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# ON EQUIVARIANT J-HOMOMORPHISM FOR INVOLUTIONS

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**Introduction.** Let $G$ be the cyclic group of order 2.

We denote by $\pi_S^{*,*}$ the equivariant stable cohomotopy theory [2, 3] and by $KO_G^*$ the $K$-theory of real $G$-vector bundles on $G$-spaces. For a finite pointed $G$-complex we then have an equivariant $J_G$-map $J\colon \widetilde{KO}_G^{-1}(X) \to \pi_S^{0,0}(X)$ [14], which becomes a homomorphism if $X$ is a suspension in the usual sense.

Let $R^{p,q}$ be the euclidean space $R^{p+q}$ with non trivial $G$-action on the first $p$ coordinates and $\Sigma^{p,q}$ be the one point compactification of $R^{p,q}$, with $\infty$ as base point. We have the canonical isomorphism $\pi_S^{0,0}(\Sigma^{p,q}) \approx \pi_{p,q}^S$, the $(p, q)$-th equivariant stable homotopy group of Landweber [9, 3] (which is $\pi_{p+q,p}$ of Bredon [5]), and therefore we get an induced map

$$\widetilde{KO}_G^{-1}(\Sigma^{p,q}) \to \pi_S^{0,0}(\Sigma^{p,q}) \approx \pi_{p,q}^S$$

which we also denote by $J_G$. P. Löffler [10] showed that if $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ is a torsion group then $J_G$ is a split injection. In this paper we shall study the image of $J_G$ when $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ is torsion free. And then we shall give a supplement to [11] on $\operatorname{Im} J_R$. The $J_G$ is also studied by M.C. Crabb [6].

We denote by $Z/n$ a cyclic group of order $n$, by $R\cdot x$ the free module over a ring $R$ generated by $x$. If $p \equiv i \bmod 8$ and $q \equiv j \bmod 8$, then we write $(p, q) \equiv (i, j) \bmod 8$.

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## 1. The $J$-homomorphism $J_G$

In this section we shall give the relations between various homomorphisms and collect some basic tools. Let $X$ be a finite pointed $G$-complex.

Let $KR$ denote the $K$-functor of [4]. By regarding a Real vector bundle on $X$ as a real $G$-vector bundle on $X$ we get a homomorphism $\sigma\colon \widetilde{KR}^{-1}(X) \to \widetilde{KO}_G^{-1}(X)$. We define a map $J_R\colon \widetilde{KR}^{-1}(X) \to \pi_S^{0,0}(X)$ by

$$J_R = J_G\sigma \tag{1.1}$$

which is the same as in [11].

Let $\rho$, $\psi$, $\chi$ and $\delta$ be the homomorphisms defined in [2], §§1–5 and $J_o$ be the usual real $J$-map. From the definitions of these maps it follows that

$$\psi J_G = J_o \psi,\ \chi J_G = J_G \chi,\ \delta J_o = J_G \delta \ \text{ and } \ \rho J_G = J_G \rho \tag{1.2}$$

when each map is valid.

Let $RO(G)$ denote the real representation ring of $G$. If we put $1=R^{0,1}$ and $H=R^{1,0}$ then $RO(G)=Z\cdot 1\oplus Z\cdot H$. Define a homomorphism $\varepsilon$: $RO(G)\to Z$ by the assignment $s+tH\mapsto s$ and define a homomorphism $\phi: \widetilde{KO}_G^{-1}(X)\to\widetilde{KO}^{-1}(X^G)$ to be the composite of the homomorphisms

$$\widetilde{KO}_G^{-1}(X)\xrightarrow{i^*}\widetilde{KO}_G^{-1}(X^G)\approx\widetilde{KO}^{-1}(X^G)\otimes RO(G)$$
$$\xrightarrow{1\otimes\varepsilon}\widetilde{KO}^{-1}(X^G)\otimes Z=\widetilde{KO}^{-1}(X^G)$$

where $i$ denotes the inclusion of the fixed point set $X^G$ of $X$ and the isomorphism is the same as in Remark of [13], p. 133. Then we also have

$$\phi J_G = J_o \phi \tag{1.3}$$

where $\phi$ on the left hand side denotes the homomorphism as in [2], §4.

Let $\hat{\eta}$: $\Sigma^{2,1}\to\Sigma^{1,1}$ be the $G$-map defined in [3], §8 such that $\psi(\hat{\eta})=\eta$: $S^3\to S^2$ is the Hopf map. We see that $\hat{\eta}$ and $\eta$ yield natural homomorphisms of our cohomology theories. Denote these homomorphisms by the same letters, i.e.,

$$\hat{\eta}\colon \widetilde{KO}_G^{-1}(\Sigma^{1,1}\wedge X)\to\widetilde{KO}_G^{-1}(\Sigma^{2,1}\wedge X),\ \hat{\eta}\colon \pi_S^{0,0}(\Sigma^{1,1}\wedge X)\to\pi_S^{0,0}(\Sigma^{2,1}\wedge X),$$
$$\eta\colon \widetilde{KO}^{-1}(S^2\wedge X)\to\widetilde{KO}^{-1}(S^3\wedge X),\ \ \eta\colon \pi_S^0(S^2\wedge X)\to\pi_S^0(S^3\wedge X).$$

It is clear that

$$\text{i) } \chi\hat{\eta}=1+\rho,\ \text{ ii) } \psi\hat{\eta}=\eta\psi \ \text{ and iii) } \hat{\eta}J_G=J_G\hat{\eta}. \tag{1.4}$$

Note that the first formula follows from [3], (8.1).

