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Osaka J. Math.
17 (1980), 433–438

ON ONE-SIDED QF-2 RINGS II

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(Received July 19, 1979)

We have studied the extending property on direct sums of indecomposable modules in [4]. We shall apply those results to projective modules and give characterizations of semi-perfect rings whose projective modules have the extending property of simple module. We shall deal with the dual concept of [5].

1. Preliminaries

Throughout this paper we shall denote a ring with identity by R and every R-module M is a right unitary R-module. By $S(M)$ we denote the *socle* of M. We shall recall the definition of extending property of simple module. If for every simple submodule A_α of $S(M)$ there exists a direct summand M_α of M such that $S(M_\alpha)=A_\alpha$, we say M have the *extending property of simple module*. Let $\{N_\beta\}_I$ be a set of submodules of M. If $\bigcap_{I_1} N_\gamma \supsetneqq \bigcap_{I_2} N_\delta$ for subset $I_1 \subsetneqq I_2$, $\bigcap_{I} N_\delta$ is called *irredundant*.

In this paper we shall study the dual properties to those in [5] and so we shall first introduce the dual condition to (**) in [2] and [3].

(**)* *Every indecomposable projective module contains a unique minimal submodule and is uniform.*

If further every indecomposable left projective module contains a unique minimal submodule, we call R a QF-2 ring following Thrall [7]. Hence, if R satisfies (**)*, we call R a *right* QF-2 *ring* in this note.

Let M be an R-module. If M is a homomorphic image of projective module with non-essential kernel, we call M a *non-cosmall module* [3] and [6]. Every epimorphism onto non-cosmall module has the non-essential kernel [3]. We have dealt with conditions on non-small modules in [5]. We shall consider the dual or similar conditions to them.

(*1)* *Every non-cosamll module which is contained in a projective module contains a non-zero projective summand* (dual to (*1) in [5]).
And

(**2) *For every finitely generated projective P with essential socle* $\mathrm{S}(P)$, *P/T contains a non-zero projective summand for any submodule $T \subsetneqq \mathrm{S}(P)$.*

They are weaker conditions than the following:

(*)* *Every non-cosmall module contains a non-zero projective summand* [4].

2. Right QF-2 rings

We are only interested in right QF-2 rings in this note and so from now on we always assume that R satisfies (**)* unless otherwise stated. Furthermore, we assume R is semi-perfect [1] and we shall denote the Jacobson radical of R and primitive idempotents by J and e, respectively. Let P be projective. Then $P=\sum\oplus P_\alpha$; the P_α is indecomposable. Hence, $\mathrm{S}(P)$ is essential in P by (**)* (see [8]).

Lemma 1. *Let R be a right* QF-2 *and semi-perfect ring and e a primitive idempotent. Let $eR\supset eJ^n \supsetneqq eJ^{n+k}$ be projectives. Then $eJ^n \not\approx eJ^{n+k}$ if J is nil or eR is injective.*

Proof. Since eJ^n is projective and $\mathrm{S}(eJ^n)$ is simple, $eJ^n\approx fR$ for some idempotent f. If $eJ^n\approx eJ^{n+k}$, $fR\approx fJ^k$. This isomorphism is induced by an element in fJf. If J is nil, we have a contradiction. If eR is injective, the isomorphism $eJ^n\approx eJ^{n+k}$ is extended to one on eR. Hence, $eJ^n=eJ^{n+k}$, a contradiction.

Theorem 1. *Let R be a semi-perfect and right* QF-2 *ring with nil Jacobson radical. Then the following conditions are equivalent.*

1) *R satisfies* (*1)*.

