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ACTIONS OF SPECIAL UNITARY GROUPS ON A PRODUCT OF COMPLEX PROJECTIVE SPACES

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0. Introduction

Let X be a connected closed orientable C^∞ manifold which admits a non-trivial smooth $\boldsymbol{SU}(n)$ action. Suppose

$$H^*(X; \boldsymbol{Q}) = \boldsymbol{Q}[u, v]/(u^{a+1}, v^{b+1}), \deg u = \deg v = 2,$$

that is, the cohomology ring of X is isomorphic to that of a product $P_a(\boldsymbol{C})\times P_b(\boldsymbol{C})$ of complex projective spaces, where $\boldsymbol{Q}$ is the field of rational numbers. We shall show the following result.

Theorem. *On the above situation, suppose*

$$1\leqq b\leqq a<n\leqq a+b\leqq 2n-3 .$$

Then, $a=n-1$ and X is equivariantly diffeomorphic to $P_{n-1}(\boldsymbol{C})\times Y$, where Y is a connected closed orientable manifold whose rational cohomology ring is isomorphic to that of $P_b(\boldsymbol{C})$, and $\boldsymbol{SU}(n)$ acts naturally on $P_{n-1}(\boldsymbol{C})$ and trivially on Y.

1. Preliminary lemmas

We prepare the following lemmas.

Lemma 1.1. *Let G be a closed connected proper subgroup of $\boldsymbol{SU}(n)$ such that $g=\dim \boldsymbol{SU}(n)/G\leqq 4n-6$. Then it is one of the following up to an inner automorphism of $\boldsymbol{SU}(n)$.*

(i) $\boldsymbol{SU}(n-k)\subset G\subset \boldsymbol{S}(\boldsymbol{U}(k)\times \boldsymbol{U}(n-k))$, $n\geqq 2k$; $k=1$, 2 *or* 3.

(ii)

n	G	g	$4n-6$	n	G	g	$4n-6$
6	$\boldsymbol{Sp}(3)$	14	18	4	$\boldsymbol{SO}(4)$	9	10
5	$\boldsymbol{Sp}(2)$	14	14	4	$\boldsymbol{Sp}(2)$	5	10
5	$\boldsymbol{NSp}(2)$	13	14	3	$\boldsymbol{SO}(3)$	5	6
5	$\boldsymbol{SO}(5)$	14	14	3	T^2	6	6

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Here $\boldsymbol{NSp}(2)$ denotes the normalizer of $\boldsymbol{Sp}(2)$ in $\boldsymbol{SU}(5)$.

The proof is a routine work by a standard method [2, 3], so we omit it.

Lemma 1.2. *Suppose $n \geqq 3$ and $k \leqq 4n-6$. Then a non-trivial real representation of $\boldsymbol{SU}(n)$ of degree k is equivalent to $(\mu_n)_{\boldsymbol{R}} \oplus \theta^{k-2n}$ or $\pi \oplus \theta^{k-6}$ (for $n=4$). Here $(\mu_n)_{\boldsymbol{R}}$: $\boldsymbol{SU}(n) \rightarrow \boldsymbol{O}(2n)$ is a standard inclusion, π: $\boldsymbol{SU}(4) \rightarrow \boldsymbol{SO}(6)$ is a double covering, ond θ^i is a trivial representation of degree i.*

Proof. The proof is also a routine work by a standard method [3], but we give a proof for completeness. Denote by $L_1, L_2, \cdots, L_{n-1}$ the standard fundamental weights of $\boldsymbol{SU}(n)$. Then there is a one-to-one correspondence between complex irreducible representations of $\boldsymbol{SU}(n)$ and sequences $(a_1, \cdots, a_{n-1})$ of non-negative integers such that $a_1L_1 + \cdots + a_{n-1}L_{n-1}$ is the highest weight of a corresponding representation. Denote by $d(a_1L_1 + \cdots + a_{n-1}L_{n-1})$ the degree of the complex irreducible representation of $\boldsymbol{SU}(n)$ with the highest weight $a_1L_1 + \cdots + a_{n-1}L_{n-1}$. Notice that if $a_i \geqq a_i'$ for $i=1, \cdots, n-1$, then $d(a_1L_1 + \cdots + a_{n-1}L_{n-1}) \geqq d(a_1'L_1 + \cdots + a_{n-1}'L_{n-1})$ and the equality holds only if $a_i = a_i'$ for $i=1, \cdots, n-1$. The degree can be computed by Weyl's dimension formula. We obtain

