Iriye, K.
Osaka J. Math.
19 (1982), 733–743

EQUIVARIANT STABLE HOMOTOPY GROUPS OF SPHERES WITH INVOLUTIONS, II

KOUYEMON IRIYE

(Received October 31, 1980)

Introduction. This note is the second paper in a series of papers with the same title, in which we aim at computing of the equivariant stable homotopy groups of spheres with linear involutions. In the present paper we give the computation of $\pi_{p,q}^S$ for $p+q=9$, 10 and 11. At the end of this paper we list the tables of $\pi_{p,q}^S$ for $0 \leq p+q \leq 13$ and $-1 \leq q \leq p$, and $\lambda_{p,q}^S$ for $0 \leq p+q \leq 13$. The computations of 12 and 13 stems are similar situations to 4 and 5 stems since $\pi_{12}^S=0$ and $\pi_{13}^S=\boldsymbol{Z}/3$, hence they are easily obtained by the forgetful exact sequence.

We quote the part I of this series by [I]. All other references are listed at the end of [I].

The author expresses his hearty gratitude to Professor S. Araki and Dr. A. Kono for their valuable suggestions.

1. 9 stem

The equivariant Toda bracket

$$\langle 120\mathring{\sigma}, 2, \xi_{3,2}^{*}\omega_2^{2n+1}\rangle^{\tau}$$

is well-defined. By parallel arguments to the case of $[\nu^2\omega_1^{4n+3}]_3$ (see [I], (15.2)), there exists an element

$$(1.1) \qquad [120\sigma\omega_1^{4n+3}]_3 \in \langle 120\mathring{\sigma}, 2, \xi_{3,2}^{*}\omega_2^{2n+1}\rangle^{\tau} \subset \pi_S^{4n+3,-4n-10}(S_+^{3,0})$$

such that

$$(1.1') \qquad \psi(\delta_3[120\sigma\omega_1^{4n+3}]_3) = \mu\,.$$

Put

$$(1.1'') \qquad \mathring{\mu}_{4n} = \delta_3[120\sigma\omega_1^{-4n+3}]_3 \in \langle 120\mathring{\sigma}, 2, \mathring{\eta}_{4n}\rangle^{\tau} \subset \pi_{4n,-4n+9}^S\,.$$

Then we have

Proposition 1.2. i) $\eta_{1,3}^{*}[120\sigma\omega_1^{4n+3}]_3 = 120\sigma\omega_1^{4n+3}$ *and* $[120\sigma\omega_1^{4n+3}]_3$ *is of* *order* 2.

ii) $\beta_1(\mathring{\mu}_{4n})=\mu\omega_1^{-4n}$ *and* $\chi^2\mathring{\mu}_{4n}=120\sigma\delta_1\omega_1^{-4n+3}$.

By parallel arguments to the computations in [I] we easily obtain

Proposition 1.3. $(\pi_S^{10-p,-q-1}(S_+^{10,0}),\ p+q=9)$

i) $\pi_S^{32n,-32n}(S_+^{10,0})=\boldsymbol{Z}\cdot\omega_{10}^n\oplus\boldsymbol{Z}/32\cdot(1+\rho)\omega_{10}^n\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\xi_{10,4}^*\omega_4^{4n}\bar{\omega}_4^{4n-1}$,

ii) $\pi_S^{32n+16,-32n-16}(S_+^{10,0})=\boldsymbol{Z}\cdot\delta_{10,1}\omega_1^{32n+17}\oplus\boldsymbol{Z}/32\cdot\hat{\eta}\xi_{10,9}^*\omega_9^{2n+1}$
$\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\xi_{10,4}^*\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$,

iii) $\pi_S^{16n+8,-16n-8}(S_+^{10,0})=\boldsymbol{Z}\cdot\delta_{10,1}\omega_1^{16n+9}\oplus\boldsymbol{Z}/16\cdot\hat{\eta}^2\xi_{10,8}^*\omega_8^{n+1}\bar{\omega}_8^n$
$\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\xi_{10,4}^*\omega_4^{2n+1}\bar{\omega}_4^{2n}$,

iv) $\pi_S^{8n+1,-8n-1}(S_+^{10,0})=\boldsymbol{Z}/2\cdot\beta_{10}(\chi\mathring{\eta}_{-8n})\oplus\boldsymbol{Z}/2\cdot\chi\hat{\nu}\xi_{10,8}^*\omega_8^n\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\hat{\sigma}\xi_{10,2}^*\omega_2^{4n-1}$
$\oplus\boldsymbol{Z}/2\cdot\mathring{\varepsilon}_5\xi_{10,2}^*\omega_2^{4n-1}$,

v) $\pi_S^{8n+2,-8n-2}(S_+^{10,0})=\boldsymbol{Z}\cdot\delta_{10,1}\omega_1^{8n+3}\oplus\boldsymbol{Z}/2\cdot\nu\mathring{\nu}\xi_{10,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/2\cdot\eta\sigma\xi_{10,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/2\cdot\rho\eta\sigma\xi_{10,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/2\cdot\varepsilon\xi_{10,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/2\cdot\rho\varepsilon\xi_{10,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{10,3}^*[120\sigma\omega_1^{8n-5}]_3\oplus\boldsymbol{Z}/2\cdot\rho\xi_{10,3}^*[120\sigma\omega_1^{8n-5}]_3$,

vi) $\pi_S^{8n+3,-8n-3}(S_+^{10,0})=\boldsymbol{Z}/2\cdot\beta_{10}(\chi^3\mathring{\nu}_{-8n})\oplus\boldsymbol{Z}/8\cdot\chi^5\hat{\sigma}\xi_{10,8}^*\omega_8^n\oplus\boldsymbol{Z}/2\cdot\xi_{10,1}^*(\mu\omega_1^{8n-6})$,

vii) $\pi_S^{8n+4,-8n-4}(S_+^{10,0})=\boldsymbol{Z}\cdot\delta_{10,1}\omega_1^{8n+5}\oplus\boldsymbol{Z}/8\cdot\hat{\eta}\xi_{10,7}^*[\eta^2\omega_1^{8n+2}]_7$,

viii) $\pi_S^{8n+5,-8n-5}(S_+^{10,0})=\boldsymbol{Z}/2\cdot\beta_{10}(\chi\mathring{\eta}_{-8n-4})\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\hat{\sigma}\xi_{10,2}^*\omega_2^{4n+1}$
$\oplus\boldsymbol{Z}/2\cdot\mathring{\varepsilon}_5\xi_{10,2}^*\omega_2^{4n+1}$,

ix) $\pi_S^{8n+6,-8n-6}(S_+^{10,0})=\boldsymbol{Z}\cdot\delta_{10,1}\omega_1^{8n+7}\oplus\boldsymbol{Z}/2\cdot\eta\sigma\xi_{10,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/2\cdot\varepsilon\xi_{10,2}^*\omega_2^{4n-1}$
$\oplus\boldsymbol{Z}/2\cdot(1+\rho)\eta\sigma\xi_{10,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/2\cdot\xi_{10,3}^*[120\sigma\omega_1^{8n-1}]_3\oplus\boldsymbol{Z}/2\cdot\rho\xi_{10,3}^*[120\sigma\omega_1^{8n-1}]_3$,

x) $\pi_S^{16n+7,-16n-7}(S_+^{10,0})=\boldsymbol{Z}/2\cdot\beta_{10}(\chi^7\mathring{\sigma}_{-16n})\oplus\boldsymbol{Z}/4\cdot\tilde{\nu}_3\xi_{10,4}^*\omega_4^{2n+1}\bar{\omega}_4^{2n}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{10,1}^*(\mu\omega_1^{16n-2})$,

xi) $\pi_S^{32n+15,-32n-15}(S_+^{10,0})=\boldsymbol{Z}/4\cdot\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}\oplus\boldsymbol{Z}/4\cdot\tilde{\nu}_3\xi_{10,4}^*\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$
$\oplus\boldsymbol{Z}/2\cdot\xi_{10,1}^*(\mu\omega_1^{32n+6})$,

xii) $\pi_S^{32n-1,-32n+1}(S_+^{10,0}) = \boldsymbol{Z}/8\cdot\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{2n}\bar{\omega}_8^{2n-1}$
$\oplus\boldsymbol{Z}/4\cdot(\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{2n}\bar{\omega}_8^{2n-1}-\tilde{\nu}_3\xi_{10,4}^*\omega_4^{4n}\bar{\omega}_4^{4n-1})\oplus\boldsymbol{Z}/2\cdot\xi_{10,1}^*(\mu\omega_1^{32n-10})$

for any integer n.