2) *Let $\{P_\alpha\}_I$ be a set of direct summands of a projective P such that $P=P_\alpha \oplus P_\alpha'$ and $\mathrm{S}(P_\alpha')$ is simple. If $\bigcap_I \mathrm{S}(P_\alpha)$ is irredundant, $\bigcap_K P_\alpha$ is a direct summand of P for any finite subset K of I.*

3) i) *For some primitive idempotent e, there exists a positive integer $t(e)$ such that $eR/eJ^{t(e)}$ is a serial module, $eB(=eJ^s,\ s\leqslant t(e))$ is projective for any $eR\supset eB\supset eJ^{t(e)}$ and $Z(eC)=eC$ and $eC\subsetneqq eJ^{t(e)}$ for every non-projective right ideal eC in eR.*

ii) *$\{eJ^s\}_{e,s=0}^{t(e)}$ is the representative set of indecomposable projectives, where $Z(\ )$ means the singular submodule* (dual to [5], Theorem 2).

Proof. 1)$\to$2). Let $K=\{1, 2, \cdots, n\}$ be a finite subset of I and put $P(n)=\bigcap_{i=1}^n P_i$. We shall show $P(n)$ is a direct summand of P by the induction on n. If $n=1$, it is clear by the assumption. Put $P=P_n\oplus P_n'$ with P_n' indecomposable and $\pi_n\colon P\to P_n'$ the projection. We note $\mathrm{S}(\bigcap P_\alpha)=\bigcap \mathrm{S}(P_\alpha)$. Since

$S(P(n-1))=\bigcap_{i=1}^{n-1} S(P_i)\not\subset S(P_n)$, $\pi_n(S(P(n-1))\neq 0$. Hence, $\pi_n(P(n-1))$ is non-cosmall module in P_n'. Then there exists an indecomposable summand P_0 of $\pi_n(P(n-1))$ by 1). Since $S(P_n')$ is simple, $\pi_n(P(n-1))=P_0$. Therefore, $P(n-1)=P_0'\oplus\ker\pi_n|P(n-1)=P_0'\oplus P(n)$, where $P_0'\approx P_0$. Since $P=P(n-1)\oplus P'$, $P(n)$ is a direct summand of P.

2)$\to$3). Let e be a primitive idempotent. We assume eA is projective and $eB(\subset eA)$ is non-cosmall for right ideals eA and eB. Then there exists a projective module P such that $0\leftarrow eB\xleftarrow{f}P\leftarrow K\leftarrow 0$ is exact and $S(P)\not\subset K$ by the defintion (see [3], Proposition 3.1). If $S(P)$ is simple, $K=0$ and eB is projective. We assume $P=P_1\oplus\sum_L\oplus P_\alpha$ such that the P_α is indecomposable and $S(P_1)$ is a simple module not contained in K. We put $Q=P\oplus eA$ and $P'=\{x+f(x)\mid x\in P\}\subset Q$. Then $S(P')=(S(P)\cap K)\oplus S((1+f)(P_1))$ and $S(P)=S(P_1)\oplus(S(P)\cap K)$. Since $S(P)\cap S(P')$ is irredundant, $P\cap P'=K$ is a direct summand of Q and hence of P. Accordingly, eB is projective. Now if eJ is non-cosmall, eJ is projective from the above. Hence, eJ contains a unique maximal submodule eJ^2, since eJ is indecomposable by (**)*. Repeating those arguments, we obtain a unique chain $eR\supset eJ\supset eJ^2\supset\cdots\supset eJ^t$ of projectives and eB is cosmall for any $eB\subsetneqq eJ^t$ by Lemma 1. Hence, $eB=Z(eB)$ by [3], Proposition 3.2. The remaining part is clear from the construction of eJ^i.

3)$\to$1). Let P be a projective module which contains a non-cosmall module M. Then $P=\sum\oplus e_iJ^{t_{ij}}$. Let $\pi_{ij}\colon P\to e_iJ^{t_{ij}}$ be the projection. Since $M\neq Z(M)$, $\pi_{kl}(M)\not\subset Z(e_kJ^{t_{kl}})\subsetneqq e_kR$ for some k,l. Hence, $\pi_{kl}(M)$ is projective and so $M=\ker\pi_{kl}|M\oplus M'$; $M'\approx\pi_{kl}(M)$.

Corollary. *Let R be semi-perfect. Then R satisfies* (*)* *if and only if R is right* QF-2 *and* QF-3 *and satisfies* (*1)*.

Proof. In the above proof the implication 1)$\to$2) is valid without the assumption on J. Hence, we obtain the corollary by the implication 2)$\to$3), Lemma 1 and [3], Theorems 1.3 and 3.6.

As the dual to Theorem 2′ in [5] we have

Theorem 1′. *Let R be as before. Then the following conditions are equivalent.*

1) *R is right hereditary.*

2) *Let P be projective and P_i direct summands of P for $i=1, 2$. Then $P_1\cap P_2$ is a direct summand of P.*

3) i) *For some primitive idempotent e, eR is uni-serial and eB is projective for any right ideal $eB\subseteq eR$.* ii) *$\{eB\}_{e,B}$ is the representative set of indecomposable projectives.*

In this case R is right artinian.

Proof. 1)$\rightarrow$2). We can use the same argument as before.

2)$\rightarrow$1). Let P be projective and A a submodule of P. Let $P_1 \xrightarrow{f} A \rightarrow 0$ be an exact sequence with P_1 projective. We put $F=P_1 \oplus P$ and $P_1'=\{x+f(x) \mid x \in P_1\}$. Then $F=P_1' \oplus P$ and so $K=\ker f=P_1 \cap P_1'$ is a direct summand of F. Hence, K is a direct summand of P_1. Therefore, A is projective and R is hereditary.

1)$\rightarrow$3). It is clear from Theorem 1.

3)$\rightarrow$1). We know from 3) that R is right artinian and $Z(R)=0$. Hence, every right ideal A contains a projective summand by Theorem 1. Since R is noetherian, A is projective.

Theorem 2. *Let R be a right* QF-2 *and semi-perfect ring. Then the following conditions are equivalent.*

1) *R satisfies* (**2).

2) *Every projective module has the extending property of simple module.*

3) i) *For some primitive idempotent e there exists a chain of projective right ideals eA_i such that $eR=eA_1 \supset eA_2 \supset \cdots \supset eA_t$ and* $\mathrm{Hom}_R(\mathrm{S}(eA_i), \mathrm{S}(eA_j))$ *is extended to* $\mathrm{Hom}_R(eA_i, eA_j)$ for any pair $i \geqslant j$, (see [4], Theorem 2).

ii) *$\{eA_i\}_{e,i}$ is the representative set of indecomposable projective such that* $\mathrm{S}(eR) \not\approx \mathrm{S}(e'R)$ *if $e \neq e'$.*

Proof. 1)$\rightarrow$2). Let P be projective and $P=\sum_I \oplus P_\alpha$; the P_α is uniform. Let S be a simple submodule of $\mathrm{S}(P)$. Then there exists a finite subset $K=\{1, 2, \cdots, n\}$ of I such that $S \subset \mathrm{S}(\sum_K \oplus P_i)$. If $n=1$, it is clear. Hence, we assume $S \subsetneqq \mathrm{S}(\sum_K \oplus P_i)$ and put $P(n)=\sum_{i=1}^n \oplus P_i$. Then $P^{(n)}/S=P_0 \oplus Q$ and P_0 is projective by 1). Considering an epimorphism $P^{(n)} \rightarrow P/S \rightarrow P_0$, we obtain $P^{(n)}=P_0' \oplus L$; $P_0' \approx P_0$ and $L \supset S$. Since $L=\sum_{i=1}^{n-1} \oplus P_i'$, we can use the induction argument.

2)$\rightarrow$3). Let eR and fR be uniform projectives with isomorphic socle. Then there exists a monomorphism $f: eR \rightarrow fR$ (or $fR \rightarrow eR$) by [4], Corollary 8, i.e. $eR <^* fR$ or $fR <^* eR$ (see [4]). Let eR be a maximal one among uniform projectives P with isomorphic socle with respect to the relation $<^*$. Then those P are isomorphic to right ideals eA in eR. Since the relation $<^*$ is linear on $\{eA\}$, taking repeatedly maximal ones, we get a chain of projective right ideals $eR=eA_1 \supset eA_2 \supset \cdots \supset eA_t$. The second condition is clear by [4], Corollary 8.