Let $\widetilde{Sph}_G^{-1}(X)$ be the group of stable fibre homotopy equivalence classes of spherical $G$-fibrations on $\Sigma^{0,1}X$ ([7], §7). As in the non equivariant case $J_G$ can be factorized as the composition of maps $\widetilde{KO}_G^{-1}(X)\to\widetilde{Sph}_G^{-1}(X)\to\pi_S^{0,0}(X)$. By [7], Theorem 0.4 we therefore have

**Theorem 1.5** ([7], Theo. 0.4). *If $X$ is a suspension then, for any $x\in KO_G^{-1}(X)$ there exists $e>0$ such that $3^e J_G(\psi^3-1)x=0$.*

Finally we recall the following theorems.

**Theorem 1.6** ([1, 12]).

$$\mathrm{Im}\ \{J_0\colon \widetilde{KO}^{-1}(S^{4p-1}) \to \pi^S_{4p-1}\} \approx Z/m(2p)$$

*where $m(t)$ denotes the numerical function as in* [1, II].

**Theorem 1.7** ([5], p. 272).

$$\psi\oplus\phi\colon \pi^S_{p,q}\left[\frac{1}{2}\right] \to (\pi^S_{p+q}\oplus\pi^S_q)\left[\frac{1}{2}\right] \quad \textit{for } p \textit{ even}$$

*and*

$$\phi\colon \pi^S_{p,q}\left[\frac{1}{2}\right] \to \pi^S_q\left[\frac{1}{2}\right] \quad \textit{for } p \textit{ odd}$$

*are isomorphisms.*

## 2. The equivariant $K$-theory of spheres

According to [10] $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ is an abelian 2-group or a free abelian group (with a single generator as a module over $RO(G)$). We refer the reader to Table in [10] for these groups. We extract only the cases when $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ is free.

(2.1) a) If $(p, q)\equiv(2, 1)$, $(6, 1)$, $(2, 5)$ or $(6, 5)$ mod 8 then $\widetilde{KO}_G^{-1}(\Sigma^{p,q})\approx Z$,

b) If $(p, q)\equiv(0, 3)$, $(4, 3)$, $(0, 7)$ or $(4, 7)$ mod 8 then $\widetilde{KO}_G^{-1}(\Sigma^{p,q})\approx RO(G)$ (cf. [9]),

c) If $(p, q)\equiv(i, 3)$ or $(i, 7)$ mod 8 for $i=1, 2, 3, 5, 6$ or $7$ then $\widetilde{KO}_G^{-1}(\Sigma^{p,q})\approx Z$.

In this section we provide preliminary lemmas. By [9], Lemma 4.1 and [13], $\widetilde{KO}_G^{-1}(\Sigma^{p,q}/\Sigma^{0,q})\approx KO^{-q-2}(P^{p-1})$, where $P^n$ is the real projective $n$-space. This isomorphism and the exact sequence of the pair $(\Sigma^{p,q}, \Sigma^{0,q})$ induce the exact sequence:

$$(2.2) \qquad \to KO^{-q-2}(P^{p-1}) \to \widetilde{KO}_G^{-1}(\Sigma^{p,q}) \xrightarrow{i^*} \widetilde{KO}_G^{-1}(\Sigma^{0,q}) \to KO^{-q-1}(P^{p-1}) \to$$

where $i$ is the inclusion $\Sigma^{0,q}\subset\Sigma^{p,q}$.

**Lemma 2.3.** *Let $t$ be a generator of $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$, $p\geqq 1$. Then*

i) *$\rho(t)=-t$ and $Ht=t$ in case of* (2.1), a),

ii) *$\rho(t)=-Ht$ in case of* (2.1), b),

iii) *$\rho(t)=t$ and $Ht=-t$ in case of* (2.1), c).

Proof. i) Using [2], (5.1) of $KO_G^*$ we see that $\psi\colon \widetilde{KO}_G^{-1}(\Sigma^{p,q})\to\widetilde{KO}^{-1}(S^{p+q})$ is monic. From this and the equalities $\psi\rho=-\psi$ ([2], §3) and $\psi H=\psi$ the claim follows immediately.

ii) From the equality $\rho^2=1$ it follows that $\rho(t)=\pm t$ or $\pm Ht$. Since $\psi(t)$ is a generator of $\widetilde{KO}^{-1}(S^{p+q})$, the relation $\psi\rho=-\psi$ implies that $\rho(t)=-t$ or

$-Ht$. Using [8], by (2.2) we get $i^*(t)=2^s(1-H)t_0$ for some $s>0$ up to sign, where $t_0$ denotes a generator of $\widetilde{KO}_G^{-1}(\Sigma^{0,q})$. This and the relation $i^*\rho=i^*$ (cf. [2]) show that $\rho(t)=-Ht$.

iii) Analogous to ii). To prove the second we have to use that $i^*$ is a homomorphism of $RO(G)$-modules. q.e.d.

**Lemma 2.4.** i) *In case of* (2.1), a), *if* $(p, q)\equiv(2, 1)$ *or* $(6, 5)$ mod 8 *then* $\psi\colon \widetilde{KO}_G^{-1}(\Sigma^{p,q})\approx\widetilde{KO}^{-1}(S^{p+q})$, *and if* $(p, q)\equiv(2, 5)$ *or* $(6, 1)$ mod 8 *then* $\psi$ *carries a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ *to twice a generator of* $\widetilde{KO}^{-1}(S^{p+q})$. *And* $\phi=0$ *on* $\widetilde{KO}^{-1}(S^q)\left[\frac{1}{2}\right]$,

ii) *In case of* (2.1), b), $\psi(t)$ *and* $\phi(t)$ *generate* $\widetilde{KO}^{-1}(S^{p+q})$ *and* $\widetilde{KO}^{-1}(S^q)\left[\frac{1}{2}\right]$,

iii) *In case of* (2.1), c), $\psi=0$ *on* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})\left[\frac{1}{2}\right]$ *and* $\phi(t)$ *generates* $\widetilde{KO}^{-1}(S^q)\left[\frac{1}{2}\right]$, *where t denotes a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$.

