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INDEX OF THE EXPONENTIAL MAP ON A COMPLEX SIMPLE LIE GROUP

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0. Introduction

Let $\mathfrak{G}$ be a connected Lie group with Lie algebra G. Following Goto [2], for $g\in\mathfrak{G}$, we define the index (of the exponential map) ind(g) to be the smallest positive integer q such that $g^q\in\exp G$, if it exists, otherwise, $\mathrm{ind}(g)=\infty$. The index ind($\mathfrak{G}$) of $\mathfrak{G}$ is defined to be the least common multiple of all ind(g) ($g\in\mathfrak{G}$).

Given a complex simple Lie algebra G with a Cartan subalgebra H, let $-\alpha_0=m_1\alpha_1+\cdots+m_l\alpha_l$ be the highest root of G with respect to H expressed in terms of a simple root system $\{\alpha_1, \cdots, \alpha_l\}$. Consider the center-free Lie group with Lie algebra G, which can be identified with the adjoint group of (all inner automorphisms of) G. In Lai [4], we proved the following theorem:

Theorem. $\{\mathrm{ind}(g); g\in Ad(G)\}=\{1, m_1, \cdots, m_l\}=\{d;$ *d is a factor of some* $m_j\}$.

The main purpose of this paper is to generalize the above result to an arbitrary (always assumed to be connected) complex simple Lie group $\mathfrak{G}$.

Theorem. *Let $\mathfrak{G}$ be a complex simple Lie group with Lie algebra G. We can find certain positive integers $p_0, \cdots, p_l$(depending on the center $Z(\mathfrak{G})$ of $\mathfrak{G}$, to be defined in the next section) such that*

$$\{\mathrm{ind}(g); g\in\mathfrak{G}\} = \{d;\ d \text{ is a factor of some } p_jm_j\ (0\leq j\leq l) \text{ with } m_0 = 1\}\ .$$

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1. Notation and definition of p_j's

Let G be a complex semisimple Lie algebra with a (fixed) Cartan subalgebra H. Let Δ be the root system of G with respect to H, $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ a fundamental root system of Δ, and $-\alpha_0=m_1\alpha_1+\cdots+m_l\alpha_l$ be the highest root.

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Let B be the Killing form on G. Then for each $\alpha\in\Delta$,we can find $h_\alpha\in H$ with $B(h, h_\alpha)=\alpha(h)$ for all $h\in H$, and $e_\alpha\in G$ such that

$$G = H+\textstyle\sum_{\alpha\in\Delta}Ce_\alpha$$

$$\begin{array}{lll} [h, e_\alpha] = \alpha(h)e_\alpha\,, & [e_\alpha, e_\beta] = N_{\alpha,\beta}e_{\alpha+\beta} & \text{if } \alpha+\beta\neq 0 \text{ is in } \Delta\,, \\ [e_\alpha, e_{-\alpha}] = -h_\alpha\,, & [e_\alpha, e_\beta] = 0 & \text{if } 0\neq\alpha+\beta\notin\Delta\,. \end{array}$$

Let $H_0\subset H$ be the real vector space spanned by $h_\alpha(\alpha\in\Delta)$, then $\beta|_{H_0}$ is real for any $\beta\in\Delta$. Since $\Pi=\{\alpha_1, \cdots, \alpha_l\}$ is linearly independent, we can choose $h_1, \cdots, h_l\in H_0$ such that $\alpha_i(h_j)=\delta_{ij}$ $1\leq i, j\leq l$. The lattice $\Omega=\boldsymbol{Z}2\pi ih_1+\cdots+\boldsymbol{Z}2\pi ih_l\subset iH_0$ $(i=\sqrt{-1})$ is the kernel of $\exp|_H: H\to Ad(G)$. On the other hand, let $\tilde{\mathfrak{G}}$ be the simply connected Lie group with Lie algebra G, denoting $2h_\alpha/B(h_\alpha, h_\alpha)$ by h^*_α, the lattice Ω^* generated by $\{2\pi ih^*_\alpha; \alpha\in\Delta\}$ becomes the kernel of $\exp|_H: H\to\tilde{\mathfrak{G}}, \Omega^*$ is of finite index in Ω. For simplicity, we identify Δ with a subset of iH_0 by the map $\alpha\mapsto h_\alpha/2\pi i$, and introduce an inner product in iH_0 by $(h, h')=-B(h, h')/(2\pi)^2$. Then $(\alpha, h)=\alpha(h)/2\pi i$ for $\alpha\in\Delta, h\in iH_0$.