3)$\rightarrow$2). It is clear from [4], Corollary 8.

2)$\rightarrow$1). Let $P=P_1 \oplus P_2 \oplus \cdots \oplus P_n$ be projective and the P_i uniform. Let $T \subsetneqq$

S(P) and $T=S_1\oplus S_2\oplus\cdots\oplus S_i$; the S_j is simple. Then there exists a direct summand P_1' of P such that S$(P_1')=S_1$. Let $P=P_1'\oplus K_1$. Then $T=S_1\oplus\pi_1(T)$; $\pi_1\colon P\to K_1$. Hence, S$(K_1)\supsetneqq\pi_1(T)$ and $P/T\approx P_1'/S_1\oplus K_1/\pi_1(T)$. Repeating the same argument on $K_1/\pi_1(T)$, finally we obtain $P/T\approx P_1'/S_1\oplus\cdots\oplus P_i'/S_i'\oplus K_i$ and K_i is projective, since $\pi_j(T)=0$ for some $j\leqslant n$.

3. Corollaries and examples

We shall consider some special cases of rings.

Corollary 1. *If R is a right* QF-2 *and semi-perfect ring with* Z$(R)\supset J$, *then R satisfies* (* 1)*.

Proof. It is clear from the proof of the implication 3)→1) in Theorem 1.

Corollary 2. *If R is a right* QF-2 *and semi-perfect ring with $J^2=0$, then R satsifies* (* 1)*.

Proof. Let $R=\sum\oplus e_iR\oplus\sum\oplus f_jR$, where the e_i and the f_j are primitive and the f_jR is simple. Then S$(R)=\sum\oplus e_iJ\oplus\sum\oplus f_jR$. If $e_iJf_j\neq 0$, $e_iJ\approx f_jR$. Hence, $e_iJ=$ Z(e_iJ) or e_iJ is projective. Accordingly, R satisfies (* 1)* by Theorem 1.

Corollary 3. *Let R be a right* QF-2 *and semi-perfect ring with nil Jacobson radical. Then* Z$(R)=0$ *and* (* 1)* *is satisfied if and only if R is a right generalized uniserial and right artinian hereditary ring.*

Proof. It is clear from Theorem 1.

EXAMPLES 1. Let $K\subset L$ be fields and put

$$R=\begin{pmatrix} K & 0 & L \\ 0 & L & L \\ 0 & 0 & L \end{pmatrix}.$$

Then R is a right QF-2 and hereditary artinian ring. Hence, R satisties (* 1)*. If $[L\colon K]=\infty$, R is not left artinian and does not satisfy (** 2).

2. Let $C=K\oplus M$; $M=K$, be the trivial extension and put

$$R=\begin{pmatrix} C & C \\ 0 & C \end{pmatrix} \text{ ([5], Example 2).}$$

Then R is QF-2 and $e_{11}R$ is injective and projective. Hence, R satisfies (** 2) by Theorem 2. Put $P=e_1R\oplus e_1R\oplus e_2R$, where $e_i=e_{ii}$. We have a homomorphism e_1R to e_1R by a multiplication of $m(m\in M)$ from the left side and a monomorphism ρ of e_2R into e_1R. We take an epimorphism

$$(1, m, \rho): P \to e_1R\,.$$

Then its kernel $N_1=\{(x, y, z)\,|\in P,\ x+my+\rho(z)=0\}$ is a direct summand of P. Put $N_2=\{(0, y, z)\,|\in P\}$ and $N_3=\{(x, 0, z)\,|\in P\}$. Then $N_1\cap N_2\cap N_3=0$. However, $N_1\cap N_2=\{(0,0), (a,b), (0,mb)\,|\,a\in M,\ b\in C\}\approx e_1J$ is not projective. Hence, R does not satisfy (* 1).

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