Proof. i) Making use of [2], (5.1) of $KO_G^*$ we get Im $\psi=\widetilde{KO}^{-1}(S^{p+q})$ or $2\widetilde{KO}^{-1}(S^{p+q})$ according as $p+q\equiv 3$ or 7 mod 8. The second is immediate.

ii) The claim for $\psi$ is obvious by the definition of $t$. As in the proof of Lemma 2.3, ii) we obtain $i^*(t)=2^s(1-H)t_0$, with the above notation, when $p>0$. So $\phi(t)$ is $2^s$ times of a generator of $\widetilde{KO}^{-1}(S^q)$ and hence the claim for $p>0$ follows immediately. Clear when $p=0$.

iii) Since $\widetilde{KO}^{-1}(S^{p+q})$ is a 2-group, the claim for $\psi$ is obvious. For $\phi$, same as in ii). q.e.d.

**Lemma 2.5.** *For* $\chi\colon \widetilde{KO}_G^{-1}(\Sigma^{p,q})\to\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$, $p\geqq 1$, $q\geqq 0$, *and* $\hat{\eta}\colon KO_G^{-1}(\Sigma^{p-1,q})\to\widetilde{KO}_G^{-1}(\Sigma^{p,q})$, $p\geqq 2$, $q\geqq 1$, *we have*

i) *If* $(p, q)\equiv(2, 3)$, $(3, 3)$, $(6, 7)$ *or* $(7, 7)$ mod 8 *then* $\chi\colon \widetilde{KO}_G^{-1}(\Sigma^{p,q})\approx\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$, *and* $\hat{\eta}$ *carries a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$ *to twice a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$,

ii) *If* $(p, q)\equiv(2, 7)$, $(3, 7)$, $(6, 3)$ *or* $(7, 3)$ mod 8 *then* $\chi$ *carries a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ *to twice a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$, *and* $\hat{\eta}\colon \widetilde{KO}_G^{-1}(\Sigma^{p-1,q})\approx\widetilde{KO}_G^{-1}(\Sigma^{p,q})$,

iii) *If* $(p, q)\equiv(1, 3)$, $(5, 3)$, $(1, 7)$ *or* $(5, 7)$ mod 8 *then* $\chi$ *carries a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ *to* $(1+\rho)$ *or* $(1-H)$ *times a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$ *according as* $p>1$ *or* $p=1$, *and* $\hat{\eta}$ *is epic,*

iv) *If* $(p, q)\equiv(0, 3)$, $(4, 3)$, $(0, 7)$ *or* $(4, 7)$ mod 8 *then* $\chi$ *is epic, and* $\hat{\eta}$ *carries* *a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$ *to* $(1+\rho)$ *times a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$.

Proof. For the proof we use Lemma 2.3 freely. The claims for $\chi$ follow immediately from [2], (5.1) of $KO_G^*$. Hence i)–iii) for $\hat{\eta}$ follow easily from the results for $\chi$ and (1.4), i). We now prove iv) for $\hat{\eta}$. Let $t$ be a generator of $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ and write $\chi(t)=t'$. Then $t'$ becomes a generator of $\widetilde{KO}_G^{-1}(\Sigma^{p-1,q})$ by iv) for $\chi$. Put $\hat{\eta}(t')=(a+bH)t$, then, by (1.4), i) we see that $a-b=2$. By (1.4), ii) $\psi\hat{\eta}(t')=\eta\psi(t')=0$ because $\widetilde{KO}^{-1}(S^{p+q-1})=0$. This implies that $a+b=0$ and therefore $a=-b=1$. Hence $\hat{\eta}(t')=(1-H)t=(1+\rho)t$. q.e.d.

Finally we consider

$$\sigma\colon \widetilde{KR}^{-1}(\Sigma^{p,q}) \to \widetilde{KO}_G^{-1}(\Sigma^{p,q})$$

when $\widetilde{KR}^{-1}(\Sigma^{p,q})\approx Z$.

**Lemma 2.6.** i) *If* $(p, q)\equiv(2, 1)$ *or* $(6, 5)$ mod 8 *then* $\sigma\colon \widetilde{KR}^{-1}(\Sigma^{p,q})\approx \widetilde{KO}_G^{-1}(\Sigma^{p,q})$, *and if* $(p, q)\equiv(2, 5)$ *or* $(6, 1)$ mod 8 *then* $\sigma$ *carries a generator of* $\widetilde{KR}^{-1}(\Sigma^{p,q})$ *to twice a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$,

ii) *If* $(p, q)\equiv(0, 3)$, $(4, 3)$, $(0, 7)$ *or* $(4, 7)$ mod 8 *then* $\sigma$ *carries a generator of* $\widetilde{KR}^{-1}(\Sigma^{p,q})$ *to* $(1-\rho)$ *or* $(1+H)$ *times a generator of* $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ *according as* $p>0$ *or* $p=0$.

