs of our main results.

Proof of Theorem A [resp. Theorem B]. Let $G=\mathrm{Aut}\,\theta$. Suppose the case (i) does not occur. Then it follows from Proposition 1 [resp. Proposition 2] that there exists a number $m\leqq(r^n-1)^n!$ [resp. $((r-1)^{2n}-1)^n!$] such that g^m is unipotent for every element g of G. By Proposition 4 [resp. Proposition 5], $g^m=1$. So the exponent of G divides m, in particular, it is finite. Now the finiteness follows from Proposition 6.

DEFINITION 4. Let θ be a multilinear map [resp. a multilinear form]. θ is nonsingular if θ is not in the case (i) of Theorem A [resp. Theorem B]. Moreover, we define the nonsingularity of a multilinear Hermitian form in the same way.

Using the case (ii) of Proposition 6, we have the following theorem as a corollary to the proof above.

Theorem D. *Let θ denote a nonsingular multilinear map of degree $r\geqq 2$ or a nonsingular multilinear form of degree $r\geqq 3$ on a vector space V of dimension n over an algebraically closed field of arbitrary characteristic. If $\mathrm{Aut}\,\theta$ acts irreducibly on V, the order of $\mathrm{Aut}\,\theta$ is finite.*

Since the proof of Theorem C is almost the same as that of Theorem, B we just give a sketch of the proof.

Proof of Theorem C. As we can follow the same lines of the proof of Theorem B, we need only to show the result corresponding to Proposition 5. Let σ be a unipotent automorphism of a nonsingular multilinear Hermitian form θ. Let bars denote the complex conjugates. According to a fixed basis, let $\sigma^*=(\bar{\sigma}^t)^{-1}$. Since σ and σ^* are conjugate in $GL(V)$ by an element τ, we can define θ^* in the same way as in Proposition 5. Hence we reach the desired conclusion.

Let $\mathcal{H}$ be the set of all nonsingular multilinear Hermitian forms of degree $r+1\geqq 3$ on a vector space V over the complex number field, and $\mathcal{M}$ be the set of all nonsingular multilinear maps of degree $r\geqq 2$ on the same vector space V. The following theorem gives a nice one-to-one correspondence between the elements of $\mathcal{H}$ and $\mathcal{M}$ and the automorphism groups of them.

Theorem E. *The following hold*:

(1) *Let $\theta\in\mathcal{H}$, and $\{e(i)\}$ a basis. Then there is a Hermitian positive definite matrix X such that $x\sigma=\sigma x$ for all $\sigma\in\operatorname{Aut}\theta$, where x is the linear transformation of V represented by X according to a basis above. Moreover let*

$$\Phi(\theta)(v_1, \cdots, v_r) = \sum_{i=1}^{n} \theta(v_1, \cdots, v_r, e(i)^x)e(i)\,.$$

Then $\Phi(\theta)$ belongs to $\mathcal{M}$ and

$$\operatorname{Aut}\theta = \operatorname{Aut}(\Phi(\theta))\,.$$

(2) *Let $\theta\in\mathcal{M}$, and $\{e(i)\}$ a basis. Then there is a Hermitian positive definite matrix X such that $x\sigma=\sigma x$ for all $\sigma\in\operatorname{Aut}\theta$, where x is the linear transformation of V represented by X according to a basis above. Let $\{f(i)\}$ be the dual basis and*

$$w^* = w^{x^{-1}} = \sum_{i=1}^{n} \mu_i e(i), \quad \textit{for each} \quad w = \sum_{i=1}^{n} \lambda_i e(i)\,.$$

Define

$$\begin{aligned}\Psi(\theta)(v_1, \cdots, v_r, w) &= \sum_{i=1}^{n} \bar{\mu}_i \Psi(\theta)(v_1, \cdots, v_r, e(i)^x) \\ &= \sum_{i=1}^{n} \bar{\mu}_i f(i)(\theta(v_1, \cdots, v_r))\,.\end{aligned}$$

Then $\Psi(\theta)\in\mathcal{H}$ and $\operatorname{Aut}\theta=\operatorname{Aut}(\Psi(\theta))$.

Proof. Let θ be an element of $\mathcal{H}$ [resp. $\mathcal{M}$]. Since θ is nonsingular, $\operatorname{Aut}\theta$ is a finite complex linear group. Hence as is well-known, there is a positive definite $\operatorname{Aut}\theta$ invariant Hermitian form. So we can define a correspondence above and it is easy to establish the rest of the properties of the correspondence.

7. Examples

In this section we give a couple of examples. The first example shows that the restriction given for the characteristic of the base field in Proposition 4 is best possible, Example 2 gives a counter example to Proposition 2 of [7].

In the following two examples, let V be a vector space of dimension p, which is equal to the characteristic of the base field F, and σ be a cyclic permutation of a basis $\{e(i)\}$ of V, i.e., $e(i)^{\sigma}=e(i+1)$ for $1\leqq i\leqq p-1$ and $e(p)^{\sigma}=e(1)$.

EXAMPLE 1. Let θ be a multilinear map of degree 2, defined by $\theta(e(i), e(j))=\delta_{ij}e(i)$. Then it is easy to see that θ is nonsingular and admits a unipotent automorphism σ.

EXAMPLE 2. Let θ be a symmetric multilinear form of degree 3, defined by

$$\theta(e(i), e(j), e(k)) = \delta_{ij}\delta_{jk} \cdots (\#)$$

Then it is easy to check that θ is nonsingular and admits a unipotent automorphism σ. If $p\geqq 3$, the corresponding cubic form f has the form

$$x_1^3+x_2^3+\cdots+x_p^3 .$$

Moreover, counting the number of triples which satisfy (#), we see that

$$\mathrm{Aut}\,\theta = \mathrm{Aut}f = Z_3\ wr\ \Sigma_p ,$$

where *wr* denotes the wreath product of two groups. Thus even in this case, $\mathrm{Aut}\,\theta$ is a finite group. (See [5], [6].)

EXAMPLE 3. In [3], K. Harada proved the following:

Let A be a commutative (nonassociative) algebra over some field F satisfying the following conditions:

(1) A is a vector space over F with a system of basis $x_1, x_2, \cdots$, and x_n.

(2) $x_i^2=(n-1)x_i$ for $1\leqq i\leqq n$.

(3) $x_ix_j=-x_i-x_j$ for $1\leqq i<j\leqq n$.

Then if the characteristic of F is zero or greater than $n+1$, the automorphism group of A is isomorphic to the symmetric group Σ_{n+1} of degree $n+1$.

Since a commutative algebra is nothing but the one whose product is defined by a symmetric multilinear map of degree 2 using the universality of the symmetric tensor, we can regard this theorem as a theorem on a specific symmetric multilinear map of degree 2. By a little computation, one can show that the corresponding symmetric multilinear map of degree 2 is nonsingular, if and only if $(2p, n+1)=1$, where $p=\mathrm{char}F$ if $\mathrm{char}F$ is nonzero, and $p=1$ if $\mathrm{char}F=0$. So the finiteness part of the theorem follows from Theorem A if $(2p, n+1)=1$.

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