$$
\begin{aligned}
& d(L_i) = {}_nC_i \quad \text{for} \quad 1 \leqq i \leqq n-1,\ d(2L_1) = d(2L_{n-1}) = n(n+1)/2\,, \\
& d(2L_2) = d(2L_{n-2}) = n^2(n^2-1)/12,\ d(L_1+L_{n-1}) = n^2-1\,, \\
& d(L_1+L_{n-2}) = d(L_2+L_{n-1}) = n(n+1)(n-2)/2\,, \\
& d(L_2+L_{n-2}) = n^2(n+1)\,(n-3)/4\,, \\
& d(L_1+L_2) = d(L_{n-2}+L_{n-1}) = n(n^2-1)/3\,, \\
& d(3L_1) = d(3L_{n-1}) = n(n+1)\,(n+2)/6\,.
\end{aligned}
$$

(i) Suppose $n \geqq 5$. Then a non-trivial complex irreducible representation of degree $\leqq 4n-6$ is equivalent to one of the following: μ_n, μ_n^*, $\Lambda^2(\mu_n)$, $\Lambda^2(\mu_n^*)$, where μ_n^* is the conjugate representation and $\Lambda^2(\ )$ is the second exterior product. Therefore a non-trivial self-conjugate complex representation of degree $\leqq 4n-6$ is equivalent to $\mu_n + \mu_n^* \oplus$trivial, which has a real form $(\mu_n)_{\boldsymbol{R}} \oplus$ trivial.

(ii) Suppose $n=4$. Then a non-trivial complex irreducible representation of degree $\leqq 4n-6=10$ is equivalent to one of the following: μ_4, μ_4^*, $\Lambda^2(\mu_4) = \Lambda^2(\mu_4^*)$, $S^2(\mu_4)$, $S^2(\mu_4^*)$, where $S^2(\ )$ is the second symmetric product. Therefore a non-trivial self-conjugate complex representation of degree $\leqq 10$ is equivalent to $\mu_4 \oplus \mu_4^* \oplus$ trivial or $\Lambda^2(\mu_4) \oplus$ trivial. They have a real form $(\mu_4)_{\boldsymbol{R}} \oplus$ trivial and $\pi \oplus$ trivial, respectively.

(iii) Suppose $n=3$. Then a non-trivial complex irreducible representation of degree $\leqq 4n-6=6$ is equivalent to one of the following: μ_3, μ_3^*, $S^2(\mu_3)$, $S^2(\mu_3^*)$.

Therefore a non-trivial self-conjugate complex representation of degree $\leqq 6$ is equivalent to $\mu_3 \oplus \mu_3^*$, which has a real form $(\mu_3)_{\boldsymbol{R}}$. q.e.d.

NOTATIONS. In the following sections, let K^0 denote the identity component of a closed subgroup K of $\boldsymbol{SU}(n)$, and $N(K)$ denote the normalizer of K in $\boldsymbol{SU}(n)$. Let $\chi(X)$ denote the Euler characteristic of a manifold X.

2. Smooth $\boldsymbol{SU}(n)$ actions

Throughout this section, suppose that X is a connected closed orientable manifold with a non-trivial smooth $\boldsymbol{SU}(n)$ action such that $\dim X \leqq 4n-6$. Denote by (H) the principal isotropy type.