Proof. By $r$ and $c$ we denote the realification homomorphism and the complexification homomorphism. Then it is clear that $\psi\sigma=rc$. Let $\bar{t}$, $t$ and $t_0$ be generators of $\widetilde{KR}^{-1}(\Sigma^{p,q})$, $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$ and $\widetilde{KO}^{-1}(S^{p+q})$ respectively. From the relations between $c$, $r$ and the coefficients of $K$-theories it follows that

$$rc(\bar{t}) = \begin{cases} t_0 & \text{for } (p, q)\equiv(2, 1), (6, 5) \text{ mod } 8 \\ 4t_0 & \text{for } (p, q)\equiv(2, 5), (6, 1) \text{ mod } 8 \\ 2t_0 & \text{otherwise} \end{cases}$$

up to sign.

i) Immediate from this and Lemma 2.4, i).

ii) Since $\widetilde{KR}^{-1}(\Sigma^{0,q})\approx KO^{-1}(S^q)$ the assertion for $p=0$ is obvious. So we consider the case $p>0$. Put $\sigma(\bar{t})=(a+bH)t$. Since $\psi(t)=t_0$ (up to sign) and $rc(\bar{t})=2t_0$, we get $a+b=\pm 2$. $\rho$ acts on $\widetilde{KR}^{-1}(\Sigma^{p,q})$ as $-1$ because $c$ is monic. Using this and Lemma 2.3, ii), from the equality $\sigma\rho=\rho\sigma$ it follows that $a=b$ and hence $a=b=\pm 1$. This and Lemma 2.3, ii) complete the proof. q.e.d.

## 3. Im $J_G$ and Im $J_R$

Let

$$I_{p,q}=\mathrm{Im}\,\{J_G\colon \widetilde{KO}_G^{-1}(\Sigma^{p,q})\to\pi_{p,q}^S\}\qquad\text{for } q\geqq 1\,.$$

Denote by $t$ a generator of $\widetilde{KO}_G^{-1}(\Sigma^{p,q})$, so that $\widetilde{KO}_G^{-1}(\Sigma^{p,q})=Z\cdot t$ or $RO(G)\cdot t$ in our cases. Put

$$\alpha=J_G(t)\,.$$

Let $m(s)$ be as in Theorem 1.6 and $\nu_p(s)$, $p$ prime, be the power in $s$. (We write $\rho(x)=\rho x$.)

**Theorem 3.1.** *Suppose that* $(p, q)\equiv(2, 1)$, $(6, 1)$, $(2, 5)$ *or* $(6, 5)$ mod 8. *Then*

$$I_{p,q}=Z/m\left(\frac{p+q+1}{2}\right)\cdot\alpha\,.$$

**Theorem 3.2.** *Suppose that* $(p, q)\equiv(i, 3)$ *or* $(i, 7)$ mod 8 *for* $i=1, 2, 3, 5, 6$ *or* 7. *Then*

$$I_{p,q}=Z/m\left(\frac{q+1}{2}\right)\cdot\alpha\,.$$

**Theorem 3.3.** i) *Suppose that* $q\equiv 3$ *or* 7 mod 8. *Then*

$$I_{0,q}=Z/m\left(\frac{q+1}{2}\right)\cdot\alpha\oplus Z/m\left(\frac{q+1}{2}\right)\cdot\beta$$

*where* $\beta=J_G(Ht)$,

ii) *Suppose that* $\varepsilon=0$ *or* 1, $\nu_2(2q+\varepsilon)\leqq\nu_2(p+q+\varepsilon)$ *and*

$$\left(\frac{m(4p+4q+4\varepsilon)}{2^{\nu_2(p+q+\varepsilon)+4}d},\ d\right)=1$$

*where*

$$d=\left(\frac{m(4p+4q+4\varepsilon)}{2^{\nu_2(p+q+\varepsilon)+4}},\ \frac{m(4q+2\varepsilon)}{2^{\nu_2(2q+\varepsilon)+3}}\right).$$

*Then, for* $8p+4\varepsilon>0$

$$I_{8p+4\varepsilon,8q+4\varepsilon-1}=Z/\frac{m(4p+4q+4\varepsilon)m(4q+2\varepsilon)}{2^{\nu_2(2q+\varepsilon)+2}d}\cdot\alpha\oplus Z/2^{\nu_2(2q+\varepsilon)+2}d\cdot\gamma$$

*where*

$$\gamma=\left(\frac{m(4p+4q+4\varepsilon)}{2^{\nu_2(2p+\varepsilon)+3}d}+\frac{m(4q+2\varepsilon)}{2^{\nu_2(2q+\varepsilon)+3}}\right)(1+\rho)\alpha-\frac{m(4p+4q+4\varepsilon)}{2^{\nu_2(2p+\varepsilon)+2}d}\,\alpha\,.$$

**Corollary 3.4.**

i) $I_{4p,4p-1}=Z/2m(4p)\cdot\alpha\oplus Z/\dfrac{m(2p)}{2}\cdot\gamma$

*where*

$$\gamma=\left(\frac{m(4p)}{m(2p)}+\frac{m(2p)}{2^{\nu_2(p)+3}}\right)(1+\rho)\alpha-\frac{2m(4p)}{m(2p)}\alpha\,,$$

ii) *Suppose that* $\nu_3(p)\leqq\nu_3(2p-1)$. *Then*

$$I_{4,8p-5}=Z/\frac{m(4p)m(4p-2)}{4\cdot 3^{\nu_3(p)+1}}\cdot\alpha\oplus Z/4\cdot 3^{\nu_3(p)+1}\cdot\gamma$$

*where*

$$\gamma=\left(\frac{m(4p)}{8\cdot 3^{\nu_3(p)+1}}+\frac{m(4p-2)}{8}\right)(1+\rho)\alpha-\frac{m(4p)}{4\cdot 3^{\nu_3(p)+1}}\alpha\,.$$

REMARK. We can take $\gamma$ to be the following simpler elements respectively in Corollary 3.4:

i) $\dfrac{m(4p)}{2m(2p)}((1+\rho)\alpha-4\alpha)$,

ii) $(1+\rho)\alpha-\dfrac{3^{4p}-1}{4}\alpha$ for $\nu_3(2p-1)-2\leqq\nu_3(p)\leqq\nu_3(2p-1)$. ( ii) was suggested by the computation of $\pi^S_{4,3}$ by Araki-Iriye [3] and also this suggested (5.4) of §5.)