If $\mathfrak{G}$ is a connected Lie group with G as its Lie algebra. Let Ω' be the kernel of $\exp|_H: H\to\mathfrak{G}$, then $\Omega^*\subset\Omega'\subset\Omega$, so that Ω' is an additive subgroup of finite index in Ω. For each h_j, let p_j be the smallest positive integer such that $2\pi ip_jh_j\in\Omega'$ $(j=1, \cdots, l)$. Denote by p_0 the least common multiple of $\{p_1, \cdots, p_l\}$, and $m_0=1$.

REMARK. p_0 is the smallest positive integer such that $g^{p_0}=1$ for any element g in the center $Z(\mathfrak{G})$ (which is equal to $\exp(\Omega)$). In case G is simple, computation shows that $p_0=p_j$ for some $j=1, \cdots, l$. (For this, see, e.g. Goto-Grosshans [3] Chapter 5.)

Let $Ad(\Delta)$ denote the Weyl group of Δ. Any element S of $Ad(\Delta)$, regarded as a linear transformation on iH_0, can be extended to an inner automorphism of the Lie algebra G. Let $T(\Omega^*)$ be the group of translations of the euclidean space iH_0 induced by elements in Ω^*. Then, if G is simple, the group $Ad(\Delta)\cdot T(\Omega^*)$ acts transitively on the set of all cells, see Goto-Grosshans [3] Chapter 5. We summarize as follows:

Proposition. *Let G be a complex simple Lie algebra and C_0 the fundamental cell*: $C_0=\{h\in iH_0;\ (\alpha_1, h)>0, \cdots, (\alpha_l, h)>0$ *and* $(-\alpha_0, h)<1\}$. *Let $\bar{C}_0$ denote the closure of C_0. Then for any h in iH_0, we can find $U\in Ad(\Delta)\cdot T(\Omega^*)=Afd(\Delta)$ such that $h\in U\bar{C}_0$.*

In the following, we assume $\mathfrak{G}$ is a connected simple complex Lie group.

2. Upper bound for ind(g)

Theorem. *For any $g\in\mathfrak{G}$,* ind(g) *is a factor of p_jm_j for some $j=0, \cdots, l$.*

Any element g in $\mathfrak{G}$ has a decomposition $g=g_0\cdot\exp N$ into semisimple part g_0 and unipotent part $\exp N$ such that $g_0\cdot\exp N=\exp N\cdot g_0$. Let $G(1, Ad\, g_0)$ denote the 1-eigenspace of $Ad\, g_0$ in G. Then $G(1, Ad\, g_0)$ is a subalgebra of G and $N\in G(1, Ad\, g_0)$.

By Gantmacher [1], g_0 is conjugate to some element in $\exp H$. Hence, to prove our theorem, it suffices to consider elements g whose semisimple part lies in $\exp H$, i.e., $g=\exp h_0\cdot\exp N$, $h_0\in H$ and $N\in G(1, Ad\exp h_0)$. Let $\Delta(h_0)=\{\alpha\in\Delta;\, Ad\exp h_0\cdot e_\alpha=e_\alpha\}=\{\alpha\in\Delta;\, \alpha(h_0)\in 2\pi i\boldsymbol{Z}\}$. Then $G(1, Ad\exp h_0)=H+\sum_{\alpha\in\Delta(h_0)}Ce_\alpha$, and $\Delta(h_0)$ is a subsystem of Δ, we can choose a simple root system $\Pi(h_0)=\{\beta_1, \cdots, \beta_r\}$ for $\Delta(h_0)$.

Lemma 1. *To find an upper bound for* ind(g) $(g\in\mathfrak{G})$, *it suffices to consider elements with semisimple part* $\exp h_0$, *where* $h_0\in iH_0$ *and* $\Pi(h_0)$ *has cardinality* $l=$rank *of* G.