Proposition 1.4. *There holds the relation*

$$(1+\rho)\varepsilon\xi_{10,2}^*\omega_2^{4n-1} = (1+\rho)\eta\sigma\xi_{10,2}^*\omega_2^{4n-1}$$

for any integer n.

Computation of $\pi_S^{11-p,-q-1}(S_+^{11,0})$ for $p+q=9$.

Proposition 1.5. i) $\delta_{10,1}\omega_1^{64n}=0$, ii) $\delta_{10,1}\omega_1^{64n+32}=4\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{4n+2}\bar{\omega}_8^{4n+1}$, iii) $\delta_{10,1}\omega_1^{32n+16}=2\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}$, iv) $\delta_{10,1}\omega_1^{16n+8}=\beta_{10}(\chi^7\mathring{\sigma}_{-16n})$, v) $\delta_{10,1}\omega_1^{8n+4}=\beta_{10}(\chi^3\mathring{\nu}_{-8n})$, vi) $\delta_{10,1}\omega_1^{4n+2}=\beta_{10}(\chi\mathring{\eta}_{-4n})$, vii) $\delta_{10,1}\omega_1^{64n+1}=2\omega_{10}^{2n}-(1+\rho)\omega_{10}^{2n}$ *and* viii) $\delta_{10,1}\omega_1^{64n+33}=2\omega_{10}^{2n+1}-(1+\rho)\omega_{10}^{2n+1}$.

Proof. We prove only the relation iii). The other relations can be proved by similar arguments to [I], Proposition 10.7.

Since $\delta_{9,1}\omega_1^{32n+16}=2\chi^6\mathring{\sigma}\xi_{9,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}$, there exists $\alpha\in\pi_9^S$ satisfying

$$\delta_{10,1}\omega_1^{32n+16}=2\chi^5\mathring{\sigma}\xi_{10,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}+\xi_{10,1}^*(\alpha\omega_1^{32n+6})\,.$$

Therefore there exists $\beta\in\pi_{10}^S$ satisfying

$$\delta_{11,1}\omega_1^{32n+16}=2\chi^4\mathring{\sigma}\xi_{11,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}+\xi_{11,2}^*(\alpha\omega_2^{16n+3})+\xi_{11,1}^*(\beta\omega_1^{32n+5})\,.$$

Apply $\delta_{1,11}$ to both sides of this equality. Then

$$(\eta\alpha+2\beta)\omega_1^{32n+4}=0$$

Hence $\alpha\in \boldsymbol{Z}/2\cdot\eta^2\rho\oplus\boldsymbol{Z}/2\cdot\eta\varepsilon$, which implies $\xi_{10,1}^*(\alpha\omega_1^{32n+6})=0$. This completes the proof. $\square$

Thus we obtain

Proposition 1.6. $(\pi_S^{11-p,-q-1}(S_+^{11,0}),\ p+q=9)$

i) $\pi_S^{64n+1,-64n}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{64}\cdot\chi\omega_{11}^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\nu}\mathring{\nu}\xi_{11,4}^*\omega_4^{8n}\bar{\omega}_4^{8n-1}$,

ii) $\pi_S^{64n+33,-64n-32}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{64}\cdot\xi_{11,10}^*\omega_{10}^{2n+1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\nu}\mathring{\nu}\xi_{11,4}^*\omega_4^{8n+4}\bar{\omega}_4^{8n+3}$,

iii) $\pi_S^{32n+17,-32n-16}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{32}\cdot\hat{\eta}\xi_{11,9}^*\omega_9^{2n+1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\nu}\mathring{\nu}\xi_{11,4}^*\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$,

iv) $\pi_S^{16n+9,-16n-8}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{16}\cdot\hat{\eta}^2\xi_{11,8}^*\omega_8^{n+1}\bar{\omega}_8^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\nu}\mathring{\nu}\xi_{11,4}^*\omega_4^{2n+1}\bar{\omega}_4^{2n}$,

v) $\pi_S^{8n+2,-8n-1}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi\hat{\nu}\xi_{11,8}^*\omega_8^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\eta}\hat{\sigma}\xi_{11,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\varepsilon}_5\xi_{11,2}^*\omega_2^{4n-1}$,

vi) $\pi_S^{8n+3,-8n-2}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\nu\mathring{\nu}\xi_{11,4}^*\omega_4^n\bar{\omega}_4^{n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\sigma\xi_{11,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\eta\sigma\xi_{11,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\varepsilon\xi_{11,2}^*\omega_2^{4n-3}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\varepsilon\xi_{11,2}^*\omega_2^{4n-3}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,3}^*[120\sigma\omega_1^{8n-5}]_3\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\xi_{11,3}^*[120\sigma\omega_1^{8n-5}]_3$,

vii) $\pi_S^{8n+4,-8n-3}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\chi^5\hat{\sigma}\xi_{11,8}^*\omega_8^n\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,1}^*(\mu\omega_1^{8n-6})$,

viii) $\pi_S^{8n+5,-8n-4}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\hat{\eta}\xi_{11,7}^*[\eta^2\omega_1^{8n+2}]_7$,

ix) $\pi_S^{8n+6,-8n-5}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\hat{\eta}\hat{\sigma}\xi_{11,2}^*\omega_2^{4n+1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\mathring{\varepsilon}_5\xi_{11,2}^*\omega_2^{4n+1}$,

x) $\pi_S^{8n+7,-8n-6}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\eta\sigma\xi_{11,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\varepsilon\xi_{11,2}^*\omega_2^{4n-1}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot(1+\rho)\eta\sigma\xi_{11,2}^*\omega_2^{4n-1}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,3}^*[120\sigma\omega_1^{8n-1}]_3\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\rho\xi_{11,3}^*[120\sigma\omega_1^{8n-1}]_3$,

xi) $\pi_S^{16n+8,-16n-7}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\tilde{\nu}_3\xi_{11,4}^*\omega_4^{2n+1}\bar{\omega}_4^{2n}\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,1}^*(\mu\omega_1^{16n-2})$,

xii) $\pi_S^{32n+16,-32n-15}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{2}\cdot\chi^5\mathring{\sigma}\xi_{11,8}^*\omega_8^{2n+1}\bar{\omega}_8^{2n}\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\tilde{\nu}_3\xi_{11,4}^*\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,1}^*(\mu\omega_1^{32n+6})$,

xiii) $\pi_S^{64n+32,-64n-31}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{4}\cdot\chi^5\mathring{\sigma}\xi_{11,8}^*\omega_8^{4n+2}\bar{\omega}_8^{4n+1}\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot\tilde{\nu}_3\xi_{11,4}^*\omega_4^{8n+4}\bar{\omega}_4^{8n+3}$
$\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,1}^*(\mu\omega_1^{64n+22})$,

xiv) $\pi_S^{64n,-64n-1}(S_+^{11,0})=\boldsymbol{Z}/\boldsymbol{8}\cdot\chi^5\mathring{\sigma}\xi_{11,8}^*\omega_8^{4n}\bar{\omega}_8^{4n-1}$
$\oplus\boldsymbol{Z}/\boldsymbol{4}\cdot(\chi^5\mathring{\sigma}\xi_{11,8}^*\omega_8^{4n}\bar{\omega}_8^{4n-1}-\tilde{\nu}_3\xi_{11,4}^*\omega_4^{8n}\bar{\omega}_4^{8n-1})\oplus\boldsymbol{Z}/\boldsymbol{2}\cdot\xi_{11,1}^*(\mu\omega_1^{64n-10})$

for any integer n.

Proposition 1.7. *There hold the relations* $(1+\rho)\varepsilon\xi_{11,2}^*\omega_2^{4n+1}=(1+\rho)\eta\sigma\xi_{11,2}^*\omega_2^{4n+1}$ *and* $\rho\xi_{11,10}^*\omega_{10}^{2n+1}=\xi_{11,10}^*\omega_{10}^{2n+1}$ *for any integer n.*

Proposition 1.6 describes $\lambda_{p,q}^S$ for $p+q=9$. By [I], Proposition 4.8, the groups $\pi_{p,q}^S$ for $p+q=9$ are determined if $(p,q)\neq(5,4)$, $(6,3)$, $(7,2)$, $(8,1)$, $(9,0)$ or $(10,-1)$.