Let $o(x)$ denote the order of an element $x$ of a finite group. Then we obtain

**Proposition 3.5.** *Let* $\alpha\in\pi^S_{8p+4\varepsilon,8q+4\delta-1}$, $8p+4\varepsilon>0$, *where* $\varepsilon,\delta=0$ *or* 1. *Then*

i) $o((1-\rho)\alpha)=m(4p+4q+2\varepsilon+2\delta)$,

$o((1+\rho)\alpha)=m(4q+2\delta)$,

ii) $o(\alpha)=\dfrac{m(4p+4q+2\varepsilon+2\delta)m(4q+2\delta)}{2^{\zeta}d}$

*where*

$$d=\left(\frac{m(4p+4q+2\varepsilon+2\delta)}{2^{\nu_2(2p+2q+\varepsilon+\delta)+3}},\ \frac{m(4q+2\delta)}{2^{\nu_2(2q+\delta)+3}}\right)$$

*and* $\zeta$ *is the following integer*:

$$\begin{array}{ll}
\nu_2(2q+\varepsilon)+2 & \text{if } \varepsilon=\delta \text{ and } \nu_2(2q+\varepsilon)\leqq\nu_2(p+q+\varepsilon)\,,\\
\nu_2(2q+\varepsilon)+2 \text{ or } \nu_2(2q+\varepsilon)+3 & \\
 & \text{if } \varepsilon=\delta \text{ and } \nu_2(2q+\varepsilon)=\nu_2(p+q+\varepsilon)+1\,,\\
\nu_2(p+q+\varepsilon)+3 & \text{if } \varepsilon=\delta \text{ and } \nu_2(2q+\varepsilon)\geqq\nu_2(p+q+\varepsilon)+2\,,\\
2 \text{ or } 3 & \text{if } \varepsilon=0 \text{ and } \delta=1\,,\\
2 & \text{if } \varepsilon=1 \text{ and } \delta=0\,.
\end{array}$$

For a finite abelian group $A$, we denote by $A_p$ the $p$-component of $A$ and by $\tilde{x}\in A_p$ the $p$-component of $x\in A$.

**Proposition 3.6.** i) *Suppose that* $\nu_2(2q+\varepsilon)\leqq\nu_2(p+q+\varepsilon)$, $\varepsilon=0$ *or* 1. *Then*

$$(I_{8p+4\varepsilon,8q+4\varepsilon-1})_2 = Z/2^{\nu_2(p+q+\varepsilon)+5}\cdot\tilde{\alpha}\oplus Z/2^{\nu_2(2p+\varepsilon)+2}\cdot\tilde{\gamma}\,.$$

*for* $8p+4\varepsilon>0$ *where*

$$\gamma = (1+\rho)\alpha-2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}\alpha\,,$$

ii) *Suppose that* $\nu_2(2q+\varepsilon)\geqq\nu_2(p+q+\varepsilon)+2$, $\varepsilon=0$ *or* 1. *Then*

$$(I_{8p+4\varepsilon,8p+4\varepsilon-1})_2 = Z/2^{\nu_2(2q+\varepsilon)+4}\cdot\tilde{\alpha}\oplus Z/2^{\nu_2(2p+\varepsilon)+2}\cdot\tilde{\gamma}$$

*for* $8p+4\varepsilon>0$ *where*

$$\gamma = \frac{3^{4p+2\varepsilon}-1}{2^{\nu_2(2p+\varepsilon)+3}}(1+\rho)\alpha-\frac{3^{4p+4q+4\varepsilon}-1}{2^{\nu_2(2p+\varepsilon)+2}}\alpha\,,$$

iii) $(I_{8p+4,8q-1})_2=Z/2^{\nu_2(q)+5}\cdot\tilde{\alpha}\oplus Z/4\cdot\tilde{\gamma}$
*for* $p\geqq0$ *where*

$$\gamma = \frac{3^{4p+2}-1}{8}(1+\rho)\alpha-\frac{3^{4p+4q+2}-1}{4}\alpha\,.$$

Let

$$\bar{I}_{p,q} = \mathrm{Im}\,\{J_R\colon \widetilde{KR}^{-1}(\Sigma^{p,q})\to\pi^S_{p,q}\}\quad\text{for } q\geqq1\,.$$

Suppose that $\widetilde{KR}^{-1}(\Sigma^{p,q})\approx Z$, generated by $\bar{t}$, and set

$$\bar{\alpha} = J_R(\bar{t})\,.$$

Then we have

**Theorem 3.7.**

$$\bar{I}_{p,q} = \begin{cases} Z/\dfrac{m(2s)}{2}\cdot\bar{\alpha} & \textit{if } (p,q)\equiv(2,5) \textit{ or } (6,1) \bmod 8 \\ Z/m(2s)\cdot\bar{\alpha} & \textit{if } (p,q)\equiv(2,1),(6,5),(0,3),(4,3) \\ & \qquad (0,7) \textit{ or } (4,7) \bmod 8 \end{cases}$$

*where* $s=\dfrac{p+q+1}{2}$.

## 4. Proofs of Theos. 3.1, 3.2 and 3.3, i)

Proof of Theorem 3.1. Put $m=m\left(\dfrac{p+q+1}{2}\right)$ for brevity.