Proposition 2.1. *Suppose $n=5$ and $H^0 = \boldsymbol{NSp}(2)$. Then $\chi(X)=0$. In fact, X has only one orbit type $\boldsymbol{SU}(5)/\boldsymbol{NSp}(2)$.*

Proof. Since $N(\boldsymbol{NSp}(2)) = \boldsymbol{NSp}(2)$, it follows that $H = \boldsymbol{NSp}(2)$ and X has no exceptional orbits. Now we shall show that X has no singular orbits. It is clear for $\dim X = 13$. Suppose that $\dim X = 14$ and X has a singular orbit. Then the orbit type must be $\boldsymbol{SU}(5)/\boldsymbol{S}(\boldsymbol{U}(1)\times \boldsymbol{U}(4))$ by Lemmas 1.1 and 1.2. Considering the slice representation, we obtain a covering projection of $\boldsymbol{SU}(4)/$center onto $\boldsymbol{SO}(6)$. But, there is no injection of $\pi_1(\boldsymbol{SU}(4)/\text{center}) = \boldsymbol{Z}_4$ into $\pi_1(\boldsymbol{SO}(6)) = \boldsymbol{Z}_2$, and hence there is no covering projection of $\boldsymbol{SU}(4)/$center onto $\boldsymbol{SO}(6)$. Therefore, X has no singular orbits. q.e.d.

The next three propositions can be easily proved.

Proposition 2.2. *Suppose that H^0 is one of the following: $\boldsymbol{Sp}(3)$, $n=6$; $\boldsymbol{Sp}(2)$, $n=5$; $\boldsymbol{Sp}(2)$, $n=4$; $\boldsymbol{SO}(5)$, $n=5$; $\boldsymbol{SO}(4)$, $n=4$; $\boldsymbol{SO}(3)$, $n=3$. Then, X has no singular orbits and $\chi(X)=0$.*

Proposition 2.3. *Suppose $n=3$ and $H^0 = T^2$. Then $\boldsymbol{SU}(3)$ acts transitively on X.*

Proposition 2.4. *Suppose $n \geqq 6$ and $\boldsymbol{SU}(n-3) \subset H^0 \subset \boldsymbol{S}(\boldsymbol{U}(3)\times \boldsymbol{U}(n-3))$. Then $n=6$ and $X = \boldsymbol{SU}(6)/\boldsymbol{S}(\boldsymbol{U}(3)\times \boldsymbol{U}(3))$.*

The remaining possibilities are the followings:

$$\boldsymbol{SU}(n-k) \subset H^0 \subset \boldsymbol{S}(\boldsymbol{U}(k)\times \boldsymbol{U}(n-k)); \; k = 1, 2 .$$

In these cases, considering the slice representation, we can prove that $\boldsymbol{SU}(n-j) \subset K^0 \subset \boldsymbol{S}(\boldsymbol{U}(j)\times \boldsymbol{U}(n-j))$; $j=0$, 1 or 2, for any singular isotropy type (K). Denote

$$F_{(k)} = \{x \in X \mid \boldsymbol{SU}(n-k) \subset \boldsymbol{SU}(n)_x^0 \subset \boldsymbol{S}(\boldsymbol{U}(k)\times \boldsymbol{U}(n-k))\} ,$$
$$X_{(k)} = \boldsymbol{SU}(n)\cdot F_{(k)} .$$

Then $X=X_{(0)}\cup X_{(1)}\cup X_{(2)}$ for the remaining cases.

Proposition 2.5. *If $X_{(2)}$ is non-empty, then $X_{(0)}$ and $X_{(1)}$ are empty.*

Proof. Since $X_{(2)}$ is non-empty, we have $n\geqq 4$ and

$$(*)\qquad \boldsymbol{SU}(n-2)\subset H^0\subset \boldsymbol{S}(\boldsymbol{U}(2)\times \boldsymbol{U}(n-2)).$$

Suppose that $X_{(1)}$ is non-empty. Let σ_y be the slice representation at $y\in F_{(1)}$. Then

$$\deg \sigma_y = \dim X-\dim \boldsymbol{SU}(n)\cdot y\leqq 2n-4<4(n-1)-6.$$

Hence we obtain $\sigma_y|\boldsymbol{SU}(n-1)=(\mu_{n-1})_{\boldsymbol{R}}\oplus$trivial, by (*) and Lemma 1.2. Let ρ_y be the isotropy representation at y in the orbit $\boldsymbol{SU}(n)\cdot y$. Then $\rho_y|\boldsymbol{SU}(n-1)=(\mu_{n-1})_{\boldsymbol{R}}\oplus$trivial, and hence

$$\operatorname{codim} F_{(1)} \text{ at } y = 4n-4>4n-6.$$

This is a contradiction, and hence $X_{(1)}$ is empty. Similarly we can prove that $X_{(0)}$ is empty. q.e.d.

Proposition 2.6. *Suppose $X=X_{(2)}$ and $\chi(X)\neq 0$. Then $X=\boldsymbol{SU}(n)/\boldsymbol{S}(\boldsymbol{U}(2)\times \boldsymbol{U}(n-2))$ or $X=\boldsymbol{SU}(n)/\boldsymbol{SU}(n-2)\times_W S^2$, where $W=\boldsymbol{S}(\boldsymbol{U}(2)\times \boldsymbol{U}(n-2))/\boldsymbol{SU}(n-2)=\boldsymbol{U}(2)$.*

Proof. Since $X=X_{(2)}$, we obtain an equivariant decomposition $X=\boldsymbol{SU}(n)/\boldsymbol{SU}(n-2)\times_W F_{(2)}$, where $F_{(2)}$ is a connected closed orientable manifold on which W acts smoothly. The conditions $\dim X\leqq 4n-6$ and $\chi(X)\neq 0$ imply that $\dim F_{(2)}\leqq 2$ and $\chi(F_{(2)})\neq 0$. Hence we have a desired result. q.e.d.

Put $G_{n,2}=\boldsymbol{SU}(n)/\boldsymbol{S}(\boldsymbol{U}(2)\times \boldsymbol{U}(n-2))$. For the case $X=\boldsymbol{SU}(n)/\boldsymbol{SU}(n-2)\times_W S^2$, there is a fibration: $S^2\rightarrow X\xrightarrow{\pi} G_{n,2}$. Suppose that the W action on S^2 is non-transitive. Then the W action on S^2 has a fixed point, and hence the above fibration has an equivariant cross-section s.

Proposition 2.7. *On the above situation, there is an element of $H^4(X;\boldsymbol{Q})$ which is not a linear combination of x_j^2; $x_j\in H^2(X;\boldsymbol{Q})$.*

Proof. Let c_1 and c_2 be the first and the second Chern classes of the canonical 2-plane bundle over $G_{n,2}$, respectively. Suppose that $\pi^*(c_2)$ is represented as

$$\pi^*(c_2) = \textstyle\sum_j a_j x_j^2;\ a_j\in \boldsymbol{Q},\ x_j\in H^2(X;\boldsymbol{Q}).$$

Then $c_2=s^*\pi^*(c_2)=\sum_j a_j(s^*x_j)^2=a\,c_1^2$ for some $a\in\boldsymbol{Q}$, and hence c_2 and c_1^2 are linearly dependent in $H^4(G_{n,2};\boldsymbol{Q})$. This is a contradiction. Hence $\pi^*(c_2)$ is a desired element. q.e.d.

REMARK. Suppose $H^*(X; \boldsymbol{Q}) = \boldsymbol{Q}[u, v]/(u^{a+1}, v^{b+1})$, $\deg u = \deg v = 2$. Then any element of $H^4(X; \boldsymbol{Q})$ is represented as

$$p\, u^2 + q\, uv + r\, v^2 = p\, u^2 + q'(u+v)^2 - q'(u-v)^2 + r\, v^2\,,$$

where $p, q, r \in \boldsymbol{Q}$ and $q' = q/4$.

Suppose next that the W action on S^2 is transitive. Then $X = \boldsymbol{SU}(n)/\boldsymbol{SU}(n-2) \times_W S^2 = \boldsymbol{SU}(n)/\boldsymbol{S}(\boldsymbol{U}(1) \times \boldsymbol{U}(1) \times \boldsymbol{U}(n-2))$. Define

$$X_1 = \{(x_1 : \cdots : x_n) \times (y_1 : \cdots : y_n) \in P_{n-1} \times P_{n-1} \mid \bar{x}_1 y_1 + \cdots + \bar{x}_n y_n = 0\}\,.$$

Then X_1 is invariant under the natural diagonal $\boldsymbol{SU}(n)$ action on $P_{n-1}(\boldsymbol{C}) \times P_{n-1}(\boldsymbol{C})$, and we have $X_1 = \boldsymbol{SU}(n)/\boldsymbol{S}(\boldsymbol{U}(1) \times \boldsymbol{U}(1) \times \boldsymbol{U}(n-2))$. Considering X_1 as a projective space bundle over $P_{n-1}(\boldsymbol{C})$, we have a ring structure: $H^*(X_1; \boldsymbol{Q}) = \boldsymbol{Q}[c, t]/(c^n, \sum_i c^i t^{n-i-1})$, $\deg c = \deg t = 2$.

Proposition 2.8. *Let $X_1 = \boldsymbol{SU}(n)/\boldsymbol{S}(\boldsymbol{U}(1) \times \boldsymbol{U}(1) \times \boldsymbol{U}(n-2))$ and $u \in H^2(X_1; \boldsymbol{Q})$. If $u^{n-1} = 0$, then $u = 0$.*

Proof. Any element of $H^2(X_1; \boldsymbol{Q})$ is represented as $u = p\, c + q\, t$; $p, q \in \boldsymbol{Q}$. Suppose $u^{n-1} = 0$. Then we have

$$q^{n-1} = {}_{n-1}C_k\, p^{n-k-1} q^k,\ k = 0, 1, \cdots, n-2\,,$$

Hence we obtain $p = q = 0$. q.e.d.

3. Proof of the theorem

Throughout this section, suppose that X is a connected closed orientable manifold with a non-trivial smooth $\boldsymbol{SU}(n)$ action, and $H^*(X; \boldsymbol{Q}) = \boldsymbol{Q}[u, v]/(u^{a+1}, v^{b+1})$; $\deg u = \deg v = 2$. Moreover, suppose

$$(1) \qquad 1 \leqq b \leqq a < n \leqq a + b \leqq 2n - 3\,.$$

By arguments and notations in Section 2, the possibility remains only when $X = X_{(0)} \cup X_{(1)}$.

Proposition 3.1. *$X_{(0)}$ is empty.*

Proof. Suppose that $X_{(0)}$ is non-empty. Let U be an invariant closed tubular neighborhood of $X_{(0)}$ in X, and let $E = X - \mathrm{int}\, U$. Put $Y = E \cap F_{(1)}$. Then Y is a connected compact orientable manifold with non-empty boundary ∂Y, and $\boldsymbol{U}(1)$ acts naturally on Y. Since there is a natural diffeomorphism: $E = \boldsymbol{SU}(n)/\boldsymbol{SU}(n-1) \times_{\boldsymbol{U}(1)} Y = S^{2n-1} \times_{\boldsymbol{U}(1)} Y$, we obtain

$$\text{(2)} \qquad \dim Y = 2(a+b-n+1) = 2k, \quad k \leqq b \leqq n-2\,.$$

Let i: $E \to X$ be the inclusion. Then, i^*: $H^t(X; \boldsymbol{Q}) \to H^t(E; \boldsymbol{Q})$ is an isomorphism for each $t \leqq 2n-2$, because the codimension of each connected component of $X_{(0)}$ is $2n$ by Lemma 1.2. By the Gysin sequence of the principal $\boldsymbol{U}(1)$ bundle p: $S^{2n-1} \times Y \to E$, we obtain an exact sequence:

$$\begin{array}{ccccccccccc} 0 \to & H^{2k-1}(S^{2k-1}\times Y) & \to & H^{2k-2}(E) & \xrightarrow{e} & H^{2k}(E) & \xrightarrow{p^*} & H^{2k}(S^{2k-1}\times Y) & \to 0\,. \\ & \| & & \| & & \| & & \| & \\ & H^{2k-1}(Y) & & H^{2k-2}(X) & & H^{2k}(X) & & H^{2k}(Y) & \end{array}$$

Hence we obtain rank $H^{2k}(Y)$—rank $H^{2k-1}(Y)=1$, by the cohomology structure of X. Considering the homology exact sequence of the pair $(Y, \partial Y)$ and the Poincaré-Lefschetz duality, we obtain

$$\operatorname{rank} H_0(\partial Y) \leqq \operatorname{rank} H_0(Y) + \operatorname{rank} H^{2k-1}(Y) - \operatorname{rank} H^{2k}(Y) = 0\,.$$

Therefore ∂Y is empty; this is a contradiction. q.e.d.