Consider the following exact sequence

$$\cdots \to \pi_S^{-8,-1}(S_+^{1,0}) \xrightarrow{\delta_1} \pi_{9,0}^S \xrightarrow{\chi} \pi_{8,0}^S \xrightarrow{\beta_1} \pi_S^{-8,0}(S_+^{1,0}) \to \cdots .$$

Since $\pi_S^{-8,-1}(S_+^{1,0})=\boldsymbol{Z}/2\cdot\eta^2\sigma\omega_1^{-8}\oplus\boldsymbol{Z}/2\cdot\eta\varepsilon\omega_1^{-8}\oplus\boldsymbol{Z}/2\cdot\mu\omega_1^{-8}$, $\delta_1(\eta^2\sigma\omega_1^{-8})=\delta_1\beta_1(\mathring{\eta}_8\eta\sigma)=0$, $\delta_1(\eta\varepsilon\omega_1^{-8})=\delta_1\beta_1(\mathring{\eta}_8\varepsilon)=0$ and $\delta_1(\mu\omega_1^{-8})=\delta_1\beta_1(\mathring{\mu}_8)=0$, we have $\mathrm{Im}[\delta_1\colon \pi_S^{-8,-1}(S_+^{1,0})\to\pi_{9,0}^S]=0$. Hence $\pi_{9,0}^S\approx\mathrm{Ker}[\beta_1\colon\pi_{8,0}^S\to\pi_S^{-8,0}(S_+^{1,0})]$. By [I], Theorem 15.26,iv), we have

$$\pi_{9,0}^S = \boldsymbol{Z}\cdot y_9\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_8\sigma\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\varepsilon}_8$$

where $\chi y_9=\hat{\eta}y_7$. Since $\chi\colon\pi_{9,0}^S\to\pi_{8,0}^S$ is a monomorphism, we see that $\rho y_9=y_9$. By the fixed-point exact sequence [I], (1.12) and Proposition 1.2, i), we have $\pi_{10,-1}^S=\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_8$. This implies that $\beta_1\colon\pi_{9,0}^S\to\pi_S^{-9,0}(S_+^{1,0})$ is an epimorphism. Hence

$$\psi(y_9)\in\mu+\boldsymbol{Z}/2\cdot\eta^2\sigma\oplus\boldsymbol{Z}/2\cdot\eta\varepsilon\,.$$

Thus we obtain

Proposition 1.8. i) $\pi_{9,0}^S=\boldsymbol{Z}\cdot y_9\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_8\sigma\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\varepsilon}_8$,
ii) $\chi y_9=\hat{\eta}y_7$ *and* $\rho y_9=y_9$,
iii) $\psi(y_9)\in\mu+\boldsymbol{Z}/2\cdot\eta^2\sigma\oplus\boldsymbol{Z}/2\cdot\eta\varepsilon$.

It is now easy to compute the groups $\pi_{p,q}^S$ for $(p,q)=(5,4)$, $(6,3)$, $(7,2)$ and $(8,1)$, and we obtain

Theorem 1.9. ($\pi_{p,q}^S$, $p+q=9$)
i) $\pi_{5,4}^S=\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\varepsilon}_4\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_4\sigma$,
ii) $\pi_{6,3}^S=\boldsymbol{Z}/24\cdot\hat{\eta}^2\hat{\sigma}\oplus\boldsymbol{Z}/2\cdot\hat{\nu}^3$,
iii) $\pi_{7,2}^S=\boldsymbol{Z}/2\cdot\chi\mathring{\eta}_8\mu$,
iv) $\pi_{8,1}^S=\boldsymbol{Z}/2\cdot\mathring{\eta}_8\eta\sigma\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_8\eta\sigma\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_8\varepsilon\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_8\varepsilon\oplus\boldsymbol{Z}/2\cdot\mathring{\mu}_8\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\mu}_8$,
v) $\pi_{9,0}^S=\boldsymbol{Z}\cdot y_9\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_8\sigma\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\varepsilon}_8$,
vi) $\pi_{10,-1}^S=\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_8$,
vii) $\pi_{8n,-8n+9}^S\approx\boldsymbol{Z}/2\cdot\nu\mathring{\nu}\mathring{\nu}_{8n}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n}\eta\sigma\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{8n}\eta\sigma\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n}\varepsilon\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{8n}\varepsilon$
$\oplus\boldsymbol{Z}/2\cdot\mathring{\mu}_{8n}\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\mu}_{8n}\oplus\pi_{-8n+9}^S$ *for* $n\neq 1$,
viii) $\pi_{8n+1,-8n+8}^S\approx\boldsymbol{Z}/2\cdot\chi\hat{\nu}\delta_8\omega_8^{1-n}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n-4}\hat{\eta}\hat{\sigma}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n-4}\hat{\varepsilon}_5\oplus\pi_{-8n+8}^S$ *for* $n\neq 1$,
ix) $\pi_{16n+2,-16n+7}^S\approx\boldsymbol{Z}/16\cdot\hat{\eta}^2\mathring{\sigma}_{16n}\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_{16n}\oplus\pi_{-16n+7}^S$ *for any integer* n,
x) $\pi_{32n-6,-32+15}^S\approx\boldsymbol{Z}/32\cdot\hat{\eta}\delta_9\omega_9^{1-2n}\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_{32n-8}\oplus\pi_{-32n+15}^S$ *for any integer* n,
xi) $\pi_{64n-22,-64n+31}^S\approx\boldsymbol{Z}/64\cdot\delta_{10}\omega_{10}^{1-2n}\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_{64n-24}\oplus\pi_{-64n+31}^S$ *for any integer* n,
xii) $\pi_{64n+10,-64n-1}^S\approx\boldsymbol{Z}/64\cdot\chi\delta_{11}\omega_{11}^{-n}\oplus\boldsymbol{Z}/2\cdot\hat{\nu}\mathring{\nu}\mathring{\nu}_{64n+8}\oplus\pi_{-64n-1}^S$ *for* $n\neq 0$,
xiii) $\pi_{16n+3,-16n+6}^S\approx\boldsymbol{Z}/4\cdot\tilde{\nu}_3\mathring{\nu}_{16n}\oplus\boldsymbol{Z}/2\cdot\chi\mathring{\eta}_{16n+4}\mu\oplus\pi_{-16n+6}^S$ *for any integer* n,
xiv) $\pi_{32n-5,-32n+14}^S\approx\boldsymbol{Z}/2\cdot\chi^5\mathring{\sigma}\mathring{\sigma}_{32n}\oplus\boldsymbol{Z}/4\cdot\tilde{\nu}_3\mathring{\nu}_{32n-8}\oplus\boldsymbol{Z}/2\cdot\chi\mathring{\eta}_{32n-4}\mu\oplus\pi_{-32n+14}^S$ *for any integer* n,

xv) $\pi^S_{64n-21,-64n+30} \approx \boldsymbol{Z}/4 \cdot \chi^5 \mathring{\sigma}\mathring{\sigma}_{64n-16} \oplus \boldsymbol{Z}/4 \cdot \tilde{\nu}_3 \mathring{\nu}_{64n-24} \oplus \boldsymbol{Z}/2 \cdot \chi \mathring{\eta}_{64n-20}\mu \oplus \pi^S_{-64n+30}$ *for any integer n,*

xvi) $\pi^S_{64n+11,-64n-2} \approx \boldsymbol{Z}/8 \cdot \chi^5 \mathring{\sigma}\mathring{\sigma}_{64n+16} \oplus \boldsymbol{Z}/4 \cdot (\chi^5 \mathring{\sigma}\mathring{\sigma}_{64n+16} - \tilde{\nu}_3 \mathring{\nu}_{64n+8}) \oplus \boldsymbol{Z}/2 \cdot \chi \mathring{\eta}_{64n+12}\mu$
$\oplus \pi^S_{-64n-2}$ *for any integer n,*

xvii) $\pi^S_{8n+4,-8n+5} \approx \boldsymbol{Z}/2 \cdot \mathring{\eta}_{8n+4}\eta\sigma \oplus \boldsymbol{Z}/2 \cdot \mathring{\eta}_{8n+4}\varepsilon \oplus \boldsymbol{Z}/2 \cdot (1+\rho)\mathring{\eta}_{8n+4}\eta\sigma \oplus \boldsymbol{Z}/2 \cdot \mathring{\mu}_{8n+4}$
$\oplus \boldsymbol{Z}/2 \cdot \rho\mathring{\mu}_{8n+4} \oplus \pi^S_{-8n+5}$ *for any integer n,*

xviii) $\pi^S_{8n+5,-8n+4} \approx \boldsymbol{Z}/2 \cdot \mathring{\eta}_{8n}\hat{\eta}\hat{\sigma} \oplus \boldsymbol{Z}/2 \cdot \mathring{\eta}_{8n}\mathring{\varepsilon}_5 \oplus \pi^S_{-8n+4}$ *for* $n \neq 0$,

xix) $\pi^S_{8n+6,-8n+3} \approx \boldsymbol{Z}/8 \cdot \hat{\eta}\mathring{\varepsilon}_{8n+5} \oplus \pi^S_{-8n+3}$ *for* $n \neq 0$,

xx) $\pi^S_{8n+7,-8n+2} \approx \boldsymbol{Z}/8 \cdot \chi^5 \hat{\sigma}\delta_8 \omega_8^{-n} \oplus \boldsymbol{Z}/2 \cdot \chi \mathring{\eta}_{8n+8}\mu \oplus \pi^S_{-8n+2}$ *for* $n \neq 0$.

Proposition 1.10. *There hold the relations* $(1+\rho)\mathring{\eta}_{8n+4}\varepsilon = (1+\rho)\mathring{\eta}_{8n+4}\eta\sigma$ *and* $\rho\delta_{10}\omega_{10}^{2n+1} = \delta_{10}\omega_{10}^{2n+1}$ *for any integer n.*

2. 10 stem

Routine computations yield the followings.

Proposition 2.1. $(\pi_S^{12-p,-q-1}(S_+^{12,0}),\ p+q=10)$

i) $\pi_S^{64n+1,-64n}(S_+^{12,0}) = \boldsymbol{Z}/128 \cdot \chi\omega_{12}^n$,

ii) $\pi_S^{64n+33,-64n-32}(S_+^{12,0}) = \boldsymbol{Z}/64 \cdot \hat{\eta}\xi^*_{12,10}\omega_{10}^{2n+1}$,

iii) $\pi_S^{32n+17,-32n-16}(S_+^{12,0}) = \boldsymbol{Z}/32 \cdot \hat{\eta}^2\xi^*_{12,9}\omega_9^{2n+1}$,

iv) $\pi_S^{16n+9,-16n-8}(S_+^{12,0}) = \boldsymbol{Z}/16 \cdot \hat{\eta}^3\xi^*_{12,8}\omega_8^{n+1}\bar{\omega}_8^n$,

v) $\pi_S^{8n+2,-8n-1}(S_+^{12,0}) = \boldsymbol{Z}/2 \cdot \hat{\nu}\xi^*_{12,8}\omega_8^n \oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{8n-9})$,

vi) $\pi_S^{8n+3,-8n-2}(S_+^{12,0}) = \boldsymbol{Z}/2 \cdot \chi^3\hat{\nu}^2\xi^*_{12,8}\omega_8^n \oplus \boldsymbol{Z}/2 \cdot \hat{\eta}\eta\sigma\xi^*_{12,2}\omega_2^{4n-3} \oplus \boldsymbol{Z}/2 \cdot \hat{\eta}\varepsilon\xi^*_{12,2}\omega_2^{4n-3}$
$\oplus \boldsymbol{Z}/2 \cdot y_9\xi^*_{12,2}\omega_2^{4n+1}$,

vii) $\pi_S^{8n+4,-8n-3}(S_+^{12,0}) = \boldsymbol{Z}/8 \cdot \chi^4\hat{\sigma}\xi^*_{12,8}\omega_8^n \oplus \boldsymbol{Z}/2 \cdot \mu\xi^*_{12,2}\omega_2^{4n-3} \oplus \boldsymbol{Z}/2 \cdot \rho\mu\xi^*_{1,22}\omega_2^{4n-3}$
$\oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{8n-7})$,

viii) $\pi_S^{8n+5,-8n-4}(S_+^{12,0}) = \boldsymbol{Z}/8 \cdot y_7\xi^*_{12,4}\omega_4^{n+1}\bar{\omega}_4^n$,

ix) $\pi_S^{8n+6,-8n-5}(S_+^{12,0}) = \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{8n-5})$,

x) $\pi_S^{8n+7,-8n-6}(S_+^{12,0}) = \boldsymbol{Z}/2 \cdot \hat{\eta}\eta\sigma\xi^*_{12,2}\omega_2^{4n-1} \oplus \boldsymbol{Z}/2 \cdot \hat{\eta}\varepsilon\xi^*_{12,2}\omega_2^{4n-1}$,

xi) $\pi_S^{16n+8,-16n-7}(S_+^{12,0}) = \boldsymbol{Z}/2 \cdot \hat{\eta}\tilde{\nu}_3\xi^*_{12,4}\omega_4^{2n+1}\bar{\omega}_4^{2n} \oplus \boldsymbol{Z}/2 \cdot \mu\xi^*_{12,2}\omega_2^{8n-1}$
$\oplus \boldsymbol{Z}/2 \cdot \rho\mu\xi^*_{12,2}\omega_2^{8n-1} \oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{16n-3})$,

xii) $\pi_S^{32n+16,-32n-15}(S_+^{12,0}) = \boldsymbol{Z}/2 \cdot \chi^4\mathring{\sigma}\xi^*_{12,8}\omega_8^{2n+1}\bar{\omega}_8^{2n} \oplus \boldsymbol{Z}/2 \cdot \hat{\eta}\tilde{\nu}_3\xi^*_{12,4}\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$
$\oplus \boldsymbol{Z}/2 \cdot \mu\xi^*_{12,2}\omega_2^{16n+3} \oplus \boldsymbol{Z}/2 \cdot \rho\mu\xi^*_{12,2}\omega_2^{16n+3} \oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{32n+5})$,

xiii) $\pi_S^{64n+32,-64n-31}(S_+^{12,0}) = \boldsymbol{Z}/4 \cdot \chi^4\mathring{\sigma}\xi^*_{12,8}\omega_8^{4n+2}\bar{\omega}_8^{4n+1} \oplus \boldsymbol{Z}/2 \cdot \hat{\eta}\tilde{\nu}_3\xi^*_{12,4}\omega_4^{8n+4}\bar{\omega}_4^{8n+3}$
$\oplus \boldsymbol{Z}/2 \cdot \mu\xi^*_{12,2}\omega_2^{32n+11} \oplus \boldsymbol{Z}/2 \cdot \rho\mu\xi^*_{12,2}\omega_2^{32n+11} \oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{64n+21})$,

xiv) $\pi_S^{64n,-64n-1}(S_+^{12,0}) = \boldsymbol{Z}/8 \cdot \chi^4\mathring{\sigma}\xi^*_{12,8}\omega_8^{4n}\bar{\omega}_8^{4n-1} \oplus \boldsymbol{Z}/2 \cdot (2\chi^4\mathring{\sigma}\xi^*_{12,8}\omega_8^{4n}\bar{\omega}_8^{4n-1}$
$- \hat{\eta}\tilde{\nu}_3\xi^*_{12,4}\omega_4^{8n}\bar{\omega}_4^{8n-1}) \oplus \boldsymbol{Z}/2 \cdot \mu\xi^*_{12,2}\omega_2^{32n-5} \oplus \boldsymbol{Z}/2 \cdot \rho\mu\xi^*_{12,2}\omega_2^{32n-5}$
$\oplus \boldsymbol{Z}/3 \cdot \xi^*_{12,1}(\beta_1\omega_1^{64n-11})$

for any integer n.

Theorem 2.2. $(\pi^S_{p,q},\ p+q=10)$

i) $\pi^S_{6,4} = \boldsymbol{Z}/3 \cdot \delta_1(\beta_1\omega_1^{-5})$,

ii) $\pi^S_{7,3} = \boldsymbol{Z}/24 \cdot \hat{\eta}^3\hat{\sigma}$,

iii) $\pi^S_{8,2}=\boldsymbol{Z}/2\cdot\mathring{\eta}_8\mu\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_8\mu\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-7})$,

iv) $\pi^S_{9,1}=\boldsymbol{Z}/2\cdot\mathring{\eta}y_9\oplus\boldsymbol{Z}/2\cdot\eta\mathring{\eta}_8\eta\sigma\oplus\boldsymbol{Z}/2\cdot\eta\mathring{\eta}_8\varepsilon$,

v) $\pi^S_{10,0}=\boldsymbol{Z}\cdot\hat{\eta}y_9\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-9})$,

vi) $\pi^S_{11,-1}=0$,

vii) $\pi^S_{8n,-8n+10}\approx\boldsymbol{Z}/8\cdot\chi^4\hat{\sigma}\delta_8\omega_8^{1-n}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n}\mu\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{8n}\mu\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{1-8n})$
$\oplus\pi^S_{-8n+10}$ *for* $n\neq1$,