Proof. Assume that $h_0=x_1h_1+\cdots+x_lh_l$ for some complex numbers x_i. For each $j=1, \cdots, r$, since $(Ad\exp h_0-1)\cdot e_{\beta_j}=0$, we have $\beta_j(h_0)=2\pi ik_j$ for some $k_j\in\boldsymbol{Z}$. If k_j are all zero, then $[h_0, N]=0$ for any $N\in G(1, Ad\exp h_0)$, so that $\exp h_0\cdot\exp N=\exp(h_0+N)$, and $\operatorname{ind}(\exp h_0\cdot\exp N)=1$. So we assume that some $k_j\neq 0$, after this.

Since $\exp h_0=\exp(h_0+\Omega')$, if we can find a positive integer d and integers $n_1, \cdots, n_l$ such that for $h=dh_0+\sum_{j=1}^{l}2\pi in_jp_jh_j$, $[h, dN]=0$, then ind$(\exp h_0\cdot\exp N)$ divides d. For this, it suffices to choose d and n_j with $\alpha(h)=0$ for all $\alpha\in\Delta(h_0)$, or equivalently, for all $\alpha\in\Pi(h_0)$. Therefore, the problem reduces to finding d so that $\beta_i(\sum_{j=1}^{l} n_jp_jh_j)=-dk_i$ has integral solutions $n_1, \cdots, n_l$.

Choose $\beta_{r+1}, \cdots, \beta_l\in\Delta$ so that $\{\beta_1, \cdots, \beta_l\}$ is a maximal linearly independent subset of Δ. We write $\beta_i=\sum_{j=1}^{l} q_{ij}\alpha_j$ where $q_{ij}\in\boldsymbol{Z}$. Consider the following system of linear equations:

$$\begin{aligned} q_{i1}p_1n_1+\cdots+q_{il}p_ln_l &= -k_i \qquad & i &= 1, \cdots, r;\\ q_{i1}p_1n_1+\cdots+q_{il}p_ln_l &= 0 \qquad & i &= r+1, \cdots, l\,. \end{aligned}$$

Since $(q_{ij}p_j)$ is a nonsingular integral matrix with determinant $p_1\cdots p_l\cdot\det(q_{ij})$ (which is not zero by the choice of β_j's and the fact that p_j are positive), and k_i are integers, this has a rational solution, say, $r_1, \cdots, r_l$.

Let $h_0'=\sum_{j=1}^{l}2\pi ir_jp_jh_j\in iH_0$, then $\beta_1, \cdots, \beta_l\in\Delta(h_0')$. Suppose we can find a positive integer d' and integers $n_1', \cdots, n_l'$ such that $\beta(d'h_0'+\sum_{j=1}^{l}2\pi in_j'p_jh_j)=0$ for all $\beta\in\Delta(h_0')$, then $(n_1, \cdots, n_l)=(n_1', \cdots, n_l')$ is the solution for the following system of linear equations:

$$\begin{aligned} \textstyle\sum_{j=1}^{l} q_{ij}p_jn_i &= -d'k_i \qquad & i &= 1, \cdots, r;\\ \textstyle\sum_{j=1}^{l} q_{ij}p_jn_j &= 0 \qquad & i &= r+1, \cdots, l\,. \end{aligned}$$

Thus we can find $n_j \in \boldsymbol{Z}$ such that $\beta_i(\sum_{j=1}^{l} 2\pi i n_j p_j h_j) = -2\pi i d' k_i$ $(i=1, \cdots, r)$. Hence for $h = d' h_0 + \sum_{j=1}^{l} 2\pi i n_j p_j h_j$, we have $\beta_i(h) = 0$ $(i=1, \cdots, r)$ and so $\beta(h) = 0$ for all $\beta \in \Delta(h_0)$.

We have proved that $\operatorname{ind}(\exp h_0 \cdot \exp N)$ is a factor of $\operatorname{ind}(\exp h_0' \cdot \exp N)$. Therefore, we may replace h_0 by h_0' which satisfies Lemma 1. ||

Let S be in the Weyl group $Ad(\Delta)$. Then S can be extended to an inner automorphism σ of the Lie algebra G, which can be extended to an inner automorphism of the Lie group $\mathfrak{G}$. Clearly $\operatorname{ind}(g) = \operatorname{ind}(\sigma g)$. Therefore, to find an upper bound for $\operatorname{ind}(g)$ $(g \in \mathfrak{G})$, we may replace g (whose semisimple part is $\exp h_0$) by an element whose semisimple part is $\exp Sh_0 (S \in Ad(\Delta))$.

On the other hand, $\exp h_0 = \exp(h_0 + \Omega^*)$ (because $\Omega^* \subset \Omega'$), so we may replace h_0 by $T(\Omega^*) h_0$. We get the following lemma by applying the proposition we stated at the end of section 1.

Lemma 2. *Let $-\alpha_0 = m_1\alpha_1 + \cdots + m_l\alpha_l$ be the highest root. To find an upper bound for* $\operatorname{ind}(g)$ $(g \in \mathfrak{G})$, *it suffices to consider elements whose semisimple part have the form* $\exp h$, $h \in iH_0$ *with* $0 \leq (\alpha_1, h), \cdots, 0 \leq (\alpha_l, h)$ *and* $(-\alpha_0, h) \leq 1$.

Let $\tilde{\Pi} = \{\alpha_0, \alpha_1, \cdots, \alpha_l\}$ be the extended simple root system. The following two lemmas, proved in [4], being properties of simple Lie algebras, can be applied in the present case too. For a proof, please see [4] or Goto-Grosshans [3] Chapter 8.

Lemma 3. *Let $h \in \bar{C}_0$ be an element satisfying Lemma* 1, *then $\Pi' = \tilde{\Pi} \cap \Delta(h)$ is a simple root system for $\Delta(h)$ with respect to a suitable ordering.*

Since $\Pi(h) = \tilde{\Pi} \cap \Delta(h)$ has cardinality l. If $\Pi(h) = \Pi$, then $\Delta(h) = \Delta$ and $h \in \Omega$,in this case, $\operatorname{ind}(\exp h \cdot \exp N)$ is a factor of p_0 $(= p_0 m_0)$ because $p_0 h \in \Omega'$.

Lemma 4. *If $\Pi(h) \neq \Pi$ has cardinality l, then $h = 2\pi i h_j / m_j$ for some $j = 1, \cdots, l$ such that $m_j > 1$.*

In the case $m_j = 1$, we have $\Pi(2\pi i h_j / m_j) = \Pi$.

Conclusion. Let $\mathfrak{G}$ be a connected complex simple Lie group. To find an upper bound for $\{\operatorname{ind}(g);\, g \in \mathfrak{G}\}$, it suffices to consider elements $g \in \mathfrak{G}$ whose semisimple part has the form $\exp 2\pi i h_j / m_j$ for some $j = 0, 1, \cdots, l$. i.e. $g = \exp 2\pi i h_j / m_j \cdot \exp N$.

Clearly, $g^{p_j m_j} = \exp(p_j m_j N)$ because $2\pi i p_j h_j \in \Omega'$.

Theorem. *For any $g \in \mathfrak{G}$, there exists j $(0 \leq j \leq l)$ such that $g^{p_j m_j} \in \exp G$. In other words,* $\operatorname{ind}(g)$ *is a factor of some $p_j m_j$ $(0 \leq j \leq l)$.*

3. Existence of elements with index exactly equal to $p_j m_j$

An element x in a semisimple Lie algebra G is said to be regular if the centralizer $z_G(x)=\{y\in G; [x,y]=0\}$ of x has minimal dimension. If H is a Cartan subalgebra of G with root system Δ and $U=\sum_{\alpha>0} Ce_\alpha$, then $B=H+U$ is a Borel subalgebra (i.e. a maximal solvable subalgebra). The following proposition is a consequence of the Lie algebra analogous of Theorem 1 and its corollary in Steinberg [5] (pp. 110–112).

Proposition. *If $x=\sum_{\alpha>0} c_\alpha e_\alpha \in U$ $(c_\alpha \in \boldsymbol{C})$ is a nilpotent element in G, then x is regular if and only if $c_\alpha \neq 0$ for any simple root α. In such case, $z_G(x)\subset U$, in particular, $z_G(x)$ consists only of nilpotent elements.*

Retaining the notation used in the previous sections, consider $h_0=2\pi i h_j/m_j$ $(1\leq j\leq l)$. Then $\Pi=\tilde{\Pi}-\{\alpha_j\}$ is a simple root system for $\Delta(h_0)$ and $G(1, Ad\exp h_0)=H+\sum_{\alpha\in\Delta(h_0)} Ce_\alpha$ is a semisimple subalgebra of G. Let $N=\sum_{i=0,\cdots,l;\, i\neq j} e_{\alpha_i}$, then N is a regular element in $G(1, Ad\exp h_0)$, so that any element of $G(1, Ad\exp h_0)$ which commutes with N must be nilpotent.

Let $g=\exp h_0\cdot\exp N$, and $\mathfrak{G}_1$ be the connected subgroup of $\mathfrak{G}$ corresponding to the subalgebra $G_1=G(1, Ad\, g)=G(1, Ad\exp h_0)$. Clearly, $g\in\mathfrak{G}_1$ because $h_0, N\in G_1$. Therefore $g^q\in\mathfrak{G}_1$ for any positive integer q.

If for certain q, $g^q=\exp x$ for some $x\in G$, then x lies in G_1 (because $G_1=\{y\in G; \exp y\in\mathfrak{G}_1\}$). We know that x has a decomposition $x=x_0+N$, where x_0 is semisimple and $[x_0, N]=0$. Since x, $N\in G_1$, we have $x_0\in G_1= G(1, Ad\exp h_0)$. But $[x_0, N]=0$, the above argument implies that x_0 is nilpotent. Thus $x_0=0$ because x_0 is also semisimple. This implies that $\exp x_0=\exp qh_0=1$, or $qh_0\in\Omega'$. This cannot happen if $q<p_j m_j$.

Therefore $\mathrm{ind}(g)=p_j m_j$.

In case $j=0$, let $h_0=\sum_{j=1}^{l} 2\pi i h_j$, then $qh_0\notin\Omega'$ unless q is a multiple of p_0. Let $N=\sum_{j=1}^{l} e_{\alpha_j}$, which is regular in G. The same argument as above proves that $\mathrm{ind}(\exp h_0\cdot\exp N)=p_0=p_0 m_0$. Q.E.D.

The results in sections 2 and 3 give the following:

Theorem. *Let $\mathfrak{G}$ be a connected complex simple Lie group. Retaining the above notation. Then $\{\mathrm{ind}(g); g\in\mathfrak{G}\}=\{q;\, q$ is a factor of some $p_j m_j$, $0\leq j\leq l\}=\{q;\, q$ is a factor of some $p_j m_j$, $1\leq j\leq l\}$.*

Corollary. $\mathrm{ind}(\mathfrak{G})$ *is the least common multiple of* $\{p_1 m_1, \cdots, p_l m_l\}$.

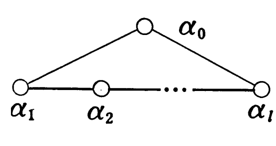
4. List of ind(g) when $\mathfrak{G}$ is simply connected

In this case, p_j can be found by using the inverse matrix of Cartan matrix of G, please see e.g. Goto-Grosshans [3] Chapter 5.

(a) G is of type A_l

The highest root is $-\alpha_0=\alpha_1+\cdots+\alpha_l$.

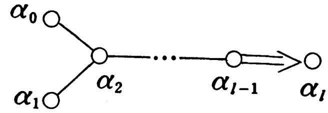
$p_1=\cdots=p_l=l+1$.



Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{q;\, q \text{ divides } l+1\}$ and $\mathrm{ind}(\mathfrak{G})=l+1$.

In fact, for any connected complex simple Lie group of type A, $\mathrm{ind}(\mathfrak{G})=$ order of the center $Z(\mathfrak{G})$.

(b) G is of type B_l

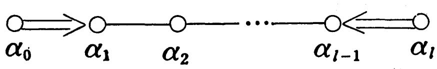


The highest root is $-\alpha_0=\alpha_1+2(\alpha_2+\cