Konno, Y.
Osaka J. Math.
18 (1981), 583–619

MULTIPLICITY FORMULAS FOR DISCRETE SERIES OF $Spin(1,2m)$ AND $SU(1,n)$

Dedicated to Professor Yozo Matsushima on his sixtieth birthday

YASUKO KONNO

(Received April 2, 1980)

Introduction. Let G be a connected semisimple Lie group with finite center. Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. Assume that Γ has no elements with finite order other than the identity. Fix a Haar measure dg on G. Then dg induces the G-invariant measure on $\Gamma\backslash G$ and we can construct the right regular representation π_Γ of G on $L^2(\Gamma\backslash G)$. It is well-known that π_Γ decomposes into the direct sum of irreducible unitary representations with finite multiplicity, up to unitarily equivalence. Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. For $U\in\hat{G}$, denote by $N_\Gamma(U)$ the multiplicity of U in π_Γ.

Let $\hat{G}_d\subset\hat{G}$ be the discrete series of G. Assume that $\hat{G}_d$ is not empty. In this paper, we shall consider the multiplicity $N_\Gamma(U)$ of every class U in $\hat{G}_d$. Langlands [8] showed that, if $U\in\hat{G}_d$ is integrable, we have the generic formula

$$N_\Gamma(U)=d(U)\mathrm{vol}(\Gamma\backslash G)$$

where $d(U)$ is the formal degree of U and $\mathrm{vol}(\Gamma\backslash G)$ is the volume of $\Gamma\backslash G$. On the other hand, it has been known that there are examples of non-integrable classes in $\hat{G}_d$ for which the above Langlands' formula breaks down. Also Hotta-Parthasarathy [6] obtained a sufficient condition for U to satisfy Langlands' formula.

Now, in [3], DeGeorge-Wallach proved a certain formula about $N_\Gamma(U)$ ($U\in\hat{G}_d$) which seems to explain the reason why Langlands' formula breaks down for non-generic classes (c.f. Theorem 3.1.1). In fact, the formula of DeGeorge-Wallach shows that, in the case of non-generic classes, there can appear the terms of "trash" representations in the sense of Wallach [15].

The purpose of this paper is to apply the formula of DeGeorge-Wallach to the special cases of $G=Spin(1,2m)$ ($m\geqq 2$) and $G=SU(1,n)$ ($n\geqq 2$) and provide the concrete formulas about $N_\Gamma(U)$ ($U\in\hat{G}_d$). The most part of our work is devoted to finding all classes in $\hat{G}$ with given infinitesimal character. In [1], Borel and Wallach make the same arguments for the special infinitesimal character. Their method depends on Langlands' classification of admissible representations of reductive groups. In this paper, we depend on detailed results by Thieleker (c.f. [12], [13], [14]) and Kraljevic (c.f. [7]).

Main results are Theorem 7.6.1 and Theorem 8.7.1. It will be interesting which classes in $\hat{G}$ appear as "trash" representations. Wallach [15] and Ragozin-Warner [11] have obtained the similar formulas in the low dimensional cases.

1. The discrete series

In this section, we shall recall some known results about the discrete series of semisimple Lie groups.

1.1. Let G be a connected noncompact semisimple Lie group with finite center. Throughout this paper, we assume that G is a real form of a simply connected complex semisimple Lie group. We fix a Haar measure dg on G, once and for all.

Let $\hat{G}$ be the set of all equivalence classes of irreducible unitary representations of G. We call $U\in\hat{G}$ a *discrete* class if a representation in the class U is equivalent to a subrepresentation of the regular representation of G on $L^2(G)$. Let us denote by $\hat{G}_d$ the set of all discrete classes in $\hat{G}$. We call $\hat{G}_d$ *the discrete series* of G.

Let K be a maximal compact subgroup of G. It is known that G has a discrete class if and only if there exists a Cartan subgroup H of G which is contained in K (c.f. [17], II, p. 401). Hereafter we always assume that G has such a subgroup H.

1.2. Let $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be the Lie algebras of G, K and H, respectively. Let $\mathfrak{p}_0$ be the orthogonal complement of $\mathfrak{k}_0$ in $\mathfrak{g}_0$ with respect to the Killing form of $\mathfrak{g}_0$. For any subalgebra $\mathfrak{u}_0$ of $\mathfrak{g}_0$, we shall denote by $\mathfrak{u}$ the complexification of $\mathfrak{u}_0$.

Now $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$ which is contained in $\mathfrak{k}$. Let $\mathfrak{h}_R^*$ be the dual space of $\mathfrak{h}_R=\sqrt{-1}\mathfrak{h}_0$ and let $\mathfrak{h}^*$ be the dual space of $\mathfrak{h}$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha\in\Delta$, denote by $\mathfrak{g}_\alpha$ the corresponding root space. A root α is called to be *compact* (resp. *noncompact*) if $\mathfrak{g}_\alpha$ is contained in $\mathfrak{k}$ (resp. $\mathfrak{p}$). Denote by Δ_k (resp. Δ_n) the set of all compact (resp. noncompact) roots. The Killing form of $\mathfrak{g}_0$ induces the inner product $\langle\ ,\ \rangle$ on $\mathfrak{h}_R^*$. Denote by $\mathfrak{F}$ the set of all integral linear forms on $\mathfrak{h}_R$; i.e.

$$\mathfrak{F}=\left\{\lambda\in\mathfrak{h}_R^*\ \middle|\ \frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\in \boldsymbol{Z} \qquad (\alpha\in\Delta)\right\}$$

and define

$$\mathfrak{F}_0=\{\lambda\in\mathfrak{F}\,|\,\langle\lambda, \alpha\rangle\neq 0 \qquad (\alpha\in\Delta)\}\,.$$

An element $\lambda\in\mathfrak{F}_0$ is called *a regular integral form* on $\mathfrak{h}_R$.

Let W be the Weyl group of $(\mathfrak{g}, \mathfrak{h})$. Let W_G be the Weyl group of (G, H); i.e. the quotient group of the normalizer of H in G modulo H. The group W and W_G act on $\mathfrak{h}_R^*$ and we can identify W_G with the subgroup of W generated by reflections with respect to compact roots.

When we choose an ordering in $\mathfrak{h}_R^*$, put

$$\Delta^+ = \{\alpha\in\Delta \mid \alpha>0\}\ , \quad \Delta_k^+ = \Delta^+\cap\Delta_k\,, \quad \Delta_n^+ = \Delta^+\cap\Delta_n\,,$$
$$\rho = \frac{1}{2}\sum_{\alpha\in\Delta^+}\alpha, \qquad \rho_k = \frac{1}{2}\sum_{\alpha\in\Delta_k^+}\alpha, \quad \rho_n = \frac{1}{2}\sum_{\alpha\in\Delta_n^+}\alpha,$$
$$\mathfrak{F}^+ = \{\lambda\in\mathfrak{F} \mid \langle\lambda, \alpha\rangle\geqq 0 \qquad (\alpha\in\Delta^+)\}\ ,$$
$$\mathfrak{F}_0^+ = \mathfrak{F}_0\cap\mathfrak{F}^+\,.$$

If an integral form λ belongs to $\mathfrak{F}^+$, we say that λ is *dominant* with respect to Δ^+. Also define the subset W^1 of W by

$$W^1 = \{s\in W \mid s\Delta^+\supset\Delta_k^+\}\ .$$

Then the mapping $W_G\times W^1\ni(s_1, s_2)\to s_1s_2\in W$ is a bijection.

Harish-Chandra showed that there is a distinguished surjection D; $\mathfrak{F}_0\to\hat{G}_d$ (c.f. [17], II, p. 407). For $\Lambda\in\mathfrak{F}_0$, denote by D_Λ the image of Λ by D. Then, for $\Lambda, \Lambda'\in\mathfrak{F}_0$, we have $D_\Lambda=D_{\Lambda'}$ if and only if there exists an element $s\in W_G$ such that $\Lambda'=s\Lambda$ (c.f. [17], II, p. 407). Hence, if we choose a positive root system Δ^+ of Δ, $\hat{G}_d$ corresponds bijectively to the subset $\bigcup_{s\in W^1} s\mathfrak{F}_0^+$ of $\mathfrak{F}_0$.

1.3. Denote by $\hat{K}$ the set of all equivalence classes of irreducible K-modules. Let π be a continuous representation of G on a Hilbert space E. For convenience' sake, we shall often denote by the pair (π, E) the representation π on E. We say that π is *K-finite* if it satisfies the following two conditions;

(1). The restriction $\pi|_K$ of π to K is a unitary representation of K.

(2). Each $\tau\in\hat{K}$ occurs with finite multiplicity in $\pi|_K$.

Let (π, E) be a K-finite representation of G. Let $\tau\in\hat{K}$. Denote by $m(\pi; \tau)$ the multiplicity with which τ occurs in $\pi|_K$. Let $\Phi(\pi)$ be the set of all $\tau\in\hat{K}$ with $m(\pi; \tau)\neq 0$, which is repeated as many times as its multiplicity $m(\pi; \tau)$. We call $\Phi(\pi)$ *the K-spectrum* of π.

When we choose a positive root system Δ_k^+ of Δ_k, each $\tau\in\hat{K}$ is characterized by its highest weight. Put

$$\mathfrak{F}_k^+ = \left\{\lambda\in\mathfrak{h}_R^* \,\middle|\, \frac{2\langle\lambda, \alpha\rangle}{\langle\alpha, \alpha\rangle}\in \boldsymbol{Z} \quad \text{and} \quad \geqq 0 \quad (\alpha\in\Delta_k^+)\right\}.$$

For $\gamma\in\mathfrak{F}\cap\mathfrak{F}_k^+$, denote by τ_γ the irreducible K-module with highest weight γ and also denote by τ_γ the equivalence class of τ_γ. Similarly, for $\gamma\in\mathfrak{F}_k^+$, we denote by τ_γ the irreducible $\mathfrak{k}$-module with highest weight γ.

1.4. Now every irreducible unitary representation π of G is K-finite (c.f. [17], I, p. 318). In particular, if π belongs to a discrete class, the K-spectrum of π is known under the assumption that G is a linear group. We shall review it precisely.

Let $\Lambda\in\mathfrak{F}_0$ and π_Λ be a representation of G in the class D_Λ. Choose a positive root system Δ^+ with respect to which Λ is dominant. For $\lambda\in\mathfrak{h}_R^*$, let $Q(\lambda)$ be the number of distinct ways in which λ can be written as a sum of positive noncompact roots. Then we have the following theorem.

Theorem 1.4.1 (Hecht-Schmid, [5]). *Assume that G has a faithful finite dimensional representation. Let* $\gamma\in\mathfrak{F}\cap\mathfrak{F}_k^+$. *Then we have*

$$m(\pi_\Lambda;\tau_\gamma)=\sum_{s\in W_G}\varepsilon(s)Q(s(\gamma+\rho_k)-\Lambda-\rho_n),$$

where $\varepsilon(s)$ *is the sign of* $s\in W_G$.

1.5. Finally, we refer to formal degrees of the discrete series. Let (π, E) be an irreducible unitary representation of G. If π belongs to a discrete class, there is a positive real number $d(\pi)$ such that

$$\int_G|(\pi(g)u, v)|^2dg=d(\pi)^{-1}(u, u)(v, v)\qquad(u, v\in E)$$

where (,) is the inner product in E (c.f. [17], I, p. 351). We call $d(\pi)$ *the formal degree* of π.

For $\Lambda\in\mathfrak{F}_0$, denote by $d(D_\Lambda)$ the formal degree of a representation in the class $D_\Lambda\in\hat{G}_d$. If we choose a positive root system Δ^+ as in 1.4., we have

$$d(D_\Lambda)=c\,|\prod_{\alpha\in\Delta^+}\langle\Lambda,\alpha\rangle| \tag{1.1}$$

where c is the positive constant depending on the measure dg (c.f. [17], II, p. 407).

2. Infinitesimal characters

In this section, we shall review certain infinitesimal properties of K-finite representations of G. Let G be as in § 1.

2.1. Let (π, E) be a K-finite representation of G. An element $u\in E$ is called *K-finite* if $\pi(K)u$ is contained in a finite dimensional subspace of E. Denote by E_f the set of all K-finite elements of E. For $u\in E_f$ and $X\in\mathfrak{g}_0$, we can define

$$\pi_f(X)(u)=\lim_{t\to0}\frac{\pi(\exp tX)u-u}{t}$$

and $\pi_f(X)(u)$ belongs to E_f (c.f. [17], I, p. 326). In this way, we can induce the representation π_f of $\mathfrak{g}_0$ on E_f. Denote by $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. The representation π_f of $\mathfrak{g}_0$ is canonically extended to the representation of $U(\mathfrak{g})$. This representation of $U(\mathfrak{g})$ is also denoted by π_f and is called *the infinitesimal representation* of π.

Let π^1 and π^2 be K-finite representations of G. If π^1_f is equivalent to π^2_f as representations of $U(\mathfrak{g})$, we say that π^1 is *infinitesimally equivalent* to π^2. In particular, let π^1 and π^2 be irreducible unitary representations. Then π^1 is unitarily equivalent to π^2 if and only if π^1 is infinitesimally equivalent to π^2 (c.f. [17], I, p. 329).

Let (π, E) be a K-finite representation of G. We say that π is *infinitesimally unitary* if E_f has an inner product with respect to which $\pi_f(\sqrt{-1}X)$ is a symmetric operator for each $X\in\mathfrak{g}_0$. If π is unitary, π is clearly infinitesimally unitary. Conversely, if G is simply connected, an irreducible infinitesimally unitary representation of G is infinitesimally equivalent to a unique irreducible unitary representation of G, up to unitarily equivalence (c.f. [17], I, p. 331).

2.2. Let $Z(\mathfrak{g})$ be the center of $U(\mathfrak{g})$. A K-finite representation (π, E) is said to be *quasisimple* if there exists a homomorphism χ of $Z(\mathfrak{g})$ into $\boldsymbol{C}$ such that $\pi_f(z)u=\chi(z)u$ for all $z\in Z(\mathfrak{g})$ and $u\in E_f$. Here χ is called *the infinitesimal character* of π. As it is well-known, an irreducible unitary representation of G is quasisimple (c.f. [17], I, p. 318). For $U\in\hat{G}$, denote by χ_U the infinitesimal character of a representation in the class U.

We shall state some preliminary facts about infinitesimal characters. Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ in § 1 and $U(\mathfrak{h})$ the universal enveloping algebra of $\mathfrak{h}$. Since $U(\mathfrak{h})$ is the symmetric tensor algebra of $\mathfrak{h}$, $U(\mathfrak{h})$ is naturally identified with the algebra of complex-valued polynomial functions on $\mathfrak{h}^*$. Choose a positive root system Δ^+ of Δ and put $\mathfrak{n}^+=\sum_{\alpha\in\Delta^+}\mathfrak{g}_\alpha$. Then $Z(\mathfrak{g})$ is contained in the direct sum $U(\mathfrak{h})\oplus U(\mathfrak{g})\mathfrak{n}^+$. Let φ be the projection of $Z(\mathfrak{g})$ into $U(\mathfrak{h})$. For $\lambda\in\mathfrak{h}^*$, define the homomorphism χ_λ; $Z(\mathfrak{g})\to\boldsymbol{C}$ by

$$\chi_\lambda(z)=\varphi(z)(\lambda-\rho)\qquad (z\in Z(\mathfrak{g}))\,.$$

Then it turns out that χ_λ does not depend on the choice of Δ^+ (c.f. [4], p. 231). By a well-known result of Harish-Chandra, every homomorphism χ; $Z(\mathfrak{g})\to\boldsymbol{C}$ is of the form χ_λ for some $\lambda\in\mathfrak{h}^*$. Moreover, for $\lambda_1, \lambda_2\in\mathfrak{h}^*$, we have $\chi_{\lambda_1}=\chi_{\lambda_2}$ if and only if $\lambda_2=s\lambda_1$ for some $s\in W$ (c.f. [4], p. 232). In particular, if $\lambda\in\mathfrak{h}^*$ is a dominant regular integral form with respect to Δ^+, χ_λ is the central character of the irreducible $\mathfrak{g}$-module with highest weight $\lambda-\rho$ (c.f. [4], p. 230).

2.3. Now let $\mathfrak{h}'$ be another Cartan subalgebra of $\mathfrak{g}$. For $\lambda'\in(\mathfrak{h}')^*$, let $\chi'_{\lambda'}$; $Z(\mathfrak{g})\to\boldsymbol{C}$ be the homomorphism constructed from λ' in the same way as in 2.2. We shall study the relation between χ_λ and $\chi'_{\lambda'}$. Let θ be an automorphism of $\mathfrak{g}$ such that $\theta(\mathfrak{h}')=\mathfrak{h}$ and let θ^*; $\mathfrak{h}^*\to(\mathfrak{h}')^*$ be the linear isomorphism induced by θ; $\mathfrak{h}'\to\mathfrak{h}$.