We divide the proof into two cases: (i) $(p, q)\equiv(2, 1)$ or $(6, 5)$ mod 8 and (ii) $(p, q)\equiv(2, 5)$ or $(6, 1)$ mod 8.

Let $t_0$ be a generator of $\widetilde{KO}^{-1}(S^{p+q})$. By Lemma 2.4, i), (1.2) and (1.3) we have

$$\psi(\alpha) = \begin{cases} J_o(t_0) & \text{for (i)} \\ 2J_o(t_0) & \text{for (ii)} \end{cases}$$

up to sign and $\phi=0$ on $\pi^S_{p,q}\left[\frac{1}{2}\right]$ in any case. Therefore, by Theorems 1.6 and 1.7 we see that the odd components of $o(\alpha)$ are equal to those of $m$, and also $m\,|\,o(\alpha)$ for (i) and $\frac{m}{2}\,|\,o(\alpha)$ for (ii). Since $\psi$ of $KO^*_G$ is monic, $\psi^3(t)=3^{(p+q+1)/2}t$. From Theorem 1.5 and the equality $\nu_2(3^{(p+q+1)/2}-1)=\nu_2(m)$ ([1, II], Lemma (2.12)) it follows that $\nu_2(m)\geqq\nu_2(o(\alpha))$. Hence $o(\alpha)=m$ for (i).

For (ii) itsuffices to prove that $\frac{m}{2}\alpha\neq0$. Observe [2], (5.1) of $KO^*_G$. Then, using (1.2) we have $\delta(J_o(t_0))=\alpha$ (up to sign). So if $\frac{m}{2}\alpha=0$, then, by [2], (10.5) there exists an element $\beta$ of $\pi^S_{p-1,q+1}$ such that $\psi(\beta)=\frac{m}{2}J_o(t_0)$. Using the equalities $\widetilde{KO}_G(\Sigma^{8n+p-1,8n+q+1})=\widetilde{KO}_G(\Sigma^{8n+p-1,8n+q+2})=0$ and the formula for $\psi^k$ on $\widetilde{KO}_G(\Sigma^{8n,8n})$ of [9] we see that $\beta$ gives rise to a contradiction such that $e'_R\left(\frac{m}{2}J_o(t_0)\right)=0$ [1, IV]. Therefore also we have $o(\alpha)=m$ for (ii) Thus we obtain $\operatorname{Im} J_G=Z/m\cdot\alpha$ because $\widetilde{KO}^{-1}_G(\Sigma^{p,q})=Z\cdot t$.

Proof of Theorem 3.2. Let $t\in\widetilde{KO}^{-1}_G(\Sigma^{8p+4\varepsilon+i,8q+4\delta-1})$, $0\leqq\varepsilon$, $\delta\leqq1$ and $1\leqq i\leqq3$. By Lemma 2.5 we have

$$\chi^i(t) = (1+\rho)t_0 \quad\text{or}\quad (1-H)t_0$$

according as $8p+4\varepsilon>0$ or $8p+4\varepsilon=0$ for $(\varepsilon,\delta)=(0,1)$ or $(1,0)$ and

$$\hat{\eta}^{4-i}(t) = (1+\rho)t_1$$

for $(\varepsilon,\delta)=(0,0)$ or $(1,1)$, where $t_s$, $s=0,1$, is a generator of $\widetilde{KO}^{-1}_G(\Sigma^{8p+4\varepsilon+4s,8q+4\delta-1})$. Therefore, using (1.2) and (1.4), iii), we get

(*) $$m(4q+2\delta)\,|\,o(\alpha)$$

by Theorem 3.3, i) and Proposition 3.5, i), which will be proved in the following.

The proof of the inverse of (*) is analogous to that of Theorem 3.1 for (i). Then we have need of the equality $\psi^3(t)=3^{4q+2\delta}t$, which is obtained by the injectivity of $\chi^i$ or $\hat{\eta}^{4-i}$ and the equality $\psi^3((1-H)t_s)=3^{4q+2\delta}(1-H)t_s$, $0\leqq s\leqq1$ [9].

Proof of Theorem 3.3, i). Let $t_0$ be a generator of $\widetilde{KO}^{-1}(S^q)$. Then, $\psi(\alpha)$

$=\psi(\beta)=J_o(t_0)$, $\phi(\alpha)=J_o(t_0)$ (up to sign) and $\phi(\beta)=0$ clearly. We get $o(\alpha)=o(\beta)$ $=m\left(\frac{q+1}{2}\right)$ as in the above, so that it follows that $\alpha$ and $\beta$ have no relation.

## 5. Proofs of Theos. 3.3, ii), 3.7, Cor. 3.4, Props. 3.5 and 3.6

As we saw in §4, by Theorems 1.6, 1.7, Lemma 2.4, ii), (1.2) and (1.3) we can determine the odd components of the order of elements of Im $J_G$. So we observe only the 2-component. Let $o_2(\ )$ denote the 2-component of $o(\ )$. We use freely the equalities

$$\nu_2(3^{2s}-1)=\nu_2(m(2s))=\nu_2(s)+3$$

([1, II], p. 139 and Lemma (2.12)) in the following.

Let $\alpha$ and $\tilde{\alpha}$ be as in §3. By Lemma 2.3, ii) and (1.2)

$$\rho\alpha=-J_G(Ht),\quad t\in\widetilde{KO}_G^{-1}(\Sigma^{8p+4\varepsilon,8q+4\delta-1}).$$

We begin by considering $o((1-\rho)\alpha)$. Write $t_0=\psi(t)$. Then $t_0$ is a generator of $\widetilde{KO}^{-1}(S^{8p+8q+4\varepsilon+4\delta-1})$. Put $\delta(t_0)=(a+bH)t$. By the definition of $\delta$ we get $\psi\delta(t_0)=2t_0$, hence $a+b=2$. Using [9], Lemma 3.3 and Proposition 3.5, from the equality $\delta\psi\psi^2=\psi^2\delta\psi$ it follows that $a=b$. Therefore $a=b=1$, so that $\delta(t_0)=(1+H)t$. This and (1.2) show $\delta(J_o(t_0))=(1-\rho)\alpha$. Since $o(J_o(t_0))=m(4p+4q+2\varepsilon+2\delta)$ by Theorem 1.6, we therefore have

$$o((1-\rho)\alpha)\,|\,m(4p+4q+2\varepsilon+2\delta).$$

Suppose that $o((1-\rho)\alpha)=\frac{m(4p+4q+2\varepsilon+2\delta)}{2}$. Then, by [2], (10.5) we have an element $\beta\in\pi^S_{8p+4\varepsilon-1,8q+4\delta}$ such that $\psi(\beta)=\frac{m(4p+4q+2\varepsilon+2\delta)}{2}J_o(t_0)$. As in the proof of Theorem 3.1, $\beta$ yields a contradiction: $e'_R(\frac{m(4p+4q+2\varepsilon+2\delta)}{2}J_o(t_0))$ $=0$. This proves

$$o((1-\rho)\alpha)=m(4p+4q+2\varepsilon+2\delta)\,. \tag{5.1}$$

By Theorem 1.5, [9], Lemma 3.3 and Proposition 3.5, we obtain

$$\frac{3^{4p+4q+2\varepsilon+2\delta}-1}{2}(\rho-1)\tilde{\alpha}=\frac{3^{4q+2\delta}-1}{2}(\rho+1)\bar{\alpha}\,. \tag{5.2}$$

From (5.1) and (5.2) it follows that

$$o((1+\rho)\alpha)=m(4q+2\delta)\,. \tag{5.3}$$

Using (5.3), the equality (5.2) can be transformed into the following

(5.4) $$(3^{4p+4q+2\varepsilon+2\delta}-1)\tilde{\alpha}-\frac{3^{4p+2\varepsilon}-1}{2}(1+\rho)\tilde{\alpha}=0\,.$$

Proof of Proposition 3.5. i) Immediate from (5.1) and (5.3).

ii) The proof breaks up into the 4 cases.

Case 1) $\varepsilon=\delta$ and $\nu_2(2q+\varepsilon)\leqq\nu_2(p+q+\varepsilon)$. We have

(*) $$2^{\nu_2(p+q+\varepsilon)+3}(1+\rho)\tilde{\alpha}=0$$

by (5.3). By (5.1), $2^{\nu_2(p+q+\varepsilon)+4}(1-\rho)\tilde{\alpha}=0$. Hence we get

$$2^{\nu_2(p+q+\varepsilon)+5}\tilde{\alpha}=0\,.$$

If $2^{\nu_2(p+q+\varepsilon)+4}\tilde{\alpha}=0$, then

$$\begin{aligned}2^{\nu_2(p+q+\varepsilon)+3}\tilde{\alpha}&=-2^{\nu_2(p+q+\varepsilon)+3}\tilde{\alpha}\\&=2^{\nu_2(p+q+\varepsilon)+3}\rho\tilde{\alpha}\qquad\text{by (*)}\,.\end{aligned}$$

This shows $2^{\nu_2(p+q+\varepsilon)+3}(1-\rho)\tilde{\alpha}=0$, which contradicts to (5.1). Therefore we have

(5.5) $$o_2(\alpha)=2^{\nu_2(p+q+\varepsilon)+5}\,.$$

By (5.4) and (5.5) we have $o_2((1+\rho)\alpha)=2^{\nu_2(2p+\varepsilon)+3}$. Comparing this with (5.3) we get

(5.6) $$\nu_2(2p+\varepsilon)=\nu_2(2q+\varepsilon)\,.$$

Case 2) $\varepsilon=\delta$ and $\nu_2(2q+\varepsilon)=\nu_2(p+q+\varepsilon)+1$. As in Case 1) we can prove $\nu_2(o_2(\alpha))\leqq\nu_2(2q+\varepsilon)+4$. To show the inverse we have to use the fact that $(\mathrm{Im}\,J_o)_2=Z/2^{\nu_2(2q+\varepsilon)+3}\cdot\psi(\alpha)$ (Lem. 2.4, ii) and Theo. 1.6). Hence we have

(5.7) $$o_2(\alpha)=2^{\nu_2(2q+\varepsilon)+3}\ \text{ or }\ 2^{\nu_2(2q+\varepsilon)+4}\,.$$

Case 3) $\varepsilon=\delta$ and $\nu_2(2q+\varepsilon)\geqq\nu_2(p+q+\varepsilon)+2$. The same argument as in Case 1) shows

(5.8) $$o_2(\alpha)=2^{\nu_2(2q+\varepsilon)+4}$$

and

(5.9) $$\nu_2(2p+\varepsilon)=\nu_2(p+q+\varepsilon)+1\,.$$

Case 4) $(\varepsilon,\delta)=(0,1)$ or $(1,0)$. As in Cases 1) and 2) we can prove

(5.10) $$o_2(\alpha)=2^3\ \text{ or }\ 2^4\ \text{ for }\ (\varepsilon,\delta)=(0,1)\ \text{ and }\ 2^{\nu_2(q)+5}\quad\text{for }(\varepsilon,\delta)=(1,0).$$

(5.5), (5.7), (5.8) and (5.10) complete the proof of ii).

Proof of Theorem 3.3, ii). For the simplicity we put

$$m(4p+4q+4\varepsilon)=2^{\nu_2(p+q+\varepsilon)+4}ad \text{ and } m(4q+2\varepsilon) = 2^{\nu_2(2q+\varepsilon)+3}bd$$

where $d$ is as in §3. Then $a$, $b$ and $d$ are odd and $(a, b)=1$.

Since $\nu_2(2p+\varepsilon)=\nu_2(2q+\varepsilon)$ by (5.6), we have

$$2^{\nu_2(2p+\varepsilon)+2}(1+\rho)\tilde{\alpha} = 2^{\nu_2(p+q+\varepsilon)+4}\tilde{\alpha} \tag{5.11}$$

by (5.4), (5.3) and (5.5).

Define

$$\gamma = (2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+1}a+bd)\,(1+\rho)\alpha-2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}a\alpha\,.$$

We observe $o(\gamma)$. Since $\nu_2(p+q+\varepsilon)\geqq\nu_2(2p+\varepsilon)=\nu_2(2q+\varepsilon)$, $o_2((1+\rho)\alpha)=2^{\nu_2(2q+\varepsilon)+3}$ and $o_2(\alpha)=2^{\nu_2(p+q+\varepsilon)+5}$ we have

$$2^{\nu_2(2p+\varepsilon)+2}\tilde{\gamma} = 0$$

by (5.11). Note that

$$\psi(\gamma) = -2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}a\psi(\alpha),\ \phi(\gamma) = 2bd\phi(\alpha)$$

and

$$o(\psi(\alpha)) = 2^{\nu_2(p+q+\varepsilon)+4}ad, \textit{ the odd component of } o(\phi(\alpha)) = bd\,.$$

Since $\psi(2^{\nu_2(2q+\varepsilon)+1}\tilde{\gamma})=-2^{\nu_2(p+q+\varepsilon)+3}a\psi(\tilde{\alpha})\neq 0$,

$$o(\tilde{\gamma}) = 2^{\nu_2(2++\varepsilon)+2}$$

and so

$$o(\gamma) = 2^{\nu_2(2q+\varepsilon)+2}d\,. \tag{5.12}$$

From the assumption such that $(a, d)=1$ it follows that

$$(2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+1}a+bd,\ 2^{\nu_2(2p+\varepsilon)+3}bd) = 1\,. \tag{*}$$

Using (5.3), we then have

$$(1+\rho)\alpha = x(\gamma+2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}a\alpha)$$

for some integer $x$. This shows that $\alpha$ and $\gamma$ generate $\mathrm{Im}\,J_G$.

Next we prove that there is no relation between $\alpha$ and $\gamma$. Suppose that $x\alpha=y\gamma$. Then

$$(x+2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}ay)\alpha = y(2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+1}a+bd)\,(1+\rho)\alpha$$

and

$$(x+2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}ay)\psi(\alpha) = 0\,.$$

Hence

$$2^{\nu_2(p+q+\varepsilon)+4}ad \mid x+2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+2}ay\,.$$

Since $o(\alpha)=2^{\nu_2(p+q+\varepsilon)+5}abd$,

$$2by(2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+1}a+bd)\,(1+\rho)\alpha=0\,.$$

So

$$2^{\nu_2(2q+\varepsilon)+3}bd \mid 2by(2^{\nu_2(p+q+\varepsilon)-\nu_2(2p+\varepsilon)+1}a+bd)$$

since $o((1+\rho)\alpha)=2^{\nu_2(2q+\varepsilon)-3}\,bd$. From this and (*), it follows that

$$2^{\nu_2(2q+\varepsilon)+2}d \mid y\,.$$

Therefore $x\alpha=y\gamma=0$ by (5.12).

Proof of Corollary 3.4. i) Let $p=q$ and replace $2p+\varepsilon$ by $p$ in Theorem 3.3, ii). If $\nu_q(m(2p))\geqq 1$ for odd prime $q$ then $\nu_q(m(2p))=\nu_q(m(4p))$ ([1, II], p. 139). Hence, $d=\dfrac{m(2p)}{2^{\nu_2(p)+3}}$ and so the assumption holds. The remaining follows easily.

ii) Take $p=0$ and $\varepsilon=1$, and replace $q$ by $p-1$ in Theorem 3.3, ii). By the definition of $m(t)$ [1, II] we have

$$\nu_3(m(4p-2))=1+\nu_3(2p-1),\ \nu_3(m(4p))=1+\nu_3(p)$$

and if $\nu_q(m(4p-2))\geqq 1$ for a prime $q\geqq 5$, then $\nu_q(m(4p))\doteq 0$. Therefore,

$$d=3^{1+\min\{\nu_3(p),\nu_3(2p-1)\}}=3^{1+\nu_3(p)}$$

using the condition. This shows that the assumption holds. The rest is immediate.

Proof of Proposition 3.6. Similar to the proof of Theorem 3.3, ii). We make only a remark. In order to determine $o_2(\gamma)$ we have to use (5.4) and $o_2(\psi^r(\alpha))$. From $o_2(\gamma)$, $o_2(\alpha)$ and $o_2((1+\rho)\alpha)$ it follows that there is no relation between $\alpha$ and $\gamma$.

Proof of Theorem 3.7. This theorem follows immediately from Lemma 2.6, Theorem 3.1 and Proposition 3.5, i).

ADDENDUM TO THE REMARK OF §3. This is proved by the analogous argument to Theorem 3.3, ii) and Proposition 3.6. We note that in order to determine $o(\gamma)$ in ii) we have need of the inequality of [1, II], Lemma (2.12): $\nu_q(3^{2s}-1)\geqq \nu_q(m(2s))$, $q$ prime $\geqq 5$.

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