Consequently we obtain $X = X_{(1)} = S^{2n-1} \times_{U(1)} F_{(1)}$.

Proposition 3.2. *$a=n-1$ and $H^*(F_{(1)}; \boldsymbol{Q}) = H^*(P_b(\boldsymbol{C}); \boldsymbol{Q})$.*

Proof. Since $n \cdot \chi(F_{(1)}) = \chi(X) = (a+1)(b+1) \neq 0$, the $\boldsymbol{U}(1)$ action on $F_{(1)}$ has a fixed point y_0. Consider the following commutative diagram:

$$\begin{array}{ccccc} S^{2n-1} & \xrightarrow{i} & S^{2n-1}\times F_{(1)} & \xrightarrow{q} & S^{2n-1} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ P_{n-1}(\boldsymbol{C}) & \xrightarrow{\bar{i}} & X & \xrightarrow{\bar{q}} & P_{n-1}(\boldsymbol{C})\,. \end{array}$$

Here π, p are projections of the principal $\boldsymbol{U}(1)$ bundles, q is the projection to the first factor, i is an inclusion defined by $i(x)=(x, y_0)$, and $\bar{i}$, $\bar{q}$ are induced mappings. Denote by $e(\ \ )$ the Euler class of a principal $\boldsymbol{U}(1)$ bundle. We can represent as $e(p) = k\,u + j\,v$; $k, j \in \boldsymbol{Q}$. Then

$$0 = \bar{q}^*(e(\pi)^n) = e(p)^n = (ku+jv)^n = \textstyle\sum_i {}_nC_i\, k^{n-i} j^i u^{n-i} v^i\,,$$

and hence ${}_nC_i\, k^{n-i} j^i = 0$ for $n-a \leqq i \leqq b$. Hence we obtain $kj=0$. Suppose $k=0$. Then

$$0 \neq e(\pi)^{n-1} = \bar{i}^*(e(p)^{n-1}) = \bar{i}^*(j^{n-1} v^{n-1}) = 0\,,$$

because $v^{b+1}=0$ and $b \leqq n-2$. This is a contradiction. Therefore $e(p) = k\,u$ $(k \neq 0)$. Since $\bar{i}^*((ku)^{n-1}) = e(\pi)^{n-1} \neq 0$, we obtain $u^{n-1} \neq 0$ and hence $a=n-1$. Next, considering the Gysin sequence of the principal $\boldsymbol{U}(1)$ bundle p: $S^{2n-1} \times Y \to X$ and the ring structure of $H^*(X; \boldsymbol{Q})$, we obtain $H^*(F_{(1)}; \boldsymbol{Q}) = H^*(P_b(\boldsymbol{C});$

$\boldsymbol{Q}$). q.e.d.

Proposition 3.3. *The $\boldsymbol{U}(1)$ action on $F_{(1)}$ is trivial.*

Proof. Suppose that the $\boldsymbol{U}(1)$ action on $F_{(1)}$ is non-trivial, and let Y be the fixed point set. Consider the following commutative diagram:

$$\begin{array}{ccccc} H^t(S^{2n-1}\times_{U(1)}F_{(1)}) & \xleftarrow{j^*} & H^t(S^\infty\times_{U(1)}F_{(1)}) & \xrightarrow{L} & S^{-1}H^*(S^\infty\times_{U(1)}F_{(1)}) \\ \downarrow{\scriptstyle i^*} & & \downarrow{\scriptstyle i^*_\infty} & & \downarrow{\scriptstyle S^{-1}i^*_\infty} \\ H^t(S^{2n-1}\times_{U(1)}Y) & \xleftarrow{j^*_Y} & H^t(S^\infty\times_{U(1)}Y) & \xrightarrow{L_Y} & S^{-1}H^*(S^\infty\times_{U(1)}Y)\,. \end{array}$$

Here i, i_∞, j, j_Y are natural inclusions; L, L_Y are localization homomorphisms; S^{-1} is a localization by the Euler class of the universal principal $\boldsymbol{U}(1)$ bundle. It is known that $S^{-1}i^*_\infty$ is an isomorphism [1]. Since $H^{\mathrm{odd}}(F_{(1)};\boldsymbol{Q})=0$, we have that j^* is surjective and L is injective, in particular, i^*_∞ is injective. On the other hand, j^*_Y is isomorphic for each $t\leqq 2n-2$.

Now we shall show that $w^{b+1}=0$ implies $w^b=0$ for $w\in H^2(S^{2n-1}\times_{U(1)}F_{(1)};\boldsymbol{Q})$. We can represent as $i^*(w)=p_1^*(\alpha)+p_2^*(\beta)$ for some $\alpha\in H^2(P_{n-1}(\boldsymbol{C}))$, $\beta\in H^2(Y)$, where p_1, p_2 are projections from $S^{2n-1}\times_{U(1)}Y=P_{n-1}(\boldsymbol{C})\times Y$ to each factor. Then

$$0=k^*i^*(w^{b+1})=(k^*(p_1^*(\alpha)+p_2^*(\beta)))^{b+1}=\alpha^{b+1}\,,$$

where $k\colon P_{n-1}(\boldsymbol{C})\to P_{n-1}(\boldsymbol{C})\times Y$ is an inclusion defined by $k(x)=(x,*)$. Since $b\leqq n-2$, we obtain $\alpha=0$, and hence $i^*(w)=p_2^*(\beta)$. Therefore $i^*(w^b)=p_2^*(\beta^b)=0$, because $\dim Y<2b=\dim F_{(1)}$. Since j^* is surjective, there is an element $\overline{w}\in H^2(S^\infty\times_{U(1)}F_{(1)};\boldsymbol{Q})$ such that $j^*(\overline{w})=w$. Then

$$j^*_Y i^*_\infty(\overline{w}^b)=i^*j^*(\overline{w}^b)=i^*(w^b)=0\,,$$

and hence $\overline{w}^b=0$, because $j^*_Y i^*_\infty$ is injective for the degree $2b$ ($\leqq 2n-2$). Then $w^b=j^*(\overline{w}^b)=0$.

On the other hand, $X=S^{2n-1}\times_{U(1)}F_{(1)}$ and $H^*(X;\boldsymbol{Q})=\boldsymbol{Q}[u,v]/(u^{b+1},v^{b+1})$, where $a=n-1$. There is an element $v\in H^2(X;\boldsymbol{Q})$ such that $v^{b+1}=0$ but $v^b\neq 0$. This is a contradiction. q.e.d.

Summarizing the above propositions, we obtain $X=P_{n-1}(\boldsymbol{C})\times Y$ as $\boldsymbol{SU}(n)$ manifolds, where Y is a connected closed orientable manifold with trivial $\boldsymbol{SU}(n)$ action, and $H^*(Y;\boldsymbol{Q})=H^*(P_b(\boldsymbol{C});\boldsymbol{Q})$. This completes the proof of the theorem stated in Introduction.

4. Concluding remark

We give examples [2] of a manifold whose rational cohomology ring is isomorphic to that of $P_k(\boldsymbol{C})$.

EXAMPLE 1. Let p be a positive integer. There is a connected closed orientable C^∞ manifold Y_1 such that

$$H^*(Y_1; \boldsymbol{Q}) = H^*(P_k(\boldsymbol{C}); \boldsymbol{Q}) \quad \text{and} \quad \pi_1(Y_1) = \boldsymbol{Z}/p\boldsymbol{Z}$$

for each $k \geqq 3$.

EXAMPLE 2. Let G be a finitely presentable group such that $H_1(G; \boldsymbol{Z}) = H_2(G; \boldsymbol{Z}) = \{0\}$, where $\boldsymbol{Z}$ is the ring of integers. There is a connected closed orientable C^∞ manifold Y_2 such that

$$H^*(Y_2; \boldsymbol{Z}) = H^*(P_k(\boldsymbol{C}); Z) \quad \text{and} \quad \pi_1(Y_2) = G$$

for each $k \geqq 5$.

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