Lemma 2.3.1. *If $\lambda \in \mathfrak{h}^*$ is a regular integral form, then we have $\chi_\lambda = \chi'_{\theta^*\lambda}$ as homomorphisms of $Z(\mathfrak{g})$ into $\boldsymbol{C}$.*

Proof. We take a positive root system Δ^+ with respect to which λ is dominant. Let $V_{\lambda-\rho}$ be the irreducible $\mathfrak{g}$-module with highest weight $\lambda-\rho$. Then χ_λ is the central character of $V_{\lambda-\rho}$. On the other hand, if we take $\theta^*(\Delta^+)$ as a positive root system of $(\mathfrak{g}, \mathfrak{h}')$, the highest weight of $V_{\lambda-\rho}$ relative to $\theta^*(\Delta^+)$ is $\theta^*(\lambda-\rho) = \theta^*\lambda - \theta^*\rho \in (\mathfrak{h}')^*$. Hence $\chi'_{\theta^*\lambda}$ is also the central character of $V_{\lambda-\rho}$. Thus the assertion follows.

3. Basic multiplicity formulas

In this section, we shall review multiplicity formulas obtained by DeGeorge and Wallach. It plays a basic role in this paper.

3.1. Let G be as in § 1. Let Γ be a discrete subgroup of G such that $\Gamma\backslash G$ is compact. In this paper, we always assume that Γ has no elements with finite order other than the identity. We fix the G-invariant measure $d\dot{g}$ on $\Gamma\backslash G$ induced by dg. For $U \in \hat{G}$, we denote by $N_\Gamma(U)$ the multiplicity with which U occurs in the right regular representation of G on $L^2(\Gamma\backslash G)$.

Now we take the Cartan subalgebra $\mathfrak{h}$ as in § 1. Consider a special linear form $\Lambda \in \mathfrak{F}_0$ and the discrete class $D_\Lambda \in \hat{G}_d$. Let χ_Λ be the homomorphism of $Z(\mathfrak{g})$ into $\boldsymbol{C}$ constructed from Λ as in 2.2. Denote by $\hat{G}_\Lambda$ the set of all $U \in \hat{G}$ such that its infinitesimal character is χ_Λ; i.e.

$$\hat{G}_\Lambda = \{U \in \hat{G} \mid \chi_U = \chi_\Lambda\} .$$

Clearly we have $\hat{G}_\Lambda = \hat{G}_{s\Lambda}$ for all $s \in W$. Also it is known that $D_\Lambda \in \hat{G}_\Lambda$ and the number of elements in $\hat{G}_\Lambda$ is finite (c.f. [3], p. 141).

In [3], DeGeorge and Wallach obtained a formula describing the relation among numbers in $\{N_\Gamma(U) \mid U \in \hat{G}_\Lambda\}$. We shall precisely review it. Choose and fix a positive root system Δ^+ with respect to which Λ is dominant. Since rank $\mathfrak{g}_0 =$ rank $\mathfrak{k}_0$, we can construct the half-spin representations of $\mathfrak{k}_0$ as in [6], p. 144. Denote by L^+ the half-spin representation of $\mathfrak{k}_0$ with highest weight ρ_n and denote by L^- the other one. Note that $L^\pm$ depends on the choice of Δ^+. Let $\tau_{\Lambda-\rho_k}$ be the irreducible $\mathfrak{k}$-module with highest weight $\Lambda-\rho_k$. Then the tensor product $\tau_{\Lambda-\rho_k} \otimes L^\pm$ integrates to the representation of K which is also denoted by $\tau_{\Lambda-\rho_k} \otimes L^\pm$ (c.f. [3], p. 139). For K-modules τ_1 and τ_2, denote by $\dim \mathrm{Hom}_K(\tau_1, \tau_2)$ the intertwining number of τ_1 and τ_2. Let $U \in \hat{G}$ and let π_U be a representation in the class U. Then define

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, U) = \dim \mathrm{Hom}_K(\tau_{\Lambda-\lambda_k} \otimes L^\pm, \pi_U|_K) .$$

Theorem 3.1.1 (DeGeorge-Wallach, [3]). *Let $\Lambda \in \mathfrak{F}_0$. Then we have*

$$(3.1)\quad d(D_\Lambda)\operatorname{vol}(\Gamma\backslash G) = N_\Gamma(D_\Lambda) + \sum_{U\in\hat{G}_\Lambda-\hat{G}_d} N_\Gamma(U)\{\dim \operatorname{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^+, U) - \dim \operatorname{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^-, U)\}$$

where $d(D_\Lambda)$ *is the formal degree of* D_Λ *and* $\operatorname{vol}(\Gamma\backslash G)$ *is the volume of* $\Gamma\backslash G$.

Our purpose in this paper is to obtain concrete formulas about $N_\Gamma(D_\Lambda)$ by using Theorem 3.1.1. We divide our study into two steps as follows;

Step 1; To find out all elements in $\hat{G}_\Lambda - \hat{G}_d$.

Step 2; To examine $\dim \operatorname{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^\pm, U)$ for every U in $\hat{G}_\Lambda-\hat{G}_d$.

4. The nonunitary principal series

In order to find out irreducible unitary representations of G with given infinitesimal character, we need the family of representations of G which is called the nonunitary principal series. In this section, we shall make preparations about this family. Let G be as in § 1.

4.1. First we shall state some notations. Let $G=KAN$ be an Iwasawa decomposition of G. Let $\mathfrak{a}_0$ and $\mathfrak{n}_0$ be the Lie algebras of A and N. We write $g=k(g)\exp(H(g))\,n(g)$ for $g\in G$, where $k(g)\in K$, $H(g)\in\mathfrak{a}_0$ and $n(g)\in N$.

Let Σ be the restricted root system of $\mathfrak{g}_0$ with respect to $\mathfrak{a}_0$. Choose a positive root system Σ^+ of Σ which is associated with the subalgebra $\mathfrak{n}_0$ (c.f. [16], p. 164). Define $\rho_a=\frac{1}{2}\sum_{\alpha\in\Sigma^+}\alpha$.

Let M be the centralizer of A in K and let $\mathfrak{m}_0$ be the Lie algebra of M. Let $\mathfrak{t}_0$ be a maximal abelian subalgebra of $\mathfrak{m}_0$ and let $\mathfrak{t}_R^*$ be the dual space of $\mathfrak{t}_R=\sqrt{-1}\mathfrak{t}_0$. Denote by Δ_m the root system of $(\mathfrak{m}, \mathfrak{t})$. Choose a positive root system Δ_m^+ of Δ_m and put $\rho_m=\frac{1}{2}\sum_{\alpha\in\Delta_m^+}\alpha$. Also define

$$\mathfrak{F}_m^+ = \left\{\mu\in\mathfrak{t}_R^* \,\middle|\, \frac{2\langle\mu,\alpha\rangle}{\langle\alpha,\alpha\rangle}\in\boldsymbol{Z} \quad\text{and}\quad \geqq 0 \qquad (\alpha\in\Delta_m^+)\right\}$$

Let $\hat{M}$ be the set of all equivalence classes of irreducible M-modules. For $\xi\in\hat{M}$, denote by $\mu(\xi)$ the highest weight of ξ. Then $\mu(\xi)$ belongs to $\mathfrak{F}_m^+$.

If we set $\mathfrak{h}_0'=\mathfrak{t}_0\oplus\mathfrak{a}_0$, $\mathfrak{h}'$ is a Cartan subalgebra of $\mathfrak{g}$. In this section, we use this Cartan subalgebra $\mathfrak{h}'$. Let $(\mathfrak{h}_R')^*$ be the dual space of $\mathfrak{h}_R'=\sqrt{-1}\,\mathfrak{t}_0\oplus\mathfrak{a}_0$. Canonically, $\mathfrak{t}^*$ and $\mathfrak{a}^*$ can be considered as subsets of the dual space $(\mathfrak{h}')^*$ of $\mathfrak{h}'$. Denote by Δ' the root system of $(\mathfrak{g}, \mathfrak{h}')$. Then there exists a positive root system Δ'^+ of Δ' such that $\Delta_m^+\subset\Delta'^+$ and $\{\alpha\in\Delta' \mid \alpha|_{\mathfrak{a}_0}\in\Sigma^+\}\subset\Delta'^+$ (c.f. [16], p. 169). This ordering of $(\mathfrak{h}_R')^*$ is called a *compatible* ordering with respect to Σ^+. Throughout this section, we fix this ordering in $(\mathfrak{h}_R')^*$. Set $\rho'=\frac{1}{2}\sum_{\alpha\in\Delta'^+}\alpha$. Then we have $\rho'=\rho_m+\rho_a$.

4.2. Let $\xi\in\hat{M}$ and let ξ also be a representation of M on the vector space V_ξ which belongs to the class ξ. Let C_ξ be the set of all continuous mappings $f; K\rightarrow V_\xi$ such that $f(km)=\xi(m)^{-1}f(k)$ for all $k\in K$ and $m\in M$. For $f_1, f_2\in C_\xi$, define

$$(f_1, f_2)=\int_K (f_1(k), f_2(k))dk$$

where dk is the normalized Haar measure on K such that $\int_K dk=1$. Denote by E_ξ the completion of C_ξ with respect to this inner product (,).

Let $\nu\in\mathfrak{a}^*$. We define the representation $\pi_{\xi,\nu}$ of G on the Hilbert space E_ξ as follows;

$$(\pi_{\xi,\nu}(g)f)(k)=e^{-(\nu+\rho_a(H(g^{-1}k)))}f(k(g^{-1}k))$$

for $g\in G$, $k\in K$ and $f\in C_\xi$. Then $\pi_{\xi,\nu}$ is not necessarily unitary, but it is K-finite (c.f. [16], p. 232) and quasisimple (c.f. [9], p. 29). The family of representations $\{\pi_{\xi,\nu}\,|\,\xi\in\hat{M},\ \nu\in\mathfrak{a}^*\}$ is called *the nonunitary principal series.*

4.3. We shall review how the infinitesimal character of each representation in the nonunitary principal series is given. Let $\pi_{\xi,\nu}$ ($\xi\in\hat{M}$, $\nu\in\mathfrak{a}^*$) be a representation in the nonunitary principal series. For $\lambda'\in(\mathfrak{h}')^*$, denote by $\chi'_{\lambda'}$ the homomorphism of $Z(\mathfrak{g})$ into $\boldsymbol{C}$ constructed from λ' as in 2.2.. Then we have the following lemma.

Lemma 4.3.1 (Lepowsky [9]). *The infinitesimal character $\chi_{\pi_{\xi,\nu}}$ of $\pi_{\xi,\nu}$ is equal to $\chi'_{\mu(\xi)+\rho_m+\nu}$, where $\mu(\xi)$, ρ_m and ν are regarded as elements in $(\mathfrak{h}')^*$.*

Proof. In [9], $\chi_{\pi_{\xi,\nu}}$ was given explicitly. We shall translate the result in [9] into our desirable form. In Proposition 8.9 of [9], take $-\Delta_m^+=\{-\alpha\,|\,\alpha\in\Delta_m^+\}$ as a positive root system of Δ_m. Note that $\mu(\xi)$ is the lowest weight of ξ with respect to $-\Delta_m^+$. Denote by $\Delta_1'^+$ the positive root system of Δ' determined by Σ^+ and $-\Delta_m^+$. Let $\mathfrak{n}_1'^+$ (resp. $\mathfrak{n}_1'^-$) be the sum of positive (resp. negative) root spaces. Then we have

$$Z(\mathfrak{g})\subset U(\mathfrak{h}')+\mathfrak{n}_1'^+U(\mathfrak{g})\,,$$
$$Z(\mathfrak{g})\subset U(\mathfrak{h}')+U(\mathfrak{g})\mathfrak{n}_1'^-\,.$$

In the above two direct sums, the projection mappings of $Z(\mathfrak{g})$ into $U(\mathfrak{h}')$ are the same (c.f. [4], p. 230). Denote by φ' this projection mapping. Then Proposition 8.9 of [9] implies that

$$\chi_{\pi_{\xi,\nu}}(z)=\varphi'(z)\,(\mu(\xi)+\nu+\rho_a)$$

for $z\in Z(\mathfrak{g})$. On the other hand, use a positive root system $-\Delta_1'^+$ of Δ' in the construction of $\chi'_{\lambda'}$ ($\lambda'\in(\mathfrak{h}')^*$). Then we have

$$\begin{aligned}\chi'_{\mu(\xi)+\rho_m+\nu}(z) &= \varphi'(z)((\mu(\xi)+\rho_m+\nu)-(\rho_m-\rho_a)) \\ &= \varphi'(z)(\mu(\xi)+\nu+\rho_a)\,.\end{aligned}$$

Hence we have $\chi_{\pi_{\xi,\nu}}(z)=\chi'_{\mu(\xi)+\rho_m+\nu}(z)$.

4.4 Finally we shall prepare some facts about the K-spectrum of $\pi_{\xi,\nu}$. Let $\xi\in\hat{M}$ and $\tau\in\hat{K}$. Denote by $m(\tau;\xi)$ the multiplicity with which ξ occurs in the restriction of τ to M. If $m(\tau;\xi)\neq 0$, τ is called to be *ξ-admissible.* Denote by $\hat{K}(\xi)$ the set of all ξ-admissible elements in $\hat{K}$.

Let $\nu\in\mathfrak{a}^*$. From the definition of $\pi_{\xi,\nu}$, the restriction of $\pi_{\xi,\nu}$ ot K is the representation induced from the representation ξ of M. Hence, by Frobenius' reciprocity theorem, we have $m(\pi_{\xi,\nu};\tau)=m(\tau:\xi)$ for $\tau\in\hat{K}$. Thus the K-spectrum of $\pi_{\xi,\nu}$ is the set of all $\tau\in\hat{K}(\xi)$ repeated as many times as $m(\tau;\xi)$.

For $\tau\in\hat{K}$, let $E_\xi(\tau)\subset E_\xi$ be the subspace of all vectors transformed by $\pi_{\xi,\nu}|_K$ according to τ. When Φ is a subset of $\hat{K}(\xi)$, define $E_\xi(\Phi)=\bigoplus_{\tau\in\Phi} E_\xi(\tau)$, where $\oplus$ means the orthogonal direct sum. Note that $E_\xi(\Phi)$ is a K-invariant subspace of E_ξ.

5. The subquotient theorem

In this section, for the sake of **Step 1,** we shall review the subquotient theorem of Harish-Chandra. It will give a method to find out the representations in question.

5.1. Let (π, E) be a continuous representation of G. Let E have a finite sequence of closed invariant subspaces

$$E=E_0\supset E_1\supset\cdots\supset E_{r-1}\supset E_r=\{0\}$$

such that, on each quotient space E_{i-1}/E_i, π induces an irreducible representation of G. Then π is said to *have a finite composition series* and each quotient representation $(\pi, E_{i-1}/E_i)$ is called *an irreducible subquotient of π.*

It is known that every member of the nonunitary principal series has a finite composition series (c.f. [9], p. 33). The following theorem gives a method of realizing our exploring representations.

Theorem 5.1.1 (Harish-Chandra; c.f. [9], p. 29). *Every irreducible unitary representation of G is infinitesimally equivalent to an irreducible subquotient of some member of the nonunitary principal series.*

5.2. Now, let $\mathfrak{h}$ be the Cartan subalgebra in § 1. We consider a special linear form $\Lambda\in\mathfrak{F}_0$. Let Δ^+ be the positive root system with respect to which Λ is dominant. On the other hand, let $\mathfrak{h}'$ and Δ'^+ be as in 4.1. There is an automorphism θ of $\mathfrak{g}$ such that $\theta(\mathfrak{h}')=\mathfrak{h}$ and $\theta^*(\Delta^+)=\Delta'^+$. Identify $\mathfrak{h}^*$ with $(\mathfrak{h}')^*$ by this isomorphism. From Lemma 2.3.1, if $\lambda\in\mathfrak{h}^*$ is a regular integral form, we have $\chi_\lambda=\chi'_\lambda$. Hence, for $\xi\in\hat{M}$ and $\nu\in\mathfrak{a}^*$, we have $\chi'_{\mu(\xi)+\rho_m+\nu}=\chi_\Lambda$ if and only if there exists an element $s\in W$ such that $\mu(\xi)+\rho_m+\nu=s(\Lambda)$. Let $R(\Lambda)$ be the set of all pair (ξ, ν) $(\xi\in\hat{M}, \nu\in\mathfrak{a}^*)$ such that $\mu(\xi)+\rho_m+\nu$ and Λ are in the same W-orbit. From Lemma 4.4.1 and Theorem 5.1.1, we obtain the following proposition.

Proposition 5.2.2. *Let $\Lambda\in\mathfrak{F}_0$ and π be an irreducible unitary representation of G such that $\chi_\pi=\chi_\Lambda$. Then π is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi,\nu}$ for some (ξ, ν) in $R(\Lambda)$.*

6. The decomposition of $\mathfrak{k}$-modules

In this section, we shall describe the direct sum decomposition of a tensor product of two $\mathfrak{k}$-modules. Also we shall review a known result about the $\mathfrak{k}$-module $L^\pm$. These facts will be used in studying **Step 2.**

6.1. Let $\mathfrak{k}_0$ and $\mathfrak{h}_0$ be as in § 1. Let $\mathfrak{c}_0$ be the center of $\mathfrak{k}_0$ and let $\mathfrak{k}_0^1$ be the semisimple part $[\mathfrak{k}_0, \mathfrak{k}_0]$ of $\mathfrak{k}_0$. Then we have

$$\mathfrak{k}_0 = \mathfrak{c}_0\oplus\mathfrak{k}_0^1$$
$$\mathfrak{h}_0 = \mathfrak{c}_0\oplus\mathfrak{h}_0^1$$

where $\mathfrak{h}_0^1$ is the maximal abelian subalgebra of $\mathfrak{k}_0^1$. We always regard $(\mathfrak{h}^1)^*$ and $\mathfrak{c}^*$ as subsets of $\mathfrak{h}^*$.

First we shall examine the decomposition as $\mathfrak{k}^1$-modules. As the positive root system of $(\mathfrak{k}^1, \mathfrak{h}^1)$, we take Δ_k^+ in § 1. For $\gamma\in\mathfrak{F}_k^+\cap(\mathfrak{h}^1)^*$, denote by τ_γ^1 the irreducible $\mathfrak{k}^1$-module with highest weight γ. The following lemma is due to [2] (c.f. [2], Chapter VIII, § 9, Proposition 2).

Lemma 6.1.1. *Let β, γ and δ be in $\mathfrak{F}_k^+\cap(\mathfrak{h}^1)^*$. Then the multiplicity of τ_β^1 in the tensor product $\tau_\gamma^1\otimes\tau_\delta^1$ is given by*

$$\sum_{s\in W_G}\varepsilon(s)M_{\tau_\delta^1}(\beta+\rho_k-s(\gamma+\rho_k))$$

where $M_{\tau_\delta^1}(\lambda)$ $(\lambda\in(\mathfrak{h}^1)^*)$ is the multiplicity of a weight λ in τ_δ^1.

Once we know the decomposition as $\mathfrak{k}^1$-modules, we can easily obtain the decomposition as $\mathfrak{k}$-modules.

Proposition 6.1.2. *Let τ be a $\mathfrak{k}$-module. Let β and γ be in $\mathfrak{F}_k^+$ and let τ_β and τ_γ be the irreducible $\mathfrak{k}$-modules with highest weight β and γ respectively. Then the multiplicity of τ_β in $\tau_\gamma\otimes\tau$ is given by*

$$\sum_{s\in W_G}\varepsilon(s)M_\tau(\beta+\rho_k-s(\gamma+\rho_k))$$

where $M_\tau(\lambda)$ $(\lambda\in\mathfrak{h}^*)$ is the multiplicity of a weight λ in τ.

Proof. It is sufficient to prove the statement in the case that τ is an irreducible $\mathfrak{k}$-module τ_δ ($\delta\in\mathfrak{F}_k^+$). Let β be in $\mathfrak{F}_k^+$. If τ_β occurs in $\tau_\gamma\otimes\tau_\delta$ with nonzero multiplicity, the restriction of $\beta|_\mathfrak{c}$ to $\mathfrak{c}$ is equal to $(\gamma+\delta)|_\mathfrak{c}$. We put

$$\mathfrak{F}_k^+(\gamma,\delta)=\{\beta\in\mathfrak{F}_k^+ \mid \beta|_\mathfrak{c}=(\gamma+\delta)|_\mathfrak{c}\}\,.$$

Assume that $\beta\in\mathfrak{F}_k^+(\gamma,\delta)$. Then τ_β occurs in $\tau_\gamma\otimes\tau_\delta$ with multiplicity m if and only if $\tau_\beta|_{\mathfrak{k}^1}$ occurs in $\tau_\gamma|_{\mathfrak{k}^1}\otimes\tau_\delta|_{\mathfrak{k}^1}$ with multiplicity m. Hence, by Lemma 6.1.1, the multiplicity of τ_β in $\tau_\gamma\otimes\tau_\delta$ is given by

$$\sum_{s\in W_G}\varepsilon(s)M_{\tau^1_{\delta^1}}(\beta^1+\rho_k-s(\gamma^1+\rho_k))$$

where β^1, γ^1 and δ^1 are the restrictions of β, γ and δ to $\mathfrak{h}^1$, respectively. Since, for $\lambda\in\mathfrak{h}^*$ and $s\in W_G$, we have $s(\lambda)|_{\mathfrak{h}^1}=s(\lambda|_{\mathfrak{h}^1})$ and $s(\lambda)|_\mathfrak{c}=\lambda|_\mathfrak{c}$, it follows that

$$\begin{aligned}\{\beta+\rho_k-s(\gamma+\rho_k)\}|_{\mathfrak{h}^1}&=\beta^1+\rho_k-s(\gamma^1+\rho_k)\\ \{\beta+\rho_k-s(\gamma+\rho_k)\}|_\mathfrak{c}&=(\beta-\gamma)|_\mathfrak{c}\\ &=\delta|_\mathfrak{c}\,.\end{aligned}$$

Hence, $\beta+\rho_k-s(\gamma+\rho_k)$ is a weight of τ_δ if and only if $\beta^1+\rho_k-s(\gamma^1+\rho_k)$ is a weight of $\tau^1_{\delta^1}$. Therefore we have

$$M_{\tau_\delta}(\beta+\rho_k-s(\gamma+\rho_k))=M_{\tau^1_\delta}(\beta^1+\rho_k-s(\gamma^1+\rho_k))\,,$$

and the statement is proved for τ_β with $\beta\in\mathfrak{F}_k^+(\gamma,\delta)$.

Next, assume that $\beta\notin\mathfrak{F}_k^+(\gamma,\delta)$. Clearly the multiplicity of τ_β in $\tau_\gamma\otimes\tau_\delta$ is equal to 0. On the other hand, since we have $\{\beta+\rho_k-s(\gamma+\rho_k)\}|_\mathfrak{c}\neq\delta|_\mathfrak{c}$, we obtain that

$$M_{\tau_\delta}(\beta+\rho_k-s(\gamma+\rho_k))=0\qquad(s\in W_G)\,.$$

The proposition is proved.

6.2. Finally we shall state how the set of weights of $L^\pm$ is given. Choose an ordering in $\mathfrak{h}_R^*$ and appoint $L^\pm$ with respect to this ordering. Let Q be a subset of Δ_n^+. Denote by $\langle Q\rangle$ the sum of all elements in Q and denote by $|Q|$ the number of elements in Q. The following lemma is known (c.f. [6], p. 144).

Lemma 6.2.1. *The set of all weights of the $\mathfrak{k}$-module L^+* (resp. L^-) *is given by*

$$\{\rho_n-\langle Q\rangle \mid Q\subset\Delta_n^+,\ (-1)^{|Q|}=+1\ (\text{resp. }-1)\}$$

and the multiplicity of each weight is the number of ways in which it can be expressed in the above form.

7. The case of $G=Spin(1, 2m)$

In this section, we shall study **Step 1** and **Step 2** in the case of $G=Spin(1, 2m)$ and we shall obtain concrete multiplicity formulas. In [12], [13] and [14], Thieleker classified all irreducible unitary representations of $Spin(1, 2m)$. We shall often use his results.

7.1. Let m be an integer such that $m\geqq 2$. Let G be the group $Spin(1, 2m)$; i.e. the universal covering group of the identity component of $SO(1, 2m)$. The subgroup K is isomorphic to the universal covering group $Spin(2m)$ of $SO(2m)$. As usual, we realize $\mathfrak{g}_0$, $\mathfrak{k}_0$ and $\mathfrak{p}_0$ as Lie algebras of matrices;

$$\mathfrak{g}_0=\left\{\left(\begin{array}{c|c} 0 & u \\ \hline {}^t u & X \end{array}\right) \middle| \begin{array}{l} u\in \boldsymbol{R}^{2m} \\ X\in \mathfrak{so}(2m, \boldsymbol{R}) \end{array}\right\}$$

$$\mathfrak{k}_0=\left\{K(X)=\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & X \end{array}\right) \middle| X\in \mathfrak{so}(2m, \boldsymbol{R})\right\}$$

$$\mathfrak{p}_0=\left\{P(u)=\left(\begin{array}{c|c} 0 & u \\ \hline {}^t u & 0 \end{array}\right) \middle| u=(u_1, u_2, \cdots, u_{2m})\in \boldsymbol{R}^{2m}\right\}.$$

Take $\mathfrak{h}_0$ as follows;

$$\mathfrak{h}_0=\left\{H(h)=\left(\begin{array}{c|c|c} 0 & \multicolumn{2}{c}{0} \\ \hline & 0 & \begin{matrix} -h_1 & & 0 \\ & \ddots & \\ 0 & & -h_m \end{matrix} \\ \cline{2-3} 0 & \begin{matrix} h_1 & & 0 \\ & \ddots & \\ 0 & & h_m \end{matrix} & 0 \end{array}\right) \middle| h=(h_1, h_2, \cdots, h_m)\in \boldsymbol{R}^m\right\}$$

For $1\leqq i\leqq m$, define a linear form e_i on $\mathfrak{h}$ by

$$e_i(H(h_1, \cdots, h_m))=\sqrt{-1}\,h_i\,.$$

Then $\mathfrak{h}_{\boldsymbol{R}}^*$ is spanned by $\{e_1, \cdots, e_m\}$ over $\boldsymbol{R}$ and the inner product in $\mathfrak{h}_{\boldsymbol{R}}^*$ is given by

$$\langle e_i, e_j\rangle=\frac{1}{2(2m-1)}\delta_{ij} \qquad (1\leqq i, j\leqq m)\,.$$

Choose and fix a lexicographic ordering in $\mathfrak{h}_{\boldsymbol{R}}^*$ with respect to the basis $\{e_1, \cdots, e_m\}$. Under these situations, the following facts are easily seen;

$$\Delta=\{\pm e_i\ (1\leqq i\leqq m),\quad \pm(e_i+e_j)\ (1\leqq i<j\leqq m)\}$$
$$\Delta^+=\{e_i\ (1\leqq i\leqq m),\quad e_i\pm e_j\ (1\leqq i<j\leqq m)\}$$

$$\Delta_k^+ = \{e_i \pm e_j \ (1 \leqq i < j \leqq m)\}$$
$$\Delta_n^+ = \{e_i \ (1 \leqq i \leqq m)\}$$
$$\rho = \frac{1}{2}\sum_{i=1}^{m}(2m-2i+1)e_i$$
$$\rho_k = \sum_{i=1}^{m}(m-i)e_i$$
$$\rho_n = \frac{1}{2}\sum_{i=1}^{m}e_i$$
$$\mathfrak{F} = \{\lambda = \sum_{i=1}^{m}\lambda_i e_i \,|\, \lambda_i - \lambda_{i+1} \in \boldsymbol{Z} \ (1 \leqq i \leqq m-1),\ 2\lambda_m \in \boldsymbol{Z}\}$$
$$\mathfrak{F}_0 = \{\lambda \in \mathfrak{F} \,|\, \lambda_1, \cdots, \lambda_m \neq 0,\ \lambda_i \neq \lambda_j \ (i \neq j)\}$$
$$\mathfrak{F}_0^+ = \{\lambda \in \mathfrak{F} \,|\, \lambda_1 > \lambda_2 > \cdots > \lambda_m > 0\}$$
$$\mathfrak{F}_k^+ = \{\lambda \in \mathfrak{F} \,|\, \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_{m-1} \geqq |\lambda_m|\} .$$

Note that, if $\lambda = \sum_{i=1}^{m}\lambda_i e_i \in \mathfrak{F}$, $\{\lambda_1, \cdots, \lambda_m\}$ are either all integers or all half odd integers.

Let $\mathfrak{S}_m$ be the symmetric group of degree m and let $\sigma \in \mathfrak{S}_m$. Let $\varepsilon = \{\varepsilon_i\}$ be a set of m signs; i.e. $\varepsilon_i = 1$ or $\varepsilon_i = -1$ for $1 \leqq i \leqq m$. Define the orthogonal transformation $s_{\sigma,\varepsilon}$ of $\mathfrak{h}_R^*$ by

$$s_{\sigma,\varepsilon}(e_i) = \varepsilon_i e_{\sigma(i)} \qquad (1 \leqq i \leqq m) .$$

The Weyl groups W and W_G are given as follows;

$$W = \{s_{\sigma,\varepsilon} \,|\, \sigma \in \mathfrak{S}_m,\ \varepsilon_i = 1 \text{ or } -1 \quad (1 \leqq i \leqq m)\}$$
$$W_G = \{s_{\sigma,\varepsilon} \,|\, \sigma \in \mathfrak{S}_m,\ \varepsilon_i = 1 \text{ or } -1 \quad (1 \leqq i \leqq m),\ \prod_{i=1}^{m}\varepsilon_i = 1\}$$

Define $s_0 \in W$ by $s_0(e_i) = e_i$ for $1 \leqq i \leqq m-1$ and $s_0(e_m) = -e_m$. Then we have $W^1 = \{1, s_0\}$, where 1 denotes the identity element of W. Hence the mapping D; $\mathfrak{F}_0^+ \cup s_0\mathfrak{F}_0^+ \to \hat{G}_d$ is bijective.

Take a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$ defined by

$$\mathfrak{a}_0 = \{P(a, 0, \cdots, 0) \,|\, a \in \boldsymbol{R}\} .$$

Then M is isomorphic to $Spin(2m-1)$ and we have

$$\mathfrak{m}_0 = \left\{ K(X) \,\middle|\, X = \left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & Y \end{array}\right) \,\middle|\, Y \in \mathfrak{so}(2m-1, \boldsymbol{R}) \right\} .$$

Also take a maximal abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{m}_0$ defined by

$$\mathfrak{t}_0 = \{H(0, h_2, \cdots, h_m) \,|\, h_i \in \boldsymbol{R} \qquad (2 \leqq i \leqq m)\} .$$

Two Cartan subalgebras $\mathfrak{h}'=\mathfrak{t}\oplus\mathfrak{a}$ and $\mathfrak{h}$ can be identified by the following isomorphism θ;

$$\theta(P(a,0,\cdots,0)+H(0,h_2,\cdots,h_m))=H(a,h_2,\cdots,h_m).$$

Moreover, by this identification, the above ordering in $\mathfrak{h}_R^*$ corresponds to a compatible ordering of $(\mathfrak{h}'_R)^*$. In this way, we always identify $(\mathfrak{h}'_R)^*$ with $\mathfrak{h}_R^*$, together with their orderings.

As for $(\mathfrak{m},\mathfrak{t})$, $\mathfrak{t}_R^*$ is spanned by $\{e_2,\cdots,e_m\}$ and, in $\mathfrak{t}_R^*$, the lexicographic ordering with respect to this basis is given. Hence Δ_m^+, ρ_m and $\mathfrak{F}_m^+$ are given as follows;

$$\Delta_m^+=\{\,e_i\quad(2\leqq i\leqq m),\qquad e_i+e_j\quad(2\leqq i<j\leqq m)\}$$

$$\rho_m=\sum_{i=2}^m\frac{2m-2i+1}{2}e_i$$

$$\mathfrak{F}_m^+=\left\{\mu=\sum_{i=2}^m\mu_ie_i\,\middle|\,\begin{array}{l}\mu_i-\mu_{i+1}\in\boldsymbol{Z}\text{ and }\geqq 0\quad(2\leqq i\leqq m-1)\\ 2\mu_m\in\boldsymbol{Z}\text{ and }\geqq 0\end{array}\right\}$$

Since M is simply connected and semisimple, the mapping μ; $\hat{M}\ni\xi\rightarrow\mu(\xi)\in\mathfrak{F}_m^+$ is bijective. Also we have $\mathfrak{a}^*=\boldsymbol{C}e_1\subset\mathfrak{h}^*$. We shall identify $\nu e_1\in\mathfrak{a}^*$ with the complex number $\nu\in\boldsymbol{C}$.

7.2. Now we shall set about **Step 1.** Consider and fix a special linear form $\Lambda=\sum_{i=1}^m\Lambda_ie_i\in\mathfrak{F}_0^+$. Then the numbers $\{\Lambda_1,\cdots,\Lambda_m\}$ are either all integers or all half odd integers and satisfy the inequalities

$$\Lambda_1>\Lambda_2>\cdots>\Lambda_m>0. \tag{7.1}$$

Depending on Proposition 5.2.2, we shall first determine the set $R(\Lambda)\subset\hat{M}\times\mathfrak{a}^*$. For $\lambda\in\mathfrak{h}^*$, denote by $\lambda|_\mathfrak{t}$ and $\lambda|_\mathfrak{a}$ the restrictions of λ to $\mathfrak{t}$ and to $\mathfrak{a}$, respectively. If $(\xi,\nu)\in R(\Lambda)$, we have $\mu(\xi)=s(\Lambda)|_\mathfrak{t}-\rho_m$ and $\nu=s(\Lambda)|_\mathfrak{a}$ for some $s\in W$. Hence it is necessary to find out all $s\in W$ such that $s(\Lambda)|_\mathfrak{t}-\rho_m$ belongs to $\mathfrak{F}_m^+$. Let j be an integer such that $1\leqq j\leqq m$. Define $\sigma_j\in\mathfrak{S}_m$ and $\varepsilon_j^\pm$ by

$$\sigma_j=(1,2)(2,3)\cdots(j-1,j)$$

$$\varepsilon_j^\pm=\{1,\cdots,\underset{\stackrel{\wedge}{j}}{\pm1},\cdots,1\}$$

and put $s_j^\pm=s_{\sigma_j,\varepsilon_j^\pm}\in W$.

Lemma 7.2.1. *Let $s\in W$. Then we have $s(\Lambda)|_\mathfrak{t}-\rho_m\in\mathfrak{F}_m^+$ if and only if s is either s_j^+ or s_j^- for some $1\leqq j\leqq m$.*

Proof. Let $\sigma\in\mathfrak{S}_m$ and $\varepsilon=\{\varepsilon_i\}$. We have

$$s_{\sigma,\varepsilon}(\Lambda)|_\mathfrak{t}-\rho_m=\sum_{i=2}^m\left(\varepsilon_{\sigma^{-1}(i)}\Lambda_{\sigma^{-1}(i)}-\frac{2m-2i+1}{2}\right)e_i.$$

Hence $s_{\sigma,\varepsilon}(\Lambda)|_{\mathfrak{t}}-\rho_m\in\mathfrak{F}_m^+$ if and only if

$$(7.2)\qquad \begin{cases} \varepsilon_{\sigma^{-1}(i)}\Lambda_{\sigma^{-1}(i)}\geqq\varepsilon_{\sigma^{-1}(i+1)}\Lambda_{\sigma^{-1}(i+1)}+1 & (2\leqq i\leqq m-1)\\ \varepsilon_{\sigma^{-1}(m)}\Lambda_{\sigma^{-1}(m)}\geqq\dfrac{1}{2}\,. \end{cases}$$

Clearly, σ_j and $\varepsilon_j^{\pm}$ satisfy (7.2). Conversely assume that $s_{\sigma,\varepsilon}(\Lambda)|_{\mathfrak{t}}-\rho_m\in\mathfrak{F}_m^+$. Then the left hand sides of (7.2) are positive. Since $\Lambda_{\sigma^{-1}(i)}$ is positive for all i, $\varepsilon_{\sigma^{-1}(i)}$ must be 1 for $2\leqq i\leqq m$. Moreover, from (7.2), we have $\Lambda_{\sigma^{-1}(i)}>\Lambda_{\sigma^{-1}(i+1)}$ for $2\leqq i\leqq m-1$. Hence, from (7.1), we obtain $\sigma^{-1}(2)<\sigma^{-1}(3)<\cdots<\sigma^{-1}(m)$. If we put $\sigma^{-1}(1)=j$, we have necessarily $\sigma=(1,2)(2,3)\cdots(j-1,j)$. The lemma is proved.

Denote by ξ_j the element of $\hat{M}$ with highest weight $s_j^{\pm}(\Lambda)|_{\mathfrak{t}}-\rho_m$; i.e. the highest weight $\mu(\xi_j)$ of ξ_j is given by

$$\begin{cases} \mu(\xi_1)=\sum\limits_{i=2}^{m}\Big(\Lambda_i-\dfrac{2m-2i+1}{2}\Big)e_i\\ \mu(\xi_j)=\sum\limits_{i=2}^{j}\Big(\Lambda_{i-1}-\dfrac{2m-2i+1}{2}\Big)e_i+\sum\limits_{i=j+1}^{m}\Big(\Lambda_i-\dfrac{2m-2i+1}{2}\Big)e_i & (2\leqq j\leqq m-1)\\ \mu(\xi_m)=\sum\limits_{i=2}^{m}\Big(\Lambda_{i-1}-\dfrac{2m-2i+1}{2}\Big)e_i \end{cases}$$

Also note that $s_j^{\pm}(\Lambda)|_{\mathfrak{a}}=\pm\Lambda_j$. After all, we obtain that

$$R(\Lambda)=\{\,(\xi_j,\Lambda_j),(\xi_j,-\Lambda_j)\ |\quad 1\leqq j\leqq m\}\,.$$

Now consider the representations π_{ξ_j,Λ_j} and $\pi_{\xi_j,-\Lambda_j}$ $(1\leqq j\leqq m)$. From Proposition 5.2.2, an irreducible unitary representation π such that $\chi_\pi=\chi_\Lambda$ is infinitesimally equivalent to an irreducible subquotient of π_{ξ_j,Λ_j} or $\pi_{\xi_j,-\Lambda_j}$ for some $1\leqq j\leqq m$.

For convenience' sake, we put $\Lambda-\rho=\sum\limits_{i=1}^{m}\bar{\Lambda}_ie_i$; i.e.

$$\bar{\Lambda}_i=\Lambda_i-\frac{2m-2i+1}{2}\qquad(1\leqq i\leqq m)\,.$$

Hence we have

$$(7.3)\qquad \bar{\Lambda}_1\geqq\bar{\Lambda}_2\geqq\cdots\geqq\bar{\Lambda}_m\geqq 0\,,$$

and we can write

$$(7.4)\qquad \begin{cases} \mu(\xi_1)=\sum\limits_{i=2}^{m}\bar{\Lambda}_ie_i\\ \mu(\xi_j)=\sum\limits_{i=2}^{j}(\bar{\Lambda}_{i-1}+1)e_i+\sum\limits_{i=j+1}^{m}\bar{\Lambda}_ie_i & (2\leqq j\leqq m-1)\\ \mu(\xi_m)=\sum\limits_{i=2}^{m}(\bar{\Lambda}_{i-1}+1)e_i \end{cases}$$

7.3. Next we shall find out all irreducible subquotients of $\pi_{\xi_j,\pm\Lambda_j}$ $(1\leqq j\leqq m)$, up to infinitesimally equivalence. In [13], Thieleker determined all irreducible subquotients of every representation in the nonunitary principal series. Let us rely on his results.

Let ξ be in $\hat{M}$ and τ in $\hat{K}$. The highest weight $\mu(\xi)$ of ξ and the highest weight $\gamma(\tau)$ of τ can be written as follows;

$$\mu(\xi) = \sum_{i=2}^{m} \mu(\xi)_i e_i$$
$$\gamma(\tau) = \sum_{i=1}^{m} \gamma(\tau)_i e_i \,.$$

By Lemma 4 in § 5 of [12], τ is ξ-admissible if and only if $\gamma(\tau)$ satisfies the following conditions;

$$\text{(7.5)}\qquad \begin{cases} \mu(\xi)_i-\gamma(\tau)_i\in \boldsymbol{Z} & (2\leqq i\leqq m) \\ \gamma(\tau)_1\geqq\mu(\xi)_2\geqq\gamma(\tau)_2\geqq\mu(\xi)_3\geqq\cdots\geqq\mu(\xi)_m\geqq|\gamma(\tau)_m|. \end{cases}$$

Moreover, for every ξ-admissible τ, we have $m(\tau;\xi)=1$. Hence, by the statement in 4.4, we have

$$\begin{aligned}\Phi(\pi_{\xi,\nu}) &= \hat{K}(\xi) \\ &= \{\tau\in\hat{K}\,|\,\gamma(\tau) \text{ satisfies (7.5)}\}\,.\end{aligned}$$

Also, note that every K-invariant subspace of E_ξ, and hence every G-invariant subspace of E_ξ, is of the form $E_\xi(\Phi)$ for a certain subset Φ of $\hat{K}(\xi)$.

Following Thieleker, we shall define some K-invariant subspaces of E_{ξ_j} $(1\leqq j\leqq m)$. In the case that $1\leqq j\leqq m-1$, we have

$$\text{(7.6)}\qquad \hat{K}(\xi_j) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{array}{l} \gamma_i-\bar{\Lambda}_i\in\boldsymbol{Z} \qquad (1\leqq i\leqq m) \\ \gamma_1\geqq\bar{\Lambda}_1+1\geqq\gamma_2\geqq\cdots\geqq\gamma_{j-1}\geqq\bar{\Lambda}_{j-1}+1 \\ \qquad\geqq\gamma_j\geqq\bar{\Lambda}_{j+1}\geqq\cdots\geqq\bar{\Lambda}_m\geqq|\gamma_m| \end{array}\right\},$$

where $\gamma=\sum_{i=1}^{m}\gamma_i e_i\in\mathfrak{F}\cap\mathfrak{F}_k^+$. Let Φ_j^+ and Φ_j^- be subsets of $\hat{K}(\xi_j)$ defined as follows;

$$\text{(7.7)}\qquad \Phi_j^+ = \{\tau_\gamma\in\hat{K}(\xi_j)\,|\,\gamma_j\geqq\bar{\Lambda}_j+1\}$$
$$\text{(7.8)}\qquad \Phi_j^- = \{\tau_\gamma\in\hat{K}(\xi_j)\,|\,\gamma_j<\bar{\Lambda}_j+1\}\,,$$

and put $E_j^+=E_{\xi_j}(\Phi_j^+)$ and $E_j^-=E_{\xi_j}(\Phi_j^-)$. In the case that $j=m$, we have

$$\hat{K}(\xi_m) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{array}{l} \gamma_i-\bar{\Lambda}_i\in\boldsymbol{Z} \qquad (1\leqq i\leqq m) \\ \gamma_1\geqq\bar{\Lambda}_1+1\geqq\gamma_2\geqq\cdots\geqq\gamma_{m-1}\geqq\bar{\Lambda}_{m-1}+1\geqq|\gamma_m| \end{array}\right\}.$$

Let Φ_m^+, Φ_m^- and Φ_m^F be subsets of $\hat{K}(\xi_m)$ defined as follows;

$$\Phi_m^+ = \{\tau_\gamma \in \hat{K}(\xi_m) \mid \gamma_m \geqq \bar{\Lambda}_m + 1\}$$
$$\Phi_m^- = \{\tau_\gamma \in \hat{K}(\xi_m) \mid \gamma_m \leqq -(\bar{\Lambda}_m + 1)\}$$
$$\Phi_m^F = \{\tau_\gamma \in \hat{K}(\xi_m) \mid -(\bar{\Lambda}_m + 1) < \gamma_m < \bar{\Lambda}_m + 1\}\ ,$$

and put $E_m^+ = E_{\xi_m}(\Phi_m^+)$, $E_m^- = E_{\xi_m}(\Phi_m^-)$ and $E_m^F = E_{\xi_m}(\Phi_m^F)$.

REMARK 7.3.1. In [14], our representation $(\pi_{\xi,\nu}, E_\xi)$ is denoted by $(\pi_\nu, L_\xi^2(K))$. Hence $E_j^\pm$ $(1 \leqq j \leqq m)$ corresponds to $D_j^\pm(\Lambda_j, \xi_j)$ in [14] and E_m^F corresponds ot $D_m^F(\Lambda_m, \xi_m)$.

Under these notations, apply Theorem 2 and Theorem 3 in [14] to our representations $\pi_{\xi_j, \pm\Lambda_j} (1 \leqq j \leqq m)$. For $1 \leqq j \leqq m-1$, E_j^+ (resp. E_j^-) is an irreducible G-invariant subspace of E_{ξ_j} under the action π_{ξ_j, Λ_j} (resp. $\pi_{\xi_j, -\Lambda_j}$). Also E_m^+ and E_m^- are irreducible G-invariant subspaces of E_{ξ_m} under the action π_{ξ_m, Λ_m} and E_m^F is an irreducible G-invariant subspace of E_{ξ_m} under the action $\pi_{\xi_m, -\Lambda_m}$. There are no more irreducible G-invariant subspaces of E_{ξ_j} $(1 \leqq j \leqq m)$ than the above.

Among these subrepresentations and quotient representations by them, there are following infinitesimal equivalences;

$$(\pi_{\xi_j, \Lambda_j}, E_j^+) \sim (\pi_{\xi_{j+1}, -\Lambda_{j+1}}, E_{j+1}^-) \qquad (1 \leqq j \leqq m-2)$$
$$(\pi_{\xi_{m-1}, \Lambda_{m-1}}, E_{m-1}^+) \sim (\pi_{\xi_m, -\Lambda_m}, E_m^F)$$
$$(\pi_{\xi_j, \Lambda_j}, E_{\xi_j}/E_j^+) \sim (\pi_{\xi_j, -\Lambda_j}, E_j^-) \qquad (1 \leqq j \leqq m-1)$$
$$(\pi_{\xi_j, -\Lambda_j}, E_{\xi_j}/E_j^-) \sim (\pi_{\xi_j, \Lambda_j}, E_j^+) \qquad (1 \leqq j \leqq m-1)$$
$$(\pi_{\xi_m, \Lambda_m}, E_{\xi_m}/E_m^+ + E_m^-) \sim (\pi_{\xi_m, -\Lambda_m}, E_m^F)$$
$$(\pi_{\xi_m, -\Lambda_m}, E_{\xi_m}/E_m^F) \sim (\pi_{\xi_m, \Lambda_m}, E_m^+ \oplus E_m^-)$$

where the symbol $\sim$ denotes the infinitesimal equivalence. Moreover, there is no more infinitesimal equivalences than the above. For the simplification, denote by $\pi_j^\pm(\Lambda)$ $(1 \leqq j \leqq m-1)$ the representation $(\pi_{\xi_j, \pm\Lambda_j}, E_j^\pm)$ and denote by $\pi_m^\pm(\Lambda)$ the representation $(\pi_{\xi_m, \Lambda_m}, E_m^\pm)$. After all, we have the following proposition.

Proposition 7.3.2. *Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to* χ_Λ *are given by*

$$\{\pi_1^-(\Lambda), \pi_1^+(\Lambda), \pi_2^+(\Lambda), \cdots, \pi_{m-1}^+(\Lambda), \pi_m^+(\Lambda), \pi_m^-(\Lambda)\}\ .$$

Moreover these representations are not infinitesimally equivalent to one another.

7.4. We shall examine which subquotients in Proposition 7.3.2 are infinitesimally unitary. Using Theorem 4 in [14], we have the following proposition.

Proposition 7.4.1.

(1). *The representation $\pi_1^-(\Lambda)$ is infinitesimally unitary if and onld if $\Lambda=\rho$. Moreover, $\pi_1^-(\rho)$ is the trivial representation 1_G of G.*

(2). *For $1\leqq j\leqq m-1$, $\pi_j^+(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_{j+1}=\bar{\Lambda}_{j+2}=\cdots=\bar{\Lambda}_m=0$.*

(3). *The representation $\pi_m^{\pm}(\Lambda)$ is always infinitesimally unitary.*

Proof. By (3) of Theorem 4 in [14], $\pi_1^-(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_2=0$ and $\Lambda_1=\dfrac{2m-1}{2}$. If $\Lambda_1=\dfrac{2m-1}{2}$, we have $\bar{\Lambda}_1=0$ and hence, by (7.3), $\bar{\Lambda}_1=\bar{\Lambda}_2=\cdots=\bar{\Lambda}_m=0$. This implies $\Lambda=\rho$. Also, by the definition of E_1^-, $\pi_1^-(\rho)$ is clearly the one-dimensional trivial representation of G. Hence the assertion (1) follows.

The assertion (2) follows immediately by (3) of Theorem 4 in [1].

In the case of $j=m$, $\pi_m^{\pm}(\Lambda)$ is infinitesimally unitary if and only if $\bar{\Lambda}_{m-1}+1>0$. This inequality is always satisfied. Thus the assertion (3) is proved. The proof is completed.

Now, if π is an irreducible infinitesimally unitary representation of G, π is infinitesimally equivalent to a unique irreducible unitary representation of G, up to unitarily equivalence. Hence π determines a class in $\hat{G}$. When Λ satisfies the condition that $\bar{\Lambda}_{j+1}=\cdots=\bar{\Lambda}_m=0$, denote by $U_j(\Lambda)\in\hat{G}$ the class which is determined by $\pi_j^+(\Lambda)$. Also denote by $U_m^{\pm}(\Lambda)$ the class which is determined by $\pi_m^{\pm}(\Lambda)$.

By Theorem 5 in [14], $U_m^{\pm}(\Lambda)$ is a discrete class. Note that, by Theorem 6 in [14], $U_j(\Lambda)$ $(1\leqq j\leqq m-1)$ is not a discrete class. In effect, we have the following corollary and complete **Step 1** for $\Lambda\in\mathfrak{F}_0^+$.

Corollary 7.4.2. *Let $\Lambda\in\mathfrak{F}_0^+$. The subset $\hat{G}_\Lambda-\hat{G}_d$ of $\hat{G}$ is given as follows;*

(1). *If $\bar{\Lambda}_m\neq0$, we have $\hat{G}_\Lambda-\hat{G}_d=\phi$.*

(2). *If $\bar{\Lambda}_{j+1}=\cdots=\bar{\Lambda}_m=0$ and $\bar{\Lambda}_j\neq0$ for some $1\leqq j\leqq m-1$, we have $\hat{G}_\Lambda-\hat{G}_d=\{U_j(\Lambda), U_{j+1}(\Lambda), \cdots, U_{m-1}(\Lambda)\}$.*

(3). *If $\bar{\Lambda}_1=\bar{\Lambda}_2=\cdots=\bar{\Lambda}_m=0$ i.e. $\Lambda=\rho$, we have $\hat{G}_\Lambda-\hat{G}_d=\{1_G, U_1(\Lambda), U_2(\Lambda), \cdots, U_{m-1}(\Lambda)\}$, where 1_G is the class of the trivial representation of G.*

REMARK 7,4.3. In fact, we have $U_m^+(\Lambda)=D_\Lambda\in\hat{G}_d$ and $U_m^-(\Lambda)=D_{s_0\Lambda}\in\hat{G}_d$. This is shown by comparing K-spectra of these representations.

7.5. We shall go forward **Step 2.** As before, let Λ be in $\mathfrak{F}_0^+$. Our purpose is to compute $\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^{\pm}, U_j(\Lambda))$ $(1\leqq j\leqq m-1)$ and $\dim \mathrm{Hom}_K(\tau_{\rho_n}\otimes L^{\pm}, 1_G)$.