viii) $\pi^S_{8n+1,-8n+9}\approx\boldsymbol{Z}/2\cdot\chi^3\hat{\nu}^2\delta_8\omega_8^{-n}\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_{8n}\eta\sigma\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_{8n}\varepsilon\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{8n-8}y_9$
$\oplus\pi^S_{-8n+9}$ *for* $n\neq1$,

ix) $\pi^S_{8n+2,-8n+8}\approx\boldsymbol{Z}/2\cdot\hat{\nu}\delta_8\omega_8^{1-n}\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-8n-1})\oplus\pi^S_{-8n+8}$ *for* $n\neq1$,

x) $\pi^S_{16n+3,-16n+7}\approx\boldsymbol{Z}/16\cdot\hat{\eta}^3\mathring{\sigma}_{16n}\oplus\pi^S_{-16n+7}$ *for any integer* n,

xi) $\pi^S_{32n-5,-32n+15}\approx\boldsymbol{Z}/32\cdot\hat{\eta}^2\delta_9\omega_9^{1-2n}\oplus\pi^S_{-32n+15}$ *for any integer* n,

xii) $\pi^S_{64n-21,-64n+31}\approx\boldsymbol{Z}/64\cdot\hat{\eta}\delta_{10}\omega_{10}^{1-2n}\oplus\pi^S_{-64n+31}$ *for any integer* n,

xiii) $\pi^S_{64n+11,-64n-1}\approx\boldsymbol{Z}/128\cdot\chi\delta_{12}\omega_{12}^{-n}\oplus\pi^S_{-64n-1}$ *for* $n\neq0$,

xiv) $\pi^S_{16n+4,-16n+6}\approx\boldsymbol{Z}/2\cdot\hat{\eta}\tilde{\nu}_3\mathring{\nu}_{16n}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{16n+4}\mu\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{16n+4}\mu\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-16n-3})$
$\oplus\pi^S_{-16n+6}$ *for any integer* n,

xv) $\pi^S_{32n-4,-32n+14}\approx\boldsymbol{Z}/2\cdot\chi^4\mathring{\sigma}\mathring{\sigma}_{32n}\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\tilde{\nu}_3\mathring{\nu}_{32n-8}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{32n-4}\mu\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{32n-4}\mu$
$\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-32n+5})\oplus\pi^S_{-32n+14}$ *for any integer* n,

xvi) $\pi^S_{64n-20,-64n+30}\approx\boldsymbol{Z}/4\cdot\chi^4\mathring{\sigma}\mathring{\sigma}_{64n-16}\oplus\boldsymbol{Z}/2\cdot\hat{\eta}\tilde{\nu}_3\mathring{\nu}_{64n-24}\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{64n-20}\mu$
$\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{64n-20}\mu\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-64n+21})\oplus\pi^S_{-64n+30}$ *for any integer* n,

xvii) $\pi^S_{64n+12,-64n-2}\approx\boldsymbol{Z}/8\cdot\chi^4\mathring{\sigma}\mathring{\sigma}_{64n+16}\oplus\boldsymbol{Z}/2\cdot(2\chi^4\mathring{\sigma}\mathring{\sigma}_{64n+16}-\hat{\eta}\tilde{\nu}_3\mathring{\nu}_{64n+8})\oplus\boldsymbol{Z}/2\cdot\mathring{\eta}_{64n+12}\mu$
$\oplus\boldsymbol{Z}/2\cdot\rho\mathring{\eta}_{64n+12}\mu\oplus\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-64n-11})\oplus\pi^S_{-64n-2}$ *for any integer* n,

xviii) $\pi^S_{8n+5,-8n+5}\approx\boldsymbol{Z}/2\cdot\hat{\eta}\mathring{\eta}_{8n+4}\eta\sigma\oplus\boldsymbol{Z}/2\cdot y_9\mathring{\eta}_{8n-4}\oplus\pi^S_{-8n+5}$ *for any integer* n,

xix) $\pi^S_{8n+6,-8n+4}\approx\boldsymbol{Z}/3\cdot\delta_1(\beta_1\omega_1^{-8n-5})\oplus\pi^S_{-8n+4}$ *for any integer* n,

xx) $\pi^S_{8n+7,-8n+3}\approx\boldsymbol{Z}/8\cdot y_7\mathring{\nu}_{8n}\oplus\pi^S_{-8n+3}$ *for* $n\neq0$.

Proposition 2.3. i) $\hat{\sigma}\hat{\nu}=0$, ii) $\hat{\eta}\mathring{\eta}_{8n+4}\varepsilon=\hat{\eta}\mathring{\eta}_{8n+4}\eta\sigma$.

3. 11 stem

To give generators of $\pi_S^{r-p,-q-1}(S_+^{r,0})$ for $p+q=11$ we prepare new elements which are given by the equivariant Toda brackets.

Since $\psi\colon\pi^S_{2,1}\to\pi^S_3$ is isomorphic by [I], Theorem 10.11, i), we see that $\hat{\eta}^2\eta=12\hat{\nu}$, hence $\hat{\eta}^2\eta\hat{\sigma}=12\hat{\nu}\hat{\sigma}=0$ by Theorem 2.2, ix). Thus the equivariant Toda brackets

(3.1) $$\langle\mathring{\sigma},\ \hat{\eta}^2\eta,\ \xi^*_{3,1}\omega_1^{4n-3}\rangle^\tau,$$

(3.2) $$\langle\hat{\eta}^2,\ \eta\mathring{\sigma},\ \xi^*_{9,1}\omega_1^{16n-1}\rangle^\tau,$$

(3.3) $$\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{10,1}\omega_1^{32n+7}\rangle^\tau,$$

and

(3.4) $$\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{11,1}\omega_1^{64n+23}\rangle^\tau$$

are well-defined. Then by [I], Proposition 6.8,

$$\eta^*_{2,3}\langle\mathring{\sigma},\ \hat{\eta}^2\eta,\ \xi^*_{3,1}\omega_1^{4n-3}\rangle^\tau = \{\hat{\eta}\eta\mathring{\sigma}\omega_2^{2n-1}\}\ ,$$
$$\eta^*_{8,9}\langle\hat{\eta}^2,\ \eta\mathring{\sigma},\ \xi^*_{9,1}\omega_1^{16n-1}\rangle^\tau \ni \hat{\eta}^3\omega_8^{n+1}\bar{\omega}_8^n\ ,$$
$$\eta^*_{9,10}\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{10,1}\omega_1^{32n+7}\rangle^\tau \ni \hat{\eta}^2\omega_9^{2n+1}$$

and

$$\eta^*_{10,11}\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{11,1}\omega_1^{64n+23}\rangle^\tau \ni \hat{\eta}\omega_{10}^{2n+1}\ .$$

By [14], Lemma 9.1, we see that

$$\psi(\delta_3\langle\mathring{\sigma},\ \hat{\eta}^2\eta,\ \xi^*_{3,1}\omega_1^{4n-3}\rangle^\tau) = \langle\sigma,\ \eta^3,\ 2\rangle = \zeta + 2\pi_{11}^S\ ,$$
$$\psi(\delta_9\langle\hat{\eta}^2,\ \eta\mathring{\sigma},\ \xi^*_{9,1}\omega_1^{16n-1}\rangle^\tau) = \langle\eta^2,\ \eta\sigma,\ 2\rangle = \zeta + 2\pi_{11}^S\ ,$$
$$\psi(\delta_{10}\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{10,1}\omega_1^{32n+7}\rangle^\tau) = \langle\eta,\ \eta^2\sigma,\ 2\rangle = \zeta + 2\pi_{11}^S$$

and

$$\psi(\delta_{11}\langle\hat{\eta},\ \hat{\eta}\eta\mathring{\sigma},\ \xi^*_{11,1}\omega_1^{64n+23}\rangle^\tau) = \langle\eta,\ \eta^2\sigma,\ 2\rangle = \zeta + 2\pi_{11}^S\ .$$

Thus there exist elements

(3.1′) $$[\eta^2\sigma\omega_1^{4n-3}]_3 \in \pi_S^{4n-3,-4n-6}(S_+^{3,0})\ ,$$

(3.2′) $$[\eta^3\omega_1^{16n+5}]_9 \in \pi_S^{16n+5,-16n-8}(S_+^{9,0})\ ,$$

(3.3′) $$[\eta^2\omega_1^{32n+14}]_{10} \in \pi_S^{32n+14,-32n-16}(S_+^{10,0})$$

and

(3.4′) $$[\eta\omega_1^{64n+31}]_{11} \in \pi_S^{64n+31,-64n-32}(S_+^{11,0})$$

such that $\eta^*_{2,3}[\eta^2\sigma\omega_1^{4n-3}]_3 = \hat{\eta}\eta\mathring{\sigma}\omega_2^{2n-1}$, $\psi(\delta_3[\eta^2\sigma\omega_1^{4n-3}]_3) = \zeta$, $\eta^*_{8,9}[\eta^3\omega_1^{16n+5}]_9 = \hat{\eta}^3\omega_8^{n+1}\bar{\omega}_8^n$, $\psi(\delta_9[\eta^3\omega_1^{16n+5}]_9) = 63\zeta$, $\eta^*_{9,10}[\eta^2\omega_1^{32n+14}]_{10} = \hat{\eta}^2\omega_9^{2n+1}$, $\psi(\delta_{10}[\eta^2\omega_1^{32n+14}]_{10}) = 63\zeta$, $\eta^*_{10,11}[\eta\omega_1^{64n+31}]_{11} = \hat{\eta}\omega_{10}^{2n+1}$ and $\psi(\delta_{11}[\eta\omega_1^{64n+31}]_{11}) = 63\zeta$, respectively.

The equivariant Toda bracket

$$\langle 30\sigma,\ 8,\ 3\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n\rangle^\tau$$

is well-defined and

$$\eta^*_{1,5}\langle 30\sigma,\ 8,\ 3\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n\rangle^\tau = \{120\sigma\omega_1^{8n+5}\}\ .$$

Thus there is an element

(3.5) $$[120\sigma\omega_1^{8n+5}]_5 \in \langle 30\sigma,\ 8,\ 3\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n\rangle^\tau \subset \pi_S^{8n-3,-8n-4}(S_+^{5,0})$$

such that $\eta^*_{1,5}\langle 30\sigma,\ 8,\ 3\xi^*_{5,4}\omega_4^{n+1}\bar{\omega}_4^n\rangle^\tau = 120\sigma\omega_1^{8n+5}$. Then $[120\sigma\omega_1^{8n+5}]_5$ is of order 8 and $\psi(\delta_5[120\sigma\omega_1^{8n+5}]_5) = 63\zeta$.

Since $\hat{\nu}\hat{\sigma} = 0$ by Proposition 2.3, the equivariant Toda bracket

$$\langle\hat{\nu},\ \hat{\sigma},\ \xi^*_{9,1}\omega_1^{16n+4}\rangle^\tau$$

is well-defined and

$$\eta^*_{8,9}\langle\hat{\nu},\ \hat{\sigma},\ \xi^*_{9,1}\omega_1^{16n+4}\rangle^{\tau}\ni\hat{\nu}\omega_8^{n+1}\bar{\omega}_8^n\,.$$

Thus there exists an element

$$\text{(3.6)}\qquad [\nu\omega_1^{16n+6}]_9\in\langle\hat{\nu},\ \hat{\sigma},\ \xi^*_{9,1}\omega_1^{16n+4}\rangle^{\tau}\subset\pi_S^{16n+6,-16n-9}(S_+^{9,0})$$

such that $\eta^*_{8,9}[\nu\omega_1^{16n+6}]_9=\hat{\nu}\omega_8^{n+1}\bar{\omega}_8^n$.

Proposition 3.7. *There hold the relations* $\hat{\eta}^3\hat{\sigma}\omega_2^{2n+3}=\hat{\eta}^2\mathring{\varepsilon}_5\omega_2^{2n+3}=\hat{\eta}^3\mathring{\sigma}\omega_2^{2n+1}=\beta_2(\hat{\eta}\mathring{\eta}_{-4n}\eta\mathring{\sigma})=\xi^*_{2,1}(\zeta\omega_1^{4n-2})$ *in* $\pi_S^{4n-1,-4n-9}(S_+^{2,0})$.

Proof. By (3.1′) we see that $\xi^*_{2,1}(\zeta\omega_1^{4n-2})=\xi^*_{2,1}\delta_{1,3}[\eta^2\sigma\omega_1^{4n+1}]_3=\delta_{2,2}\eta^*_{2,3}[\eta^2\sigma\omega_1^{4n+1}]_3=\delta_{2,2}(\hat{\eta}\eta\mathring{\sigma}\omega_2^{2n+1})=\beta_2(\hat{\eta}\mathring{\eta}_{-4n}\eta\mathring{\sigma})$. Since $\beta_2(\hat{\sigma})-\mathring{\sigma}\omega_2^{-2}\in\mathrm{Im}\,[\xi^*_{2,1}\colon\pi_S^{-5,-3}(S_+^{1,0})\to\pi_S^{-4,-3}(S_+^{2,0})]$ and $\eta^2\pi_8^S=0$, we have $\hat{\eta}^3\hat{\sigma}\omega_2^{2n+3}=\hat{\eta}^3\mathring{\sigma}\omega_2^{2n+1}$. By Theorem 1.9, ii), we see that $\hat{\eta}\mathring{\varepsilon}_5=a\cdot\hat{\eta}^2\hat{\sigma}+\hat{\nu}^3$ for some $a\in\boldsymbol{Z}/\mathbf{24}$. Since $4\mathring{\varepsilon}_5=12\hat{\eta}\hat{\sigma}$ by [I], Proposition 15.24, we see that a is of order 8. Thus $\hat{\eta}^2\mathring{\varepsilon}_5\omega_2^{2n+3}=\hat{\eta}^3\hat{\sigma}\omega_2^{2n+3}$, which completes the proof. □

By the same arguments as [I] using the elements (3.1)′, (3.2′), (3.3′), (3.4′) and Proposition 3.7, we obtain the following.

Proposition 3.8. $(\pi_S^{13-p,-q-1}(S_+^{13,0}),\ p+q=11)$

i) $\pi_S^{128n+1,-128n}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{256}\cdot\chi\omega_{13}^n\oplus\boldsymbol{Z}/\mathbf{252}\cdot(32\chi\omega_{13}^n+\xi^*_{13,1}(\zeta\omega_1^{128n-11}))$,

ii) $\pi_S^{128n+65,-128n-64}(S_+^{13,0})=\boldsymbol{Z}/(\mathbf{256}\cdot\mathbf{63})\cdot\xi^*_{13,12}\omega_{12}^{n+1}\bar{\omega}_{12}^n$
$\oplus\boldsymbol{Z}/\mathbf{4}\cdot(63(17\rho-15)\xi^*_{13,12}\omega_{12}^{n+1}\bar{\omega}_{12}^n)$,

iii) $\pi_S^{64n+33,-64n-32}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{128}\cdot\xi^*_{13,11}[\eta\omega_1^{64n+31}]_{11}$
$\oplus\boldsymbol{Z}/\mathbf{252}\cdot(16\xi^*_{13,11}[\eta\omega_1^{64n+31}]_{11}+\xi^*_{13,1}(\zeta\omega_1^{64n+11}))$,

iv) $\pi_S^{32n+17,-32n-16}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{64}\cdot\xi^*_{13,10}[\eta^2\omega_1^{32n+14}]_{10}$
$\oplus\boldsymbol{Z}/\mathbf{252}\cdot(8\xi^*_{13,10}[\eta^2\omega_1^{32n+14}]_{10}+\xi^*_{13,1}(\zeta\omega_1^{32n+5}))$,

v) $\pi_S^{16n+9,-16n-8}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{32}\cdot\xi^*_{13,9}[\eta^3\omega_1^{16n+5}]_9$
$\oplus\boldsymbol{Z}/\mathbf{252}\cdot(4\xi^*_{13,9}[\eta^3\omega_1^{16n+5}]_9+\xi^*_{13,1}(\zeta\omega_1^{16n-3}))$,

vi) $\pi_S^{16n+2,-16n-1}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{2}\cdot\hat{\nu}\xi^*_{13,9}\omega_9^n$,

vii) $\pi_S^{16n+10,-16n-9}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{2}\cdot\xi^*_{13,9}[\nu\omega_1^{16n+6}]_9$,

viii) $\pi_S^{8n+3,-8n-2}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{2}\cdot\chi^2\hat{\nu}^2\xi^*_{13,8}\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\chi\tilde{\nu}_3\mathring{\nu}\xi^*_{13,4}\omega_4^n\bar{\omega}_4^{n-1}$
$\oplus\boldsymbol{Z}/\mathbf{504}\cdot\xi^*_{13,3}[\eta^2\sigma\omega_1^{8n-7}]_3$,

ix) $\pi_S^{8n+4,-8n-3}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{8}\cdot\chi^3\hat{\sigma}\xi^*_{13,8}\omega_8^n\oplus\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{4n-3}$,

x) $\pi_S^{8n+5,-8n-4}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{8}\cdot\xi^*_{13,5}[120\sigma\omega_1^{8n-3}]_5\oplus\boldsymbol{Z}/\mathbf{504}\cdot\xi^*_{13,1}(\zeta\omega_1^{8n-7})$,

xi) $\pi_S^{8n+6,-8n-5}(S_+^{13,0})=0$,

xii) $\pi_S^{8n+7,-8n-6}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{504}\cdot\xi^*_{13,3}[\eta^2\sigma\omega_1^{8n-3}]_3$,

xiii) $\pi_S^{128n,-128n+1}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{16}\cdot\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{8n}\bar{\omega}_8^{8n-1}$
$\oplus\boldsymbol{Z}/\mathbf{2}\cdot(2\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{8n}\bar{\omega}_8^{8n-1}\pm\mathring{\varepsilon}_5\xi^*_{13,4}\omega_4^{16n}\bar{\omega}_4^{16n-1})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{64n-5}$,

xiv) $\pi_S^{128n+64,-128n-63}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{8}\cdot\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{8n+4}\bar{\omega}_8^{8n+3}$
$\oplus\boldsymbol{Z}/\mathbf{2}\cdot(2\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{8n+4}\bar{\omega}_8^{8n+3}-\mathring{\varepsilon}_5\xi^*_{13,4}\omega_4^{16n+8}\bar{\omega}_4^{16n+7})\oplus\boldsymbol{Z}/\mathbf{2}\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{64n+27}$,

xv) $\pi_S^{64n+32,-64n-31}(S_+^{13,0})=\boldsymbol{Z}/\mathbf{4}\cdot\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{4n+2}\bar{\omega}_8^{4n+1}\oplus\boldsymbol{Z}/\mathbf{2}\cdot\mathring{\varepsilon}_5\xi^*_{13,4}\omega_4^{8n+4}\bar{\omega}_4^{8n+3}$

$\oplus Z/2\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{32n+11}$,

xvi) $\pi_S^{32n+16,-32n-15}(S_+^{13,0})=Z/2\cdot\chi^3\mathring{\sigma}\xi^*_{13,8}\omega_8^{2n+1}\bar{\omega}_8^{2n}\oplus Z/2\cdot\mathring{\varepsilon}_5\xi^*_{13,4}\omega_4^{4n+2}\bar{\omega}_4^{4n+1}$
$\oplus Z/2\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{16n+3}$,

xvii) $\pi_S^{16n+8,-16n-7}(S_+^{13,0})=Z/2\cdot\mathring{\varepsilon}_5\xi^*_{13,4}\omega_4^{2n+1}\bar{\omega}_4^{2n}\oplus Z/2\cdot\hat{\eta}\mu\xi^*_{13,2}\omega_2^{8n-1}$

for any integer n.

Put

(3.9) $$\mathring{\zeta}_{4n+2}=\delta_3[\eta^2\sigma\omega_1^{5-4n}]_3\in\pi^S_{4n+2,-4n+9}\,,$$

(3.10) $$\mathring{\zeta}'_{8n}=\delta_5[120\sigma\omega_1^{5-8n}]_5\in\pi^S_{8n,-8n+11}\,,$$

(3.11) $$\mathring{\zeta}'_{16n+4}=\delta_9[\eta^3\omega_1^{5-16n}]_9\in\pi^S_{16n+4,-16n+7}\,,$$

(3.12) $$\mathring{\zeta}'_{32n-4}=\delta_{10}[\eta^2\omega_1^{14-32n}]_{10}\in\pi^S_{32n-4,-32n+15}\,,$$

(3.13) $$\mathring{\zeta}'_{64n-20}=\delta_{11}[\eta\omega_1^{31-64n}]_{11}\in\pi^S_{64n-20,-64n+31}$$

and

(3.14) $$\mathring{\zeta}_{128n-52}=\delta_{12}\omega_{12}^{1-n}\bar{\omega}_{12}^{-n}\in\pi^S_{128n-52,-128n+63}\,.$$

Then by definition we have

Proposition 3.15. i) $\psi(\mathring{\zeta}_{4n+2})=\zeta$, $\chi\mathring{\zeta}_{4n+2}=\hat{\eta}\mathring{\eta}_{4n}\eta\mathring{\sigma}$, ii) $\psi(\mathring{\zeta}'_{8n})=63\zeta$, iii) $\psi(\mathring{\zeta}'_{16n+4})=63\zeta$, $\chi\mathring{\zeta}'_{16n+4}=\hat{\eta}^3\mathring{\sigma}_{16n}$, iv) $\psi(\mathring{\zeta}'_{32n-4})=63\zeta$, $\chi\mathring{\zeta}'_{32n-4}=\hat{\eta}^2\delta_9\omega_9^{1-2n}$, v) $\psi(\mathring{\zeta}'_{64n-20})=63\zeta$, $\chi\mathring{\zeta}'_{64n-20}=\hat{\eta}\delta_{10}\omega_{10}^{1-2n}$ *and* vi) $\psi(\mathring{\zeta}_{128n-52})=\zeta$.

Thus we obtain

Theorem 3.16. ($\pi^S_{p,q}\ p+q=11$)

i) $\pi^S_{6,5}=Z/504\cdot\mathring{\zeta}_6$,

ii) $\pi^S_{7,4}=0$,

iii) $\pi^S_{8,3}=Z/8\cdot\mathring{\zeta}'_8\oplus Z/8\cdot\rho\mathring{\zeta}'_8\oplus Z/63\cdot 8\zeta\delta_1\omega_1^{-7}\oplus Z/3\cdot 8\hat{\eta}^4\hat{\sigma}$,

iv) $\pi^S_{9,2}=Z/2\cdot\hat{\eta}\mathring{\eta}_8\mu$,

v) $\pi^S_{10,1}=Z/2\cdot\chi\bar{\nu}_3\mathring{\nu}\mathring{\nu}_8\oplus Z/504\cdot\mathring{\zeta}_{10}$,

vi) $\pi^S_{11,0}=Z\cdot\hat{\eta}^2y_9$,

vii) $\pi^S_{12,-1}=Z/252\cdot\zeta\delta_1\omega_1^{-11}$,

viii) $\pi^S_{8n,-8n+11}\approx Z/8\cdot\mathring{\zeta}'_{8n}\oplus Z/8\cdot\rho\mathring{\zeta}'_{8n}\oplus Z/63\cdot 8\zeta\delta_1\omega_1^{1-8n}\oplus\pi^S_{-8n+11}$ *for* $n\neq 1$,

ix) $\pi^S_{8n+1,-8n+10}\approx Z/8\cdot\chi^3\hat{\sigma}\delta_8\omega_8^{-n}\oplus Z/2\cdot\hat{\eta}\mathring{\eta}_{8n}\mu\oplus\pi^S_{-8n+10}$ *for* $n\neq 1$,

x) $\pi^S_{8n+2,-8n+9}\approx Z/504\cdot\mathring{\zeta}_{8n+2}\oplus Z/2\cdot\chi^2\hat{\nu}^2\delta_8\omega_8^{-n}\oplus Z/2\cdot\chi\bar{\nu}_3\mathring{\nu}\mathring{\nu}_{8n}\oplus\pi^S_{-8n+9}$ *for* $n\neq 1$,

xi) $\pi^S_{16n+3,-16n+8}\approx Z/2\cdot\delta_9[\nu\omega_1^{6-16n}]_9\oplus\pi^S_{-16n+8}$ *for any integer n*,

xii) $\pi^S_{16n+11,-16n}\approx Z/2\cdot\hat{\nu}\delta_9\omega_9^{-n}\oplus\pi^S_{-16n}$ *for* $n\neq 0$,

xiii) $\pi^S_{16n+4,-16n+7}\approx Z/32\cdot\mathring{\zeta}'_{16n+4}\oplus Z/252\cdot(4\mathring{\zeta}'_{16n+4}+\zeta\delta_1\omega_1^{-16n-3})\oplus\pi^S_{-16n+7}$ *for any integer n*,

xiv) $\pi^S_{32n-4,-32n+15}\approx Z/64\cdot\mathring{\zeta}'_{32n-4}\oplus Z/252\cdot(8\mathring{\zeta}'_{32n-4}+\zeta\delta_1\omega_1^{-32n+5})\oplus\pi^S_{-32n+15}$ *for any integer n*,

xv) $\pi^S_{64n-20,-64n+31}\approx Z/128\cdot\mathring{\zeta}'_{64n-20}\oplus Z/252\cdot(16\mathring{\zeta}'_{64n-20}+\zeta\delta_1\omega_1^{-64n+21})\oplus\pi^S_{-64n+31}$

for any integer n,

xvi) $\pi^S_{128n-52,-128n+63} \approx \boldsymbol{Z}/(\mathbf{256 \cdot 63}) \cdot \mathring{\zeta}_{128n-52} \oplus \boldsymbol{Z}/\mathbf{4} \cdot (63(17\rho-15)\mathring{\zeta}_{128n-52})$
$\oplus \pi^S_{-128n+63}$ *for any integer n,*

xvii) $\pi^S_{128n+12,-128n-1} \approx \boldsymbol{Z}/\mathbf{256} \cdot \chi\delta_{13}\omega_{13}^{-n} \oplus \boldsymbol{Z}/\mathbf{252} \cdot (32\chi\delta_{13}\omega_{13}^{-n} + \zeta\delta_1\omega_1^{-128n-11})$
$\oplus \pi^S_{-128n-1}$ *for* $n \neq 0$,

xviii) $\pi^S_{16n+5,-16n+6} \approx \boldsymbol{Z}/\mathbf{2} \cdot \mathring{\varepsilon}_5 \mathring{\nu}_{16n} \oplus \boldsymbol{Z}/\mathbf{2} \cdot \hat{\eta}\mathring{\eta}_{16n+4}\mu \oplus \pi^S_{-16n+6}$ *for any integer n,*

xix) $\pi^S_{32n-3,-32n+14} \approx \boldsymbol{Z}/\mathbf{2} \cdot \chi^3 \mathring{\sigma}\mathring{\sigma}_{32n} \oplus \boldsymbol{Z}/\mathbf{2} \cdot \mathring{\varepsilon}_5 \mathring{\nu}_{32n-8} \oplus \boldsymbol{Z}/\mathbf{2} \cdot \hat{\eta}\mathring{\eta}_{32n-4}\mu \oplus \pi^S_{-32n+14}$ *for any integer n,*

xx) $\pi^S_{64n-19,-64n+30} \approx \boldsymbol{Z}/\mathbf{4} \cdot \chi^3 \mathring{\sigma}\mathring{\sigma}_{32n-16} \oplus \mathbf{Z}/\mathbf{2} \cdot \mathring{\varepsilon}_5 \mathring{\nu}_{64n-24} \oplus \boldsymbol{Z}/\mathbf{2} \cdot \hat{\eta}\mathring{\eta}_{64n-20}\mu \oplus \pi^S_{-64n+30}$ *for any integer n,*

xxi) $\pi^S_{128n-51,-128n+62} \approx \boldsymbol{Z}/\mathbf{8} \cdot \chi^3 \mathring{\sigma}\mathring{\sigma}_{128n-48} \oplus \boldsymbol{Z}/\mathbf{2} \cdot (2\chi^3 \mathring{\sigma}\mathring{\sigma}_{128n-48} - \mathring{\varepsilon}_5 \mathring{\nu}_{128n-56})$
$\oplus \boldsymbol{Z}/\mathbf{2} \cdot \hat{\eta}\mathring{\eta}_{128n-52}\mu \oplus \pi^S_{-128n+62}$ *for any integer n,*

xxii) $\pi^S_{128n+13,-128n-2} \approx \boldsymbol{Z}/\mathbf{16} \cdot \chi^3 \mathring{\sigma}\mathring{\sigma}_{128n+16} \oplus \boldsymbol{Z}/\mathbf{2} \cdot (2\chi^3 \mathring{\sigma}\mathring{\sigma}_{128n+16} \pm \mathring{\varepsilon}_5 \mathring{\nu}_{128n+8})$
$\oplus \mathbf{Z}/\mathbf{2} \cdot \hat{\eta}\mathring{\eta}_{128n+12}\mu \oplus \pi^S_{-128n-2}$ *for any integer n,*

xxiii) $\pi^S_{8n+6-8n+5} \approx \boldsymbol{Z}/\mathbf{504} \cdot \mathring{\zeta}_{8n+6} \oplus \pi^S_{-8n+5}$ *for any integer n,*

xxiv) $\pi^S_{8n+7,-8n+4} \approx \pi^S_{-8n+4}$ *for any integer n.*

Proposition 3.17. i) $\rho\mathring{\zeta}_{4n+2} = -\mathring{\zeta}_{4n+2}$, ii) $\hat{\nu}\mathring{\varepsilon}_{8n+5} = 0$ *for any integer n.*

Finally we list the tables of $\pi^S_{p,q}$ for $0 \leq p+q \leq 13$ and $-1 \leq q \leq p$, and $\lambda^S_{p,q}$ for $0 \leq p+q \leq 13$. In the following tables $\infty + p^r + q^s$ indicates the direct sum

$$Z \oplus \underbrace{Z/p \oplus \cdots \oplus Z/p}_{r} \oplus \underbrace{Z/q \oplus \cdots \oplus Z/q}_{s}\,.$$

Table 1. $\pi^S_{p,q}$, $0 \leq p+q \leq 13$, $-1 \leq q \leq p$

$p+q$ \ q	-1	0	1	2	3	4	5	6
0	0	$\infty+\infty$						
1	0	∞						
2	0	∞	2^2					
3	12	∞	24					
4	0	∞	0	2				
5	0	∞	0	2				
6	2	$\infty+2$	0	2	$24+8$			
7	$120+2$	$\infty+2^2$	240	0	$480+12+4$			
8	2	$\infty+2^4$	2^3	0	$24+4$	2^4		
9	2	$\infty+2^2$	2^6	2	$24+2$	2^2		
10	0	$\infty+3$	2^3	2^2+3	24	3	2^2	
11	252	∞	$504+2$	2	8^2+63+3	0	504	
12	0	∞	2	0	$8+3$	0	0	2^2
13	3	∞	$2+3$	0	$8+3^2$	0	3	2^2

Table 2. $\lambda^S_{p,q}$, $0 \leq p+q \leq 13$

$p+q$ \ $q\equiv$	0	1	2	3	4	5	6	7	(8)
0	∞	2	∞	2	∞	2	∞	2	
1	0	2^2	2	4	0	2^2	2	4	
2	0	2	2^2	8	0	2	2^2	8	
3	2	12	2	$24+8$	2	12	2^2	$16+12$	
4	2	0	0	8	2	0	2	16	
5	2	0	0	8	2	0	2	16	
6	2^2	2	0	8	4	0	2^2	$16+2$	
7	2^3	$240+2$	2	$240+4$	2^2	240	$6(16)\ 2$ $14(16)\ 2^2$	$7(16) 240+16+2$ $15(16) 32+120+2$	
8	2^5	2^4	4	8	2^4	2^3	$6(16)\ 4$ $14(32)\ 4+2$ $30(32)\ 4^2$	$7(16)\ 16+2$ $15(16)\ 32+2$	
9	2^3	2^7	$8+2$	8	2^2	2^5	$6(16)\ 4+2$ $14(32)\ 4+2^2$ $30(64)\ 4^2+2$ $62(64) 8+4+2$	$7(16)\ 16+2$ $15(32)\ 32+2$ $31(32)\ 64+2$	
10	$2+3$	2^4	$8+2^2+3$	8	3	2^2	$6(16)\ 2^3+3$ $14(32)\ 2^4+3$ $30(64) 4+2^3+3$ $62(64) 8+2^3+3$	$7(16)\ 16$ $15(32)\ 32$ $31(64)\ 64$ $63(64)\ 128$	
11	2	$504+2^2$	$8+2$	8^2+63	0	504	$6(16)\ 2^2$ $14(32)\ 2^3$ $30(64)\ 4+2^2$ $62(128) 8+2^2$ $126(128) 16+2^2$	$7(16)\ 32+252$ $15(32)\ 64+252$ $31(64) 128+252$ $63(64) 256+252$	
12	2	2^2	8	8	0	0	$6(16)\ 2$ $14(32)\ 2^2$ $30(64)\ 4+2$ $62(128) 8+2$ $126(128) 16+2$	$7(16)\ 16$ $15(32)\ 32$ $31(64)\ 64$ $63(128) 128$ $127(128) 256$	
13	2	2^2+3	8	$8+3$	0	3	$6(16)\ 2$ $14(32)\ 2^2$ $30(64)\ 4+2$ $62(128) 8+2$ $126(128) 16+2$	$7(16)\ 16+3$ $15(32)\ 32+3$ $31(64)\ 64+3$ $63(128) 128+3$ $127(128) 256+3$	

Department of Mathematics
Kyoto University
Kyoto 606
Japan