Let j be an integer such that $1\leqq j\leqq m-1$. When Λ satisfies that $\bar{\Lambda}_{j+1}=\bar{\Lambda}_{j+2}=\cdots=\bar{\Lambda}_m=0$, a representation in the class $U_j(\Lambda)$ has the same K-spectrum as $\pi_j^+(\Lambda)$. Hence we shall examine $\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^{\pm}, \pi_j^+(\Lambda)|_K)$ and

$\dim \mathrm{Hom}_K(\tau_{\rho_n} \otimes L^{\pm}, 1_G|_K)$. Recall that the K-spectrum of $\pi_j^+(\Lambda)$ is given by

(7.9) $\Phi(\pi_j^+(\Lambda)) = \Phi_j^+$

$$= \left\{ \tau_\gamma \in \hat{K} \,\middle|\, \begin{array}{l} \gamma_i - \bar{\Lambda}_i \in \boldsymbol{Z} \qquad (1 \leqq i \leqq m) \\ \gamma_1 \geqq \bar{\Lambda}_1 + 1 \geqq \gamma_2 \geqq \bar{\Lambda}_2 + 1 \geqq \cdots \geqq \gamma_j \geqq \bar{\Lambda}_j + 1 \\ \bar{\Lambda}_{j+1} \geqq \gamma_{j+1} \geqq \bar{\Lambda}_{j+2} \geqq \cdots \geqq \gamma_{m-1} \geqq \bar{\Lambda}_m \geqq |\gamma_m| \end{array} \right\}.$$

Also the K-spectrum of 1_G is given by $\Phi(1_G) = \{1_K\}$, where 1_K is the trivial representation of K. Thus it is sufficient to compute the multiplicities of $\tau \in \Phi(\pi_j^+(\Lambda))$ in $\tau_{\Lambda - \rho_k} \otimes L^{\pm}$ and the multiplicity of 1_K in $\tau_{\rho_n} \otimes L^{\pm}$.

In order to use Proposition 6.1.2, we shall prepare the following lemma. Put $L = L^+ \oplus L^-$.

Lemma 7.5.1. *Let β and γ be in $\mathfrak{F}_k^+$.*

(1). *Let $s \in W_G$. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of L, s is the identity.*

(2). *The multiplicity of τ_β in $\tau_\gamma \otimes L$ is equal to $M_L(\beta - \gamma)$, where $M_L(\lambda)$ $(\lambda \in \mathfrak{h}_R^*)$ is the multiplicity of the weight λ in L.*

Proof. Note that, by Lemma 6.2.1, the set of weights of L^+ (resp. L^-) is given by

$$(7.10) \qquad \left\{ \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i \,\middle|\, \begin{array}{l} \varepsilon_i = 1 \text{ or } -1 \qquad (1 \leqq i \leqq m) \\ \prod_{i=1}^m \varepsilon_i = +1 \text{ (resp. } -1) \end{array} \right\},$$

and the multiplicity of each weight is equal to 1. If $\beta + \rho_k - s(\gamma + \rho_k)$ is a weight of L, we have

$$\beta + \rho_k - s(\gamma + \rho_k) = \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i$$

where $\varepsilon_i = 1$ or -1 for $1 \leqq i \leqq m$. Assume that s is not the identity. Then there is an element α of Δ_k^+ such that $s^{-1}(\alpha) \in -\Delta_k^+$. Since $\beta + \rho_k$ and $\gamma + \rho_k$ are dominant regular integral forms with respect to Δ_k^+, we have

$$\frac{2\langle \beta + \rho_k - s(\gamma + \rho_k), \alpha \rangle}{\langle \alpha, \alpha \rangle} = \frac{2\langle \beta + \rho_k, \alpha \rangle}{\langle \alpha, \alpha \rangle} + \frac{2\langle \gamma + \rho_k, -s^{-1}(\alpha) \rangle}{\langle \alpha, \alpha \rangle} \geqq 2.$$

On the other hand, if we set $\alpha = e_i \pm e_j$ for some $1 \leqq i < j \leqq m$, we have

$$\frac{2\left\langle \frac{1}{2} \sum_{i=1}^m \varepsilon_i e_i, \alpha \right\rangle}{\langle \alpha, \alpha \rangle} = \frac{\varepsilon_i \pm \varepsilon_j}{2} \leqq 1.$$

This is the contradiction. Thus the assertion (1) is proved. The assertion (2) follows immediately from Proposition 6.1.2 and the assertion (1). The lemma is proved.

The following proposition completes **Step 2** for $\Lambda \in \mathfrak{F}_0^+$.

Proposition 7.5.2.
(1). *Let j be an integer such that $1 \leqq j \leqq m-1$. Then we have*

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^+, \pi_j^+(\Lambda)|_K) = \begin{cases} 1 & \text{if } m-j \text{ is even}, \\ 0 & \text{if } m-j \text{ is odd}, \end{cases}$$

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^-, \pi_j^+(\Lambda)|_K) = \begin{cases} 0 & \text{if } m-j \text{ is even}, \\ 1 & \text{if } m-j \text{ is odd}. \end{cases}$$

(2). *We have*

$$\dim \mathrm{Hom}_K(\tau_{\rho_n} \otimes L^+, 1_G|_K) = \begin{cases} 1 & \text{if } m \text{ is even}, \\ 0 & \text{if } m \text{ is odd}, \end{cases}$$

$$\dim \mathrm{Hom}_K(\tau_{\rho_n} \otimes L^-, 1_G|_K) = \begin{cases} 0 & \text{if } m \text{ is even}, \\ 1 & \text{if } m \text{ is odd}. \end{cases}$$

Proof. Let $\tau_\gamma \in \hat{K}$. Assume that $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ and the multiplicity of τ_γ in $\tau_{\Lambda-\rho_k} \otimes L$ is not equal to 0. By (2) of Lemma 7.5.1, we have $M_L(\gamma-\Lambda+\rho_k) \neq 0$. Hence, by (7.10), we have $\gamma-\Lambda+\rho_k = \frac{1}{2}\sum_{i=1}^m \varepsilon_i e_i$ for some $\{\varepsilon_1, \cdots, \varepsilon_m\}$ such that $\varepsilon_i = 1$ or -1 $(1 \leqq i \leqq m)$; i.e.

$$\begin{aligned} \gamma_i &= \Lambda_i - m + i + \frac{\varepsilon_i}{2} \\ &= \bar{\Lambda}_i + \frac{\varepsilon_i + 1}{2} \qquad (1 \leqq i \leqq m). \end{aligned}$$

On the other hand, $\{\gamma_1, \cdots, \gamma_m\}$ satisfy the inequalities (7.9). Therefore we have necessarily

$$\varepsilon_1 = \varepsilon_2 = \cdots = \varepsilon_j = 1 \quad \text{and} \quad \varepsilon_{j+1} = \varepsilon_{j+2} = \cdots = \varepsilon_m = -1,$$

and hence we have

$$\gamma = \sum_{i=1}^{j} (\bar{\Lambda}_i + 1) e_i + \sum_{i=j+1}^{m} \bar{\Lambda}_i e_i. \tag{7.11}$$

Conversely, if γ is given by (7.11), then we have clearly that $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ and $\gamma-\Lambda+\rho_k = \frac{1}{2}(e_1 + \cdots + e_j - e_{j+1} - \cdots - e_m)$. Then, by (2) of Lemma 7.5.1, the multiplicity of τ_γ in $\tau_{\Lambda+\rho_k} \otimes L$ is equal to 1. Whether τ_γ occurs in $\tau_{\Lambda-\rho_k} \otimes L^+$ or $\tau_{\Lambda-\rho_k} \otimes L^-$ is determined by the sign of $(-1)^{m-j}$. Since each $\tau_\gamma \in \Phi(\pi_j^+(\Lambda))$ occurs with multiplicity 1 in $\pi_j^+(\Lambda)|_K$, the assertion (1) is proved.

Next, the highest weight of 1_K is 0. Since $-\rho_n$ is a weight of L, by (2) of

Lemma 7.5.1, the multiplicity of 1_K in $\tau_{\rho_n}\otimes L$ is equal to 1. Whether 1_K occurs in $\tau_{\rho_n}\otimes L^+$ or $\tau_{\rho_n}\otimes L^-$ depends on the sign of $(-1)^m$. The assertion (2) immediately follows. The proof is completed.

7.6. Now, we have completed **Step 1** and **Step 2.** From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of $G=Spin(1, 2m)$ $(m\geqq 2)$.

Let Γ be a torsion free discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then we have the following theorem.

Theorem 7.6.1. *Let* Λ *be in* $\mathfrak{F}_0^+$ *and set* $\Lambda-\rho=\sum_{i=1}^{m}\bar{\Lambda}_i e_i$. *Then we have the following formulas about the multiplicities* $N_\Gamma(D_\Lambda)$ *and* $N_\Gamma(D_{s_0\Lambda})$;

(I). $N_\Gamma(D_\Lambda)=N_\Gamma(D_{s_0\Lambda})$.

(II). (1). *If* $\bar{\Lambda}_m\neq 0$,

$$N_\Gamma(D_\Lambda) = d(D_\Lambda)\,\mathrm{vol}(\Gamma\backslash G)\,.$$

(2). *If* $\bar{\Lambda}_{j+1}=\cdots=\bar{\Lambda}_m=0$ *and* $\bar{\Lambda}_j\neq 0$ *for some* $1\leqq j\leqq m-1$,

$$N_\Gamma(D_\Lambda) = d(D_\Lambda)\,\mathrm{vol}(\Gamma\backslash G)+N_\Gamma(U_{m-1}(\Lambda))-N_\Gamma(U_{m-2}(\Lambda))+\cdots$$
$$\cdots+(-1)^{m-j-1}N_\Gamma(U_j(\Lambda))\,.$$

(3). *If* $\bar{\Lambda}_1=\bar{\Lambda}_2=\cdots=\bar{\Lambda}_m=0$, *i.e.* $\Lambda=\rho$,

$$N_\Gamma(D_\rho) = d(D_\rho)\,\mathrm{vol}(\Gamma\backslash G)+N_\Gamma(U_{m-1}(\rho))-N_\Gamma(U_{m-2}(\rho))+\cdots$$
$$\cdots+(-1)^m N_\Gamma(U_1(\rho))+(-1)^{m+1}\,.$$

Proof. The assertion (II) follows directly from Theorem 3.1.1, Corollary 7.4.2 and Proposition 7.5.2.

We shall prove the assertion (I). To begin with, note that the positive root system of Δ with respect to which $s_0\Lambda$ is dominant is given by $s_0\Delta^+$ and we have $s_0\Delta^+\cap\Delta_k=\Delta_k^+$. Compare the multiplicity formula (3.1) for D_Λ with that for $D_{s_0\Lambda}$. From the formula (1.1), we have $d(D_\Lambda)=d(D_{s_0\Lambda})$. Since $\hat{G}_\Lambda=\hat{G}_{s_0\Lambda}$, it is sufficient to prove that

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k}\otimes L^\pm, U) = \dim \mathrm{Hom}_K(\tau_{s_0\Lambda-\rho_k}\otimes L^\pm, U)$$

for $U\in\hat{G}_\Lambda-\hat{G}_d$, where $L^\pm$ in the right hand side is appointed with respect to the positive root system $s_0\Delta^+$.

Since $s_0\Delta^+\cap\Delta_n=s_0\Delta_n^+$, the set of weights of $L^\pm$ associated with $s_0\Delta^+$ is given by

$$\{s_0\langle Q\rangle-s_0\rho_n\,|\,Q\subset\Delta_n^+,\ (-1)^{m-|Q|} = \pm 1\}\,.$$

Let γ be in $\mathfrak{F}_k^+$. Then $s_0\gamma$ is also in $\mathfrak{F}_k^+$. For $s\in W_G$ and $Q\subset\Delta_n^+$, $\gamma+\rho_k-s\Lambda=\langle Q\rangle-\rho_n$ if and only if $s_0\gamma+\rho_k-(s_0 s s_0^{-1})(s_0\Lambda)=s_0\langle Q\rangle-s_0\rho_n$. Also the mapping;

$W_G \in s \to s_0 s s_0^{-1} \in W$ is a bijection of W_G onto W_G and we have $\varepsilon(s) = \varepsilon(s_0 s s_0^{-1})$. Hence, from Proposition 6.1.2, the multiplicity of τ_γ in $\tau_{\Lambda-\rho_k} \otimes L^\pm$ is equal to the multiplicity of $\tau_{s_0\gamma}$ in $\tau_{s_0\Lambda-\rho_k} \otimes L^\pm$. On the other hand, from (7.6), τ_γ belongs to the K-spectrum of $\pi_j^+(\Lambda)$ $(1 \leqq j \leqq m-1)$ if and only if $\tau_{s_0\gamma}$ belongs to it. Also we have clearly that $\tau_\gamma = 1_K$ if and only if $\tau_{s_0\gamma} = 1_K$. After all, we obtain the following equalities;

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, \pi_j^+(\Lambda)) = \dim \mathrm{Hom}_K(\tau_{s_0\Lambda-\rho_k} \otimes L^+, \pi_j^+(\Lambda)) \quad (1 \leqq j \leqq m-1),$$

$$\dim \mathrm{Hom}_K(\tau_{\Lambda-\rho_k} \otimes L^\pm, 1_G) = \dim \mathrm{Hom}_K(\tau_{s_0\Lambda-\rho_k} \otimes L^\pm, 1_G).$$

Hence the assertion (I) follows. Thus the theorem is proved.

REMARK 7.7.2.

(1). In the above theorem, the condition on Λ in (1) of (II) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).

(2). In the case that $m=2$, the above theorem gives the formula in Theorem 4.5 of [11] (c.f. [11], p. 306).

(3). The set of classes $\{1_G, U_1(\rho), U_2(\rho), \cdots, U_{m-1}(\rho), D_\rho, D_{s_0\rho}\}$ corresponds with $\Pi^\rho(G) = \{J_0, J_1, \cdots, J_{m-1}, D_1, D_2\}$ in [1], Chapter VI, § 4, 4.5. To be more precise, $U_j(\rho)$ $(1 \leqq j \leqq m-1)$ corresponds with J_j and 1_G with J_0. Also, these classes contribute to Mutsushima's formula about Betti numbers of the manifold $\Gamma \backslash G/K$ (c.f. [15], p. 174).

8. The case of $G = SU(1, n)$

In this section, we shall pursue the same argument as in § 7 in the case of $G = SU(1, n)$ $(n \geqq 2)$. It is more complicated than that in § 7, but the methods are the same. We will depend largely on the classification of all irreducible unitary representations of $SU(1, n)$ obtained by Kraljevic (c.f. [7]).

8.1. Let n be an integer such that $n \geqq 2$. Let G be the group $SU(1, n)$ and K the group $U(n)$ imbedded in $SU(1, n)$ as follows;

$$K = \left\{ \left(\begin{array}{c|c} b & 0 \\ \hline 0 & X \end{array} \right) \middle| X \in U(n), \quad b = \overline{\det X} \right\}.$$

Then the Lie algebras $\mathfrak{k}_0$ and $\mathfrak{p}_0$ are given by

$$\mathfrak{k}_0 = \left\{ \left(\begin{array}{c|c} b & 0 \\ \hline 0 & X \end{array} \right) \middle| X \in \mathfrak{u}(n), \quad b = -\mathrm{tr}\, X \right\},$$

$$\mathfrak{p}_0 = \left\{ P(u) = \left(\begin{array}{c|c} 0 & u \\ \hline {}^t\bar{u} & 0 \end{array} \right) \middle| u \in \boldsymbol{C}^n \right\}.$$

Take a Cartan subalgebra $\mathfrak{h}_0$ as follows;

$$\mathfrak{h}_0 = \left\{ H(h_0, \cdots, h_n) = \begin{pmatrix} h_0 & & 0 \\ & \ddots & \\ 0 & & h_n \end{pmatrix} \middle| h_i \in \sqrt{-1}\boldsymbol{R},\ \sum_{i=0}^{n} h_i = 0 \right\}.$$

The semisimple part $\mathfrak{k}_1 = [\mathfrak{k}, \mathfrak{k}]$ of $\mathfrak{k}$ is isomorphic to $\mathfrak{sl}(n, \boldsymbol{C})$. If we set

$$\mathfrak{c}_k = \left\{ c \begin{pmatrix} n & & & 0 \\ & -1 & & \\ & & \ddots & \\ 0 & & & -1 \end{pmatrix} \middle| c \in \boldsymbol{C} \right\},$$

$\mathfrak{c}_k$ is the center of $\mathfrak{k}$ and we have

$$\mathfrak{k} = \mathfrak{c}_k \oplus \mathfrak{k}_1, \quad \mathfrak{h} = \mathfrak{c}_k \oplus \mathfrak{h}_1$$

where $\mathfrak{h}_1$ is the Cartan subalgebra of $\mathfrak{k}_1$ which is given by

$$\mathfrak{h}_1 = \{ H(h_0, \cdots, h_n) \in \mathfrak{h} \mid h_0 = 0 \}.$$

Define a linear form e_i $(0 \leqq i \leqq n)$ on $\mathfrak{h}$ by

$$e_i(H(h_0, \cdots, h_n)) = h_i.$$

Then $\mathfrak{h}_{\boldsymbol{R}}^*$ is the hyperplane of the $(n+1)$-dimensional real vector space V spanned by $\{e_0, \cdots, e_n\}$ which is given by

$$\mathfrak{h}_{\boldsymbol{R}}^* = \{ \sum_{i=0}^{n} \lambda_i e_i \mid \lambda_i \in \boldsymbol{R},\ \sum_{i=0}^{n} \lambda_i = 0 \}.$$

Also $\mathfrak{h}^*$ and $\mathfrak{h}_1^*$ are given as follows;

$$\mathfrak{h}^* = \{ \sum_{i=0}^{n} \lambda_i e_i \mid \lambda_i \in \boldsymbol{C},\ \sum_{i=0}^{n} \lambda_i = 0 \}$$

$$\mathfrak{h}_1^* = \{ \sum_{i=0}^{n} \lambda_i e_i \in \mathfrak{h}^* \mid \lambda_0 = 0 \}.$$

The inner product in $\mathfrak{h}_{\boldsymbol{R}}^*$ is given by

$$\langle e_i, e_j \rangle = \frac{1}{2(n+1)} \delta_{ij} \qquad (0 \leqq i, j \leqq n).$$

Take a lexicographic ordering in V relative to the basis $\{e_0, \cdots, e_n\}$. This ordering in V induces an ordering in $\mathfrak{h}_{\boldsymbol{R}}^*$. Fix this ordering in $\mathfrak{h}_{\boldsymbol{R}}^*$. Then the following facts are easily seen;

$$
\begin{aligned}
\Delta &= \{e_i - e_j \mid 0 \leqq i, j \leqq n, \quad i \neq j\} \\
\Delta^+ &= \{e_i - e_j \mid 0 \leqq i < j \leqq n\} \\
\Delta_k^+ &= \{e_i - e_j \mid 1 \leqq i < j \leqq n\} \\
\Delta_n^+ &= \{e_0 - e_i \mid 1 \leqq i \leqq n\} \\
\rho &= \frac{1}{2} \sum_{i=0}^{n} (n-2i) e_i \\
\rho_k &= \frac{1}{2} \sum_{i=1}^{n} (n-2i+1) e_i \\
\rho_n &= \frac{1}{2} (n e_0 - \sum_{i=1}^{n} e_i) \\
\mathfrak{F} &= \{\lambda = \sum_{i=0}^{n} \lambda_i e_i \in \mathfrak{h}_{\boldsymbol{R}}^* \mid \lambda_i - \lambda_j \in \boldsymbol{Z} \quad (0 \leqq i, j \leqq n)\} \\
\mathfrak{F}_0 &= \{\lambda \in \mathfrak{F} \mid \lambda_i - \lambda_j \neq 0 \quad (0 \leqq i, j \leqq n, \ i \neq j)\} \\
\mathfrak{F}_0^+ &= \{\lambda \in \mathfrak{F} \mid \lambda_0 > \lambda_1 > \cdots > \lambda_{n-1} > \lambda_n\} \\
\mathfrak{F}_k^+ &= \{\lambda \in \mathfrak{h}_{\boldsymbol{R}}^* \mid \lambda_i - \lambda_j \in \boldsymbol{Z} \quad (1 \leqq i, j \leqq n), \ \lambda_1 \geqq \lambda_2 \geqq \cdots \geqq \lambda_n\}.
\end{aligned}
$$

Note that, if $\lambda = \sum_{i=0}^{n} \lambda_i e_i \in \mathfrak{F}$, we have

$$(n+1)\lambda_i \in \boldsymbol{Z}; \text{ i.e. } \lambda_i \in \frac{1}{n+1} \boldsymbol{Z} \qquad (0 \leqq i \leqq n).$$

Now we need the Weyl group W and W_G. Let $\mathfrak{S}_{n+1}$ be the group of permutations of the set $\{0, 1, 2, \cdots, n\}$. For $\sigma \in \mathfrak{S}_{n+1}$, define an orthogonal transformation s_σ of $\mathfrak{h}_{\boldsymbol{R}}^*$ by

$$s_\sigma(e_i) = e_{\sigma(i)} \qquad (0 \leqq i \leqq n).$$

Then W and W_G are given as follows;

$$
\begin{aligned}
W &= \{s_\sigma \mid \sigma \in \mathfrak{S}_{n+1}\} \\
W_G &= \{s_\sigma \mid \sigma \in \mathfrak{S}_{n+1}, \quad \sigma(0) = 0\}.
\end{aligned}
$$

In the following, we shall always identify s_σ with σ. Then we have

$$W^1 = \{1 = \sigma_0, \sigma_1, \cdots, \sigma_n\},$$

where σ_l $(0 \leqq l \leqq n)$ is the permutation $(0, 1, \cdots, l)$. Therefore, there is the bijection between $\hat{G}_d$ and $\bigcup_{l=0}^{n} \sigma_l \mathfrak{F}_0^+$.

Next, we take a maximal abelian subalgebra $\mathfrak{a}_0$ of $\mathfrak{p}_0$ as follows;

$$\mathfrak{a}_0 = \{P(0, \cdots, 0, a) \mid a \in \boldsymbol{R}\}.$$

Then M and $\mathfrak{m}_0$ are given by

$$M=\left\{\left(\begin{array}{c|c|c} b & 0 & 0 \\ \hline 0 & Y & 0 \\ \hline 0 & 0 & b \end{array}\right)\middle|\; Y\in U(n-1)\,,\quad b\in \boldsymbol{C}\,,\quad \det Y=\bar{b}^2\right\}$$

$$\mathfrak{m}_0=\left\{\left(\begin{array}{c|c|c} b & 0 & 0 \\ \hline 0 & Y & 0 \\ \hline 0 & 0 & b \end{array}\right)\middle|\; Y\in \mathfrak{u}(n-1, \boldsymbol{C})\,,\quad b\in\sqrt{-1}\boldsymbol{R}\,,\quad 2b=-\operatorname{tr} Y\right\}$$
$$\simeq \mathfrak{su}(n-1, \boldsymbol{C})\oplus \boldsymbol{R}\,.$$

The semisimple part $\mathfrak{m}_1=[\mathfrak{m}, \mathfrak{m}]$ of $\mathfrak{m}$ is isomorphic to $\mathfrak{sl}(n-1, \boldsymbol{C})$. If we put

$$\mathfrak{c}_m=\left\{c\cdot H\left(\frac{n-1}{2}, -1, \cdots, -1, \frac{n-1}{2}\right)\middle|\, c\in \boldsymbol{C}\right\},$$

$\mathfrak{c}_m$ is the center of $\mathfrak{m}$ and we have $\mathfrak{m}=\mathfrak{c}_m+\mathfrak{m}_1$. Take a maximal abelian abelian subalgebra $\mathfrak{t}_0$ of $\mathfrak{m}_0$ as follows;

$$\mathfrak{t}_0=\{H(b, h_1, \cdots, h_{n-1}, b)\,|\, h_i, b\in\sqrt{-1}\boldsymbol{R},\ \sum_{i=1}^{n-1} h_i+2b=0\}\,.$$

Then we have $\mathfrak{t}=\mathfrak{c}_m+\mathfrak{t}_1$, where $\mathfrak{t}_1$ is a Cartan subalgebra of $\mathfrak{m}_1$ defined by

$$\mathfrak{t}_1=\{H(0, h_1, \cdots, h_{n-1}, 0)\,|\, h_i\in \boldsymbol{C},\ \sum_{i=1}^{n-1} h_i=0\}\,.$$

Two Cartan subalgebras $\mathfrak{h}$ and $\mathfrak{h}'=\mathfrak{a}+\mathfrak{t}$ are identified by the following isomorphism θ;

$$\theta(P(0, \cdots, 0, a)+H(b, h_1, \cdots, h_{n-1}, b))=H(b+a, h_1, \cdots, h_{n-1}, b-a)\,.$$

Also, by θ, our ordering in $\mathfrak{h}_{\boldsymbol{R}}^*$ corresponds to a compatible ordering of $(\mathfrak{h}_{\boldsymbol{R}}')^*$ and induces an ordering in $\mathfrak{t}_{\boldsymbol{R}}^*$. In this way, we always identify $(\mathfrak{h}')^*$ and $\mathfrak{h}^*$. Regarding $\mathfrak{t}^*$, $\mathfrak{t}_1^*$ and $\mathfrak{a}^*$ as subsets of $\mathfrak{h}^*$, we have

$$\begin{aligned}
\mathfrak{t}^* &= \{\mu=\sum_{i=0}^{n} \mu_i e_i\in\mathfrak{h}^*\,|\,\mu_0=\mu_n\}\\
\mathfrak{t}_1^* &= \{\mu\in\mathfrak{t}^*\,|\,\mu_0=\mu_n=0\}\\
\mathfrak{a}^* &= \left\{\nu\frac{e_0-e_n}{2}\,\middle|\,\nu\in\boldsymbol{C}\right\}.
\end{aligned}$$

We shall identify $\nu\cdot\frac{e_0-e_n}{2}\in\mathfrak{a}^*$ with the complex number ν and hence identify $\mathfrak{a}^*$ with $\boldsymbol{C}$.

As for the Lie algebra $\mathfrak{m}$, we have

$$\begin{aligned}
\Delta_m &= \{e_i-e_j\,|\,1\leqq i, j\leqq n-1, \qquad i\neq j\}\\
\Delta_m^+ &= \{e_i-e_j\,|\,1\leqq i<j\leqq n-1\}
\end{aligned}$$

$$\rho_m = \frac{1}{2}\sum_{1=i}^{n-1}(n-2i)e_i$$

(8.1) $$\mathfrak{F}_m^+ = \left\{\mu = \sum_{i=0}^{n}\mu_i e_i \in \mathfrak{h}_R^* \,\middle|\, \begin{array}{ll}\mu_0=\mu_n,\ \mu_i-\mu_j\in \boldsymbol{Z} & (0\leqq i, j\leqq n) \\ \mu_1\geqq\mu_2\geqq\cdots\geqq\mu_{n-1} & \end{array}\right\}.$$

8.2. Following Kraljevic [7], we shall introduce some notations. Let r be a positive integer. We define the subset $\boldsymbol{R}_>^r$ of $\boldsymbol{R}^r$ by

$$\boldsymbol{R}_>^r = \left\{p = (p_1, \cdots, p_r)\in \boldsymbol{R}^r \,\middle|\, \begin{array}{ll} p_i-p_j\in \boldsymbol{Z} & (1\leqq i, j\leqq r) \\ p_1\geqq p_2\geqq\cdots\geqq p_r & \end{array}\right\}.$$

For $p=(p_1, \cdots, p_r)\in \boldsymbol{R}_>^r$ and $q=(q_1, \cdots, q_{r+1})\in \boldsymbol{R}_>^{r+1}$, we write $p<q$ if p and q satisfy that

(8.2) $$\begin{cases} q_i-p_i\in \boldsymbol{Z} & (1\leqq i\leqq r) \\ q_1\geqq p_1\geqq q_2\geqq p_2\geqq\cdots\geqq q_r\geqq p_r\geqq q_{r+1}\,. \end{cases}$$

Also, for $p, q\in \boldsymbol{R}_>^r$, we write $(p_1, \cdots, p_r)<(\infty, q_1, \cdots, q_r)$ (resp. $(p_1, \cdots, p_r)<(q_1, \cdots, q_r, -\infty)$) if $(p_1, \cdots, p_r)<(p_1, q_1, \cdots, q_r)$ (resp. $(p_1, \cdots, p_r)<(q_1, \cdots, q_r, p_r)$).

Let $\gamma=\sum_{i=0}^{n}\gamma_i e_i\in \mathfrak{h}_R^*$. Then γ belongs to $\mathfrak{F}\cap\mathfrak{F}_k^+$ if and only if $\{\gamma_1, \cdots, \gamma_n\}$ satisfy the conditions

$$\begin{cases} \gamma_i\in \dfrac{1}{n+1}\boldsymbol{Z} & (1\leqq i\leqq n) \\ \gamma_i-\gamma_j\in \boldsymbol{Z} & (1\leqq i, j\leqq n) \\ \gamma_1\geqq\gamma_2\geqq\cdots\geqq\gamma_n & \end{cases}$$

If we put

$$\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^n = \left\{p\in \boldsymbol{R}_>^n \,\middle|\, p_i\in\frac{1}{n+1}\boldsymbol{Z} \quad (1\leqq i\leqq n)\right\},$$

the mapping; $\mathfrak{F}\cap\mathfrak{F}_k^+\in\gamma\to(\gamma_1, \cdots, \gamma_n)\in\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^n$ is a bijection. Thus we shall always identify an element γ of $\mathfrak{F}\cup\mathfrak{F}_k^+$ with an element $\gamma=(\gamma_1, \cdots, \gamma_n)$ of $\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^n$. Similarly, $\mu=\sum_{i=0}^{n}\mu_i e_i\in\mathfrak{F}\cap\mathfrak{F}_m^+$ can be identified with an element $\mu=(\mu_1, \cdots, \mu_{n-1})$ of $\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^{n-1}$. Hence $\hat{K}$ is isomorphic to the set $\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^n$ and $\hat{M}$ is isomorphic to the set $\left(\frac{1}{n+1}\boldsymbol{Z}\right)_>^{n-1}$ (c.f. [7], p. 35).

8.3. Now, we shall start off with **Step 1.** Once and for all, take a special element $\Lambda=\sum_{i=0}^{n}\Lambda_i e_i\in\mathfrak{F}_0^+$ and fix. Then we have

$$(8.3)\qquad \begin{cases} \Lambda_i \in \dfrac{1}{n+1}\boldsymbol{Z} & (0 \leqq i \leqq n) \\ \Lambda_0 > \Lambda_1 > \cdots > \Lambda_n . \end{cases}$$

Lemma 8.3.1. *Let h and k be integers such that $0 \leqq h < k \leqq n$. Define $\sigma_{h,k}^{\pm} \in \mathfrak{S}_{n+1}$ by*

$$\sigma_{h,k}^{+} = \begin{pmatrix} 0, & 1, & \cdots, & h-1, & h, & h+1, & \cdots, & k-1, & k, & k+1, & \cdots, & n \\ 1, & 2, & \cdots, & h, & 0, & h+1, & \cdots, & k-1, & n, & k, & \cdots, & n-1 \end{pmatrix}$$

$$\sigma_{h,k}^{-} = \begin{pmatrix} 0, & 1, & \cdots, & h-1, & h, & h+1, & \cdots, & k-1, & k, & k+1, & \cdots, & n \\ 1, & 2, & \cdots, & h, & n, & h+1, & \cdots, & k-1, & 0, & k, & \cdots, & n-1 \end{pmatrix}$$

Then, an element $\sigma \in W$ satisfies that $\sigma\Lambda|_{\mathfrak{t}} - \rho_m \in \mathfrak{F}_m^+$ if and only if σ is either $\sigma_{h,k}^{+}$ or $\sigma_{h,k}^{-}$ for some $0 \leqq h < k \leqq n$.

Proof. For $\sigma \in \mathfrak{S}_{n+1}$, we have

$$\sigma\Lambda|_{\mathfrak{t}} - \rho_m = \sum_{i=1}^{n-1}\left(\Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2}\right)e_i + b(e_0 + e_n)$$

where $b \in \boldsymbol{R}$ is given by $2b + \sum_{i=1}^{n-1}\Lambda_{\sigma^{-1}(i)} = 0$. Assume that $\sigma\Lambda|_{\mathfrak{t}} - \rho_m$ belongs to $\mathfrak{F}_m^+$. From (8.1), we have

$$\Lambda_{\sigma^{-1}(i)} - \frac{n-2i}{2} \geqq \Lambda_{\sigma^{-1}(i+1)} - \frac{n-2(i+1)}{2} \qquad (1 \leqq i \leqq n-2),$$

and hence we have

$$\Lambda_{\sigma^{-1}(i)} - \Lambda_{\sigma^{-1}(i+1)} \geqq 1 \qquad (1 \leqq i \leqq n-2).$$

Since Λ satisfies (8.3), it follows that

$$\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(n-1).$$

Therefore, if we put h=the minimum of $\{\sigma^{-1}(0), \sigma^{-1}(n)\}$ and k=the maximum of $\{\sigma^{-1}(0), \sigma^{-1}(n)\}$, σ is necessarily either $\sigma_{h,k}^{+}$ or $\sigma_{h,k}^{-}$. The lemma is proved.

Let h and k be integers such that $0 \leqq h < k \leqq n$. We denote by $\xi_{h,k}$ the element of $\hat{M}$ with highest weight $\sigma_{h,k}^{\pm}\Lambda|_{\mathfrak{t}} - \rho_m$ and put $\nu_{h,k}^{\pm} = \sigma_{h,k}^{\pm}\Lambda|_{\mathfrak{a}}$; i.e. the highest weight of $\xi_{h,k}$ is given by

$$(8.4)\qquad \begin{aligned} \mu(\xi_{h,k}) = \Big(&\Lambda_0 - \frac{n-2}{2}, \Lambda_1 - \frac{n-4}{2}, \cdots, \Lambda_{h-1} - \frac{n-2h}{2}, \\ &\Lambda_{h+1} - \frac{n-2(h+1)}{2}, \cdots, \Lambda_{k-1} - \frac{n-2(k-1)}{2}, \\ &\Lambda_{k+1} - \frac{n-2k}{2}, \cdots, \Lambda_n - \frac{-n+2}{2}\Big). \end{aligned}$$

and we have

$$\nu^{\pm}_{h,k} = \pm(\Lambda_h - \Lambda_k)\,.$$

In the above formulas, if $h=0$, (8.4) means

$$\mu(\xi_{0,k}) = \Big(\Lambda_1 - \frac{n-2}{2},\ \Lambda_2 - \frac{n-4}{2},\ \cdots,\ \Lambda_{k-1} - \frac{n-2(k-1)}{2},$$
$$\Lambda_{k+1} - \frac{n-2k}{2},\ \cdots,\ \Lambda_n - \frac{-n+2}{2}\Big),$$

and if $k=n$, (8.4) means

$$\mu(\xi_{h,n}) = \Big(\Lambda_0 - \frac{n-2}{2},\ \Lambda_1 - \frac{n-4}{2},\ \cdots,\ \Lambda_{h-1} - \frac{n-2h}{2},$$
$$\Lambda_{h+1} - \frac{n-2(h+1)}{2},\ \cdots,\ \Lambda_{n-1} - \frac{-n+2}{2}\Big).$$

Throughout this section, the case of $h=0$ or $k=n$ should be suitably appreciated, even if it is not specially offered. Moreover, for convenience' sake, define $\bar{\Lambda}_0, \cdots, \bar{\Lambda}_n$ by

$$\Lambda - \rho = \sum_{i=0}^{n} \bar{\Lambda}_i e_i\,.$$

Then we have

$$\bar{\Lambda}_i = \Lambda_i - \frac{n-2i}{2} \qquad (0 \leqq i \leqq n)$$
$$\bar{\Lambda}_0 \geqq \bar{\Lambda}_1 \geqq \cdots \geqq \bar{\Lambda}_n\,,$$

and we can write

$$(8.5) \qquad \mu(\xi_{h,k}) = (\bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1, \bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n)$$

Now consider the representations $\pi_{\xi_{h,k}, \nu^{\pm}_{h,k}}$ $(0 \leqq h < k \leqq n)$ in the nonunitary principal series. From Proposition 5.2.2, an irreducible unitary representation π such that $\chi_\pi = \chi_\Lambda$ is infinitesimally equivalent to an irreducible subquotient of $\pi_{\xi_{h,k}, \nu^{+}_{h,k}}$ or $\pi_{\xi_{h,k}, \nu^{-}_{h,k}}$ for some $0 \leqq h < k \leqq n$.

8.4. Using results of [7], we shall determine all irreducible subquotients of $\pi_{\xi_{h,k}, \nu^{\pm}_{h,k}}$ $(0 \leqq h < k \leqq n)$, up to infinitesimally equivalence.

Let h and k be integers such that $0 \leqq h < k \leqq n$. By Proposition 6.1 in [7] and (8.5), the K-spectrum of $\pi_{\xi_{h,k} \nu^{\pm}_{h,k}}$ is given by

$$\Phi(\pi_{\xi_{h,k}, \nu^{\pm}_{h,k}}) = \hat{K}(\xi_{h,k})$$
$$= \{\tau_\gamma \in \hat{K} \,|\, (\gamma_1, \cdots, \gamma_n) > (\bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1,$$
$$\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}, \bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1)\}$$

where we use the notation $>$ in the sence of (8.2).

We shall apply Theorem 7.5 and Theorem 8.7 in [7] to our representation $\pi_{\xi_{h,k},\nu^{\pm}_{h,k}}$. Following Kraljevic, put

$$s_j(\xi) = 2\mu(\xi)_j+n-2j+\sum_{j=1}^{n-1}\mu(\xi)_i \qquad (1\leqq j\leqq n-1)$$

for $\xi\in\hat{M}$, where $\mu(\xi)=(\mu(\xi)_1, \cdots, \mu(\xi)_{n-1})\in\mathfrak{F}^{+}_{m}\cap\mathfrak{F}$ is the highest weight of ξ (c.f. [7], p. 48). In the case of $\xi=\xi_{h,k}$, we have

$$s_j(\xi_{h,k}) = \begin{cases} 2\Lambda_{j-1}-\Lambda_h-\Lambda_k & (1\leqq j\leqq h) \\ 2\Lambda_j-\Lambda_h-\Lambda_k & (h+1\leqq j\leqq k-1) \\ 2\Lambda_{j+1}-\Lambda_h-\Lambda_k & (k\leqq j\leqq n-1)\,. \end{cases}$$

Since $\nu^{\pm}_{h,k}=\pm(\Lambda_h-\Lambda_k)$, it follows that

$$\begin{aligned} &s_j(\xi_{h,k})-\nu^{\pm}_{h,k}\in 2\boldsymbol{Z} \qquad (1\leqq j\leqq n-1) \\ &s_h(\xi_{h,k})>\nu^{+}_{h,k}>s_{h+1}(\xi_{h,k}) \\ &s_{k-1}(\xi_{h,k})>\nu^{-}_{h,k} = -\nu^{+}_{h,k}>s_k(\xi_{h,k})\,. \end{aligned}$$

Hence, if $h+1=k$ (resp. $h+1<k$), $\xi_{h,k}\in\hat{M}$ and $\nu^{\pm}_{h,k}\in\boldsymbol{C}$ satisfy the condition of the case (c) (resp. (d)) in Theorem 7.5 in [7]. By Theorem 7.5 and Theorem 8.7 in [7], K-spectra of all irreducible subquotients of $\pi_{\xi_{h,k},\nu^{\pm}_{h,k}}$ are given as follows;

(1). If $h+1=k$, there can occur only the following three K-spectra;

$$\Phi_+(\xi_{h,h+1}) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{matrix} (\gamma_1, \cdots, \gamma_{h+1})<(\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_n)<(\bar{\Lambda}_{h+2}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{matrix}\right\}$$

$$\Phi_0(\xi_{h,h+1}) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{matrix} (\gamma_1, \cdots, \gamma_h)<(\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1})<(\bar{\Lambda}_h, \bar{\Lambda}_{h+1}) \\ (\gamma_{h+2}, \cdots, \gamma_n)<(\bar{\Lambda}_{h+2}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{matrix}\right\}$$

$$\Phi_-(\xi_{h,h+1}) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{matrix} (\gamma_1, \cdots, \gamma_h)<(\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_n)<(\bar{\Lambda}_{h+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{matrix}\right\}$$

(2). If $h+1<k$, there can occur only the following four K-spectra;

$$\Phi_{++}(\xi_{h,k}) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{matrix} (\gamma_1, \cdots, \gamma_{h+1})<(\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_k)<(\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_k) \\ (\gamma_{k+1}, \cdots, \gamma_n)<(\bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{matrix}\right\}$$

$$\Phi_{+-}(\xi_{h,k}) = \left\{\tau_\gamma\in\hat{K}\;\middle|\; \begin{matrix} (\gamma_1, \cdots, \gamma_{h+1})<(\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_h+1) \\ (\gamma_{h+2}, \cdots, \gamma_{k-1})<(\bar{\Lambda}_{h+1}, \cdots, \bar{\Lambda}_{k-1}) \\ (\gamma_k, \cdots, \gamma_n)<(\bar{\Lambda}_k-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{matrix}\right.$$

$$\Phi_{-+}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_k) < (\bar{\Lambda}_h, \cdots, \bar{\Lambda}_k) \\ (\gamma_{k+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{k+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$\Phi_{--}(\xi_{h,k}) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_h) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{h-1}+1) \\ (\gamma_{h+1}, \cdots, \gamma_{k-1}) < (\bar{\Lambda}_h, \cdots, \bar{\Lambda}_{k-1}) \\ (\gamma_k, \cdots, \gamma_n) < (\bar{\Lambda}_k-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}.$$

On the other hand, by Theorem 9.2 in [7], two irreducible subquotients of representations $\{\pi_{\xi_{h,k},\nu^{\pm}_{h,k}} \mid 0 \leqq h < k \leqq n\}$ are infinitesimally equivalent to each other if and only if they have the same K-spectrum. By (1) and (2) above, among K-spectra of all irreducible subquotients of $\{\pi_{\xi_{h,k},\nu^{\pm}_{h,k}} \mid 0 \leqq h < k \leqq n\}$, there are $\frac{(n+1)(n+2)}{2}$ kinds of K-spectra which are different from one another. They are given as follows;

$$(8.6) \qquad \Phi_l(\Lambda) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_l) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{l-1}+1) \\ (\gamma_{l+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{l+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

$$(8.7) \qquad \Phi_{p,q}(\Lambda) = \left\{ \tau_\gamma \in \hat{K} \;\middle|\; \begin{array}{l} (\gamma_1, \cdots, \gamma_p) < (\infty, \bar{\Lambda}_0+1, \cdots, \bar{\Lambda}_{p-1}+1) \\ (\gamma_{p+1}, \cdots, \gamma_q) < (\bar{\Lambda}_p, \cdots, \bar{\Lambda}_q) \\ (\gamma_{q+1}, \cdots, \gamma_n) < (\bar{\Lambda}_{q+1}-1, \cdots, \bar{\Lambda}_n-1, -\infty) \end{array} \right\}$$

where l runs over $\{0, 1, \cdots, n\}$ and (p, q) runs over the set $\{(p, q) \mid 0 \leqq p < q \leqq n\}$.

Denote by $\pi_l(\Lambda)$ the irreducible subquotient with K-spectrum $\Phi_l(\Lambda)$ and denote by $\pi_{p,q}(\Lambda)$ the one with K-spectrum $\Phi_{p,q}(\Lambda)$. After all we have the following proposition.

Proposition 8.4.1. *Up to infinitesimal equivalence, all irreducible subquotients of the nonunitary principal series whose infinitesimal characters are equal to χ_Λ are given as follows;*

$$\{\pi_l(\Lambda) \quad (0 \leqq l \leqq n), \qquad \pi_{p,q}(\Lambda) \quad (0 \leqq p < q \leqq n)\}.$$

Moreover these representations are not infinitesimally equivalent to one another.

8.5. We shall examine which subquotients in Proposition 8.4.1 are infinitesimally unitary. Using Proposition 11.4 in [7], we have the following proposition.

Proposition 8.5.1.

(1). *For $0 \leqq l \leqq n$, $\pi_l(\Lambda)$ is always infinitesimally unitary.*

(2). *For $0 \leqq p < q \leqq n$, $\pi_{p,q}(\Lambda)$ is infinitesimally unitary if and only if $\Lambda_i - \Lambda_{i+1} = 1$ for all $p \leqq i \leqq q-1$.*

Proof. By easy computations, we can prove that $\Phi_l(\Lambda)$ satisfies the inequalities (6) in (ii) of Proposition 11.4 in [7]. As for $\pi_{p,q}(\Lambda)$, the condition in (iii) of Proposition 11.4 in [7] is clearly equivalent to our condition.

Now, let $\tilde{G}$ be the universal covering group of G. Canonically, we can regard $\pi_l(\Lambda)$ as a representation of $\tilde{G}$. Since $\pi_l(\Lambda)$ is infinitesimally unitary, there exists an irreducible unitary representation $\pi_l(\Lambda)^U$ of $\tilde{G}$ which is infinitesimally equivalent to $\pi_l(\Lambda)$. Then $\pi_l(\Lambda)^U$ turns out to be a representation of G (c.f. Remark 8.5.2). Denote by $U_l(\Lambda)\in\hat{G}$ the class which contains $\pi_l(\Lambda)^U$. Similarly, when $\pi_{p,q}(\Lambda)$ is infinitesimally unitary, denote by $U_{p,q}(\Lambda)\in\hat{G}$ the class which is determined by $\pi_{p,q}(\Lambda)$ in the same way.

REMARK 8.5.2. Let ψ; $\tilde{G}\to G$ be the covering epimorphism and put $\tilde{K}=\psi^{-1}(K)$. Let (π^1, E^1) and (π^2, E^2) be $\tilde{K}$-finite representations of $\tilde{G}$. Assume that π^1 is infinitesimally equivalent to π^2. Then π^1 is a representation of G if and only if π^2 is so. In fact, $\pi^1|_{\tilde{K}}$ is equivalent to $\pi^2|_{\tilde{K}}$ as representations of $\tilde{K}$ on the spaces E^1_f and E^2_f, respectively. Also the kernel of ψ is contained in $\tilde{K}$. Hence, for an element $\tilde{k}$ of the kernel of ψ, $\pi^1(\tilde{k})$ is the identity operator of E^1 if and only if $\pi^2(\tilde{k})$ is the identity operator of E^2.

Proposition 8.5.3. *For $0\leqq l\leqq n$, $U_l(\Lambda)$ is a discrete class. More precisely, we have $U_l(\Lambda)=D_{\sigma_l\Lambda}\in\hat{G}_d$.*

Proof. It is sufficient to prove that the K-spectrum $\Phi_l(\Lambda)$ of $U_l(\Lambda)$ is the same as the K-spectrum of $D_{\sigma_l\Lambda}$. To begin with, we shall show that the K-spectrum of $D_{\sigma_l\Lambda}$ contains $\Phi_l(\Lambda)$.

In order to apply Theorem 1.4.1 to $D_{\sigma_l\Lambda}$, take an ordering in $\mathfrak{h}_R^*$ with respect to which $\sigma_l\Lambda$ is dominant; i.e. take $\sigma_l(\Delta^+)$ as the positive root system of Δ. Then the set of noncompact positive roots is given by

$$\{e_i-e_0 \quad (1\leqq i\leqq l), \qquad e_0-e_j \quad (l+1\leqq j\leqq n)\} \tag{8.8}$$

and we have

$$\rho_n=\frac{n-2l}{2}e_0+\frac{1}{2}(e_1+\cdots+e_l-e_{l+1}-\cdots-e_n)\,.$$

Let $\lambda=\sum_{0=i}^{n}\lambda_i e_i$ be in $\mathfrak{h}_R^*$. In our case, $Q(\lambda)$ in Theorem 1.4.1 is equal to either 1 or 0 and $Q(\lambda)=1$ if and only if

$$\begin{cases}\lambda_i\in\boldsymbol{Z} & (0\leqq i\leqq n)\\ \lambda_i\geqq 0 & \text{if } 1\leqq i \text{ and } i\leqq l\\ \lambda_i\leqq 0 & \text{if } l+1\leqq i \text{ and } i\leqq n\,.\end{cases}$$

Let τ_γ be in $\Phi_l(\Lambda)$. From (8.6), $\gamma=\sum_{i=0}^{n}\gamma_i e_i$ satisfies the following inequalities;

$$(8.9)\qquad \begin{array}{ll} \Lambda_{i-2}-1\geqq\gamma_i+\dfrac{n-2i}{2}\geqq\Lambda_{i-1} & \text{if } 1\leqq i \text{ and } i\leqq l\,, \\ \Lambda_i-1\geqq\gamma_i+\dfrac{n-2i}{2}\geqq\Lambda_{i+1} & \text{if } l+1\leqq i \text{ and } i\leqq n \end{array}$$

where Λ_{-1} should be considered ∞ and Λ_{n+1} should be considered $-\infty$. Let σ be an element of W_G. Since $\sigma(0)=0$, we have

$$(8.10)\quad \sigma(\gamma+\rho_k)-\sigma_l\Lambda-\rho_n=\left(\gamma_0-\Lambda_l-\frac{n-2l}{2}\right)e_0$$
$$+\sum_{i=1}^{l}\left(\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_{i-1}\right)e_i+\sum_{i=l+1}^{n}\left(\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_i+1\right)e_i\,.$$

If there is an integer i such that $1\leqq i$, $i\leqq l$ and $i<\sigma^{-1}(i)$, we have $\Lambda_{i-1}\geqq\Lambda_{\sigma^{-1}(i)-2}$ and hence, from (8.9), we have

$$\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_{i-1}\leqq\Lambda_{\sigma^{-1}(i)-2}-1-\Lambda_{i-1}<0\,.$$

Also, if there is an integer i such that $l+1\leqq i$, $i\leqq n$ and $i>\sigma^{-1}(i)$, we have $\Lambda_i\leqq\Lambda_{\sigma^{-1}(i)+1}$ and hence, from (8.9), we have

$$\gamma_{\sigma^{-1}(i)}+\frac{n-2\sigma^{-1}(i)}{2}-\Lambda_i+1\geqq\Lambda_{\sigma^{-1}(i)+1}-\Lambda_i+1>0\,.$$

Therefore, in these two cases, we have

$$Q(\sigma(\gamma+\rho_k)-\sigma_l\Lambda-\rho_n)=0\,.$$

If otherwise, σ satisfies the condition that

$$\begin{array}{ll} \sigma^{-1}(i)\leqq i & \text{if } 1\leqq i \text{ and } i\leqq l\,, \\ \sigma^{-1}(i)\geqq i & \text{if } l+1\leqq i \text{ and } i\leqq n\,. \end{array}$$

Hence σ must be the identity. If we put $\sigma^{-1}(i)=i$ $(1\leqq i\leqq n)$ in (8.10), we have clearly

$$Q((\gamma+\rho_k)-\sigma_l\Lambda-\rho_n)=1\,.$$

After all, by Theorem 1.4.1, the multiplicity of τ_γ in a representation in the class $D_{\sigma_l\Lambda}$ is equal to 1. Therefore $\Phi_l(\Lambda)$ is contained in the K-spectrum of $D_{\sigma_l\Lambda}$.

Now, by Proposition 5.2.2, the K-spectrum of $D_{\sigma_l\Lambda}$ is one of $\{\Phi_{l'}(\Lambda)$ $(0\leqq l'\leqq n)$, $\Phi_{p,q}(\Lambda)$ $(0\leqq p<q\leqq n)\}$. Each $\Phi_{p,q}(\Lambda)$ does not contain $\Phi_l(\Lambda)$ and, if $l'\neq l$, $\Phi_{l'}(\Lambda)$ does not also contain $\Phi_l(\Lambda)$. Hence, $\Phi_l(\Lambda)$ is necessarily the K-spectrum of $D_{\sigma_l\Lambda}$. The proposition is proved.

From Proposition 8.4.1, 8.5.1 and 8.5.3, we obtain the following corollary and complete **Step 1** for $\Lambda\in\mathfrak{F}_0^+$.

Corollary 8.5.4. *For $\Lambda\in\mathfrak{F}_0^+$, the set $\hat{G}_\Lambda-\hat{G}_d$ is given as follows;*

(1). *If $\Lambda_i-\Lambda_{i+1}\geqq 2$ for all $0\leqq i\leqq n-1$, we have $\hat{G}_\Lambda-\hat{G}_d=\phi$.*

(2). *If $\Lambda_i-\Lambda_{i+1}=1$ for some $0\leqq i\leqq n-1$, put*

$$p_\Lambda=\text{the minimum of }\{i\,|\,0\leqq i\leqq n-1,\ \Lambda_i-\Lambda_{i+1}=1\}$$

$$q_\Lambda=\text{the maximum of }\{i\,|\,1\leqq i\leqq n,\ \Lambda_{i-1}-\Lambda_i=1\}\ .$$

The we have $\hat{G}_\Lambda-\hat{G}_d=\{U_{p,q}(\Lambda)\,|\,p_\Lambda\leqq p<q\leqq q_\Lambda\}$.

(3). *In particular, if $\Lambda=\rho$, we have $\hat{G}_\Lambda-\hat{G}_d=\{U_{p,q}(\rho)\,|\,0\leqq p<q\leqq n\}$ where $U_{0,n}(\rho)$ is the class 1_G.*

8.6. Let us go forward **Step 2.** Once and for all, we fix $\Lambda\in\mathfrak{F}_0^+$ and an integer l such that $0\leqq l\leqq n$. We shall study **Step 2** for $\sigma_l\Lambda\in\mathfrak{F}_0$. Our purpose is to compute

$$\dim \operatorname{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^{\pm},\ \pi_{p,q}(\Lambda)\,|_K)$$

for all $0\leqq p<q\leqq n$.

Take an ordering in $\mathfrak{h}_R^*$ as in the proof of Proposition 8.5.3 and fix. From (8.8) and Lemma 6.2.1, the set of weights of L^+ (1esp. L^-) is given by

$$(8.11)\qquad \left\{\frac{1}{2}\sum_{i=1}^{n}\varepsilon_i(e_0-e_i)\ \middle|\ \begin{array}{l}\varepsilon_i=1 \text{ or } -1 \qquad (1\leqq i\leqq n)\\ \left(\prod_{i=1}^{n}\varepsilon_i\right)\cdot(-1)^l=+1 \text{ (resp. } -1)\end{array}\right\}.$$

and the multiplicity of each weight is equal to 1.

The following lemma is proved by the same arguments as in Lemma 7.5.1.

Lemma 8.6.1. *Let β and γ be in $\mathfrak{F}_k^+$.*

(1). *Let $\sigma\in W_G$. If $\beta+\rho_k-\sigma(\gamma+\rho_k)$ is a weight of L, σ is the identity.*

(2). *The multiplicity of τ_β in $\tau_\gamma\otimes L$ is equal to $M_L(\beta-\gamma)$, where $M_L(\lambda)$ ($\lambda\in\mathfrak{h}_R^*$) is the multiplicity of the weight λ in L.*

Using this lemma, we can obtain the following proposition and complete **Step 2** for $\sigma_l\Lambda$ $(0\leqq l\leqq n)$.

Proposition 8.6.2. *Let l, p and q be integers such that $0\leqq l\leqq n$ abd $0\leqq p<q\leqq n$.*

(1). *If $p\leqq l\leqq q$, we have*

$$\dim \operatorname{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^+,\ \pi_{p,q}(\Lambda)\,|_K)=\begin{cases}1 & \text{if } p+q \text{ is even}\,,\\ 0 & \text{if } p+q \text{ is odd}\,,\end{cases}$$

$$\dim \mathrm{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^-, \pi_{p,q}(\Lambda)|_K) = \begin{cases} 0 & \textit{if } p+q \textit{ is even}, \\ 1 & \textit{if } p+q \textit{ is odd}, \end{cases}$$

(2). *If $l<p$ or $q<l$, we have*

$$\dim \mathrm{Hom}_K(\tau_{\sigma_l\Lambda-\rho_k}\otimes L^\pm, \pi_{p,q}(\Lambda)|_K) = 0 .$$

Proof. Let $\tau_\gamma\in\hat{K}$. Assume that $\tau_\gamma\in\Phi_{p,q}(\Lambda)$ and the multiplicity of τ_γ in $\tau_{\sigma_l\Lambda-\rho_k}\otimes L$ is not equal to 0. By (2) of Lemma 8.6.1, we have $M_L(\gamma-\sigma_l\Lambda+\rho_k)\neq 0$. Hence, by (8.11), we have

$$\gamma-\sigma_l\Lambda+\rho_k = \frac{1}{2}\sum_{i=1}^{n}\varepsilon_i(e_0-e_i)$$

for some $\{\varepsilon_1, \cdots, \varepsilon_n\}$ such that $\varepsilon_i=1$ or -1 $(1\leqq i\leqq n)$; i.e.

$$(8.12)\qquad \begin{cases} \gamma_i = \bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2} & \text{if } 1\leqq i \text{ and } i\leqq l, \\ \gamma_i = \bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2} & \text{if } l+1\leqq i \text{ and } i\leqq n. \end{cases}$$

On the other hand, $\{\gamma_1, \cdots, \gamma_n\}$ satisfy the condition (8.7). Hence we have

$$(8.13)\qquad \begin{cases} \bar{\Lambda}_{i-2}+1\geqq\gamma_i\geqq\bar{\Lambda}_{i-1}+1 & \text{if } 1\leqq i \text{ and } i\leqq p \\ \bar{\Lambda}_{i-1}\geqq\gamma_i\geqq\bar{\Lambda}_i & \text{if } p+1\leqq i \text{ and } i\leqq q \\ \bar{\Lambda}_i-1\geqq\gamma_i\geqq\bar{\Lambda}_{i+1}-1 & \text{if } q+1\leqq i \text{ and } i\leqq n. \end{cases}$$

If $p>l$, there is an integer i_0 such that $l+1\leqq i_0\leqq p$. Then, by (8.12) and (8.13), we have

$$\bar{\Lambda}_{i_0}+\frac{-1-\varepsilon_{i_0}}{2}\geqq\bar{\Lambda}_{i_0-1}+1$$

and hence we have $\bar{\Lambda}_{i_0}\leqq\bar{\Lambda}_{i_0-1}+1$. This is inconsistent with the fact that $\bar{\Lambda}_{i_0}\leqq\bar{\Lambda}_{i_0-1}$. Therefore we have $p\leqq l$. Similarly, if $q<l$, we have a contradiction. Hence we have also $q\geqq l$. Thus the assertion (2) is proved.

Now, assume that $p\leqq l\leqq q$. By (8.12) and (8.13), we have the following inqualities;

$$\begin{cases} \bar{\Lambda}_{i-2}+1\geqq\bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2}\geqq\bar{\Lambda}_{i-1}+1 & \text{if } 1\leqq i \text{ and } i\leqq p, \\ \bar{\Lambda}_{i-1}\geqq\bar{\Lambda}_{i-1}+\dfrac{1-\varepsilon_i}{2}\geqq\bar{\Lambda}_i & \text{if } p+1\leqq i \text{ and } i\leqq l, \\ \bar{\Lambda}_{i-1}\geqq\bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2}\geqq\bar{\Lambda}_i & \text{if } l+1\leqq i \text{ and } i\leqq q, \\ \bar{\Lambda}_i-1\geqq\bar{\Lambda}_i+\dfrac{-1-\varepsilon_i}{2}\geqq\bar{\Lambda}_{i+1}-1 & \text{if } q+1\leqq i \text{ and } i\leqq n. \end{cases}$$

These inequalities imply that

$$(8.14)\qquad \begin{cases} \varepsilon_i = -1 & \text{if } 1\leqq i \text{ and } i\leqq p\,, \\ \varepsilon_i = 1 & \text{if } p+1\leqq i \text{ and } i\leqq l\,, \\ \varepsilon_i = -1 & \text{if } l+1\leqq i \text{ and } i\leqq q\,, \\ \varepsilon_i = 1 & \text{if } q+1\leqq i \text{ and } i\leqq n\,. \end{cases}$$

After all, by (8.12), we have

$$(8.15)\qquad \gamma = (\bar{\Lambda}_0+1, \bar{\Lambda}_1+1, \cdots, \bar{\Lambda}_{p-1}+1, \cdots, \bar{\Lambda}_p, \cdots, \bar{\Lambda}_{l-1}, \bar{\Lambda}_{l+1}, \cdots \\ \cdots, \bar{\Lambda}_q, \bar{\Lambda}_{q+1}-1, \cdots, \bar{\Lambda}_n-1)$$

$$(8.16)\qquad \prod_{i=1}^{n} \varepsilon_i = (-1)^{p+q-l}\,.$$

Conversely, if γ is given by (8.15), it is clear that $\tau_\gamma\in\Phi_{p,q}(\Lambda)$ and we have

$$\gamma-\sigma_l\Lambda+\rho_k = \frac{1}{2}\sum_{i=1}^{n}\varepsilon_i(e_0-e_i)$$

where $\{\varepsilon_1, \cdots, \varepsilon_n\}$ are given by (8.14). Then, by (2) of Lemma 8.6.1, the multiplicity of τ_γ in $\tau_{\sigma_l\Lambda-\rho_k}\otimes L$ is equal to 1. Whether τ_γ occurs in $\tau_{\sigma_l\Lambda-\rho_k}\otimes L^+$ or $\tau_{\sigma_l\Lambda-\rho_k}\otimes L^-$ is determined by the sign of $\left(\prod_{i=1}^{n}\varepsilon_i\right)\cdot(-1)^l$. By (8.16) we have $\left(\prod_{i=1}^{n}\varepsilon_i\right)\cdot(-1)^l=(-1)^{p+q}$. Since τ_γ occurs in $\pi_{p,q}(\Lambda)$ with multiplicity 1, the assertion (2) follows. The proposition is proved.

8.7. Now, we have completed **Step 1** and **Step 2.** From the basic formula in Theorem 3.1.1, we shall obtain the main theorem in the case of $G=SU(1,n)$ $(n\geqq 2)$.

Let Γ be a torsion free discrete subgroup of G such that $\Gamma\backslash G$ is compact. Then we have the following theorem.

Theorem 8.7.1. *Let Λ be in $\mathfrak{F}_0^+$ and set $\Lambda=\sum_{i=0}^{n}\Lambda_ie_i$. Let l be an integer such that $0\leqq l\leqq n$. Then we have the following formulas for the multiplicity $N_\Gamma(D_{\sigma_l\Lambda})$.*

(1). *If $\Lambda_l-\Lambda_{l+1}\geqq 2$ and $\Lambda_{l-1}-\Lambda_l\geqq 2$, we have*

$$N_\Gamma(D_{\sigma_l\Lambda}) = d(D_{\sigma_l\Lambda})\cdot\mathrm{vol}(\Gamma\backslash G)\,.$$

(2). *If either $\Lambda_l-\Lambda_{l+1}=1$ or $\Lambda_{l-1}-\Lambda_l=1$, put*

$$p_\Lambda = \text{the minimum of } \{i\,|\,0\leqq i\leqq n-1,\ \Lambda_i-\Lambda_{i+1} = 1\}$$
$$q_\Lambda = \text{the maximum of } \{i\,|\,1\leqq i\leqq n,\ \Lambda_{i-1}-\Lambda_i = 1\}\,,$$

and define

$$J_{l,\Lambda} = \{(p, q) \in \boldsymbol{Z} \times \boldsymbol{Z} \mid p_\Lambda \leqq p \leqq l,\ l \leqq q \leqq q_\Lambda,\ p \neq q\}\ .$$

Then we have

$$N_\Gamma(D_{\sigma_l \Lambda}) = d(D_{\sigma_l \Lambda}) \operatorname{vol}(\Gamma \backslash G) + \sum_{(p,q) \in J_{l,\Lambda}} (-1)^{p+q-1} N_\Gamma(U_{p,q}(\Lambda))\ .$$

(3). *In particular, if* $\Lambda_i - \Lambda_{i+1} = 1$ *for all* $0 \leqq i \leqq n-1$, *that is,* $\Lambda = \rho$, *we have*

$$N_\Gamma(D_{\sigma_l \rho}) = d(D_{\sigma_l \rho}) \operatorname{vol}(\Gamma \backslash G) + \sum_{\substack{0 \leqq p < q \leqq n \\ p \leqq l \leqq q}} (-1)^{p+q-1} N_\Gamma(U_{p,q}(\rho))$$

where $N_\Gamma(U_{0,n}(\rho)) = N_\Gamma(1_G) = 1$.

Proof. All assertions immediately follow from Theorem 3.1,1, Corollary 8.5.4 and Proposition 8.6.2.

REMARK 8.7.2.

(1). In the above theorem, the condition on Λ in (1) agree with the condition (i) of Theorem in the introduction of [6] (c.f. [6], p. 176).

(2). In the case that $n=2$, our theorem gives some formulas in [15] (c.f. [15], p. 192).

(3). In the same way as the case of $G = Spin(1, 2m)$, the set of classes $\{U_{p,q}(\rho)\ (0 \leqq p < q \leqq n),\ D_{\sigma_l(\rho)}\ (0 \leqq l \leqq n)\}$ corresponds with $\Pi^\rho(G) = \{J_{ij} \mid i, j \geqq 0, i+j \leqq n-1\} \cup \{D_0, D_1, \cdots, D_n\}$ in [1], Chapter VI, §4, 4.8; that is, $U_{p,q}(\rho)$ corresponds with $J_{p,n-q}$ and $D_{\sigma_l \rho}$ with D_l. Also, these classes contribute to Matsushima's formula about Betti numbers of the manifold $\Gamma \backslash G / K$ (c.f. [15], p. 175, p. 191).

References

[1] A. Borel, N. Wallach: Continuous cohomology, discrete subgroups and representations of reductive groups, Annals of Math. Studies 94 (1980).

[2] N. Bourbaki: Groupes et algèbres de Lie, chapitres 7 et 8, Hermann, Paris, 1975.

[3] D.L. DeGeorge, N.R. Wallach: *Limit formulas for multiplicities* in $L^2(\Gamma \backslash G)$, Ann. of Math. **107** (1978), 133–150.

[4] J. Dixmier: Algèbres enveloppantes, Gauthier-Villars, Paris, 1974.

[5] H. Hecht, W. Schmid: *A proof of Blattner's conjecture,* Invent. Math. **31** (1975), 129–154.

[6] R. Hotta, R. Parthasarathy: *Multiplicity formulae for discrete series,* Invent. Math. **26** (1974), 133–178.

[7] H. Kraljevic: *Representations of the universal convering group of the group SU (m, 1),* Glasnik Matematicki (ser. III) **8** (1973), 23–72.

[8] R.P. Langlands: *Dimension of spaces of automorphic forms,* Proc. Sympos. Pure Math., vol. **9**, Amer. Math. Soc., Providence, R.I., 1966, 253–257.

[9] J. Lepowsky: *Algebraic results on representations of semisimple Lie groups,* Trans. Amer. Math. Soc. **176** (1973), 1–44.

[10] R. Parthasarathy: *Dirac operator and the discrete series,* Ann. of Math. **96** (1972), 1–30.

[11] D. L. Ragozin, G. Warner: *On a method for computing multiplicities in* $L^2(\Gamma\backslash G)$, Symposia Mathematica (INDAM) (1977), 291–314.

[12] E. Thieleker: *On the quasi-simple irreducible representations of the Lorentz group,* Trans. Amer. Math. Soc. **179** (1973), 465–505.

[13] E. Thieleker: *The unitary representations of the generalized Lorentz groups,* Trans. Amer. Math. Soc. **199** (1974), 327–366.

[14] E. Thieleker: *On the integrable and square-integrable representations of Spin* (1, $2m$), Trans. Amer. Math. Soc. **230** (1977), 1–40.

[15] N. Wallach: *On the Selberg trace formula in the case of compact quotient,* Bull. Amer. Math. Soc. **82** (1976), 171–195.

[16] N. Wallach: Harmonic analysis on homogeneous spaces, Marcel Dekker, New York, 1973.

[17] G. Warner: Harmonic analysis on semisimple Lie groups I and II, Springer-Verlag, New York, 1972.

Department of Mathematics
University of Osaka Prefecture
Sakai, Osaka 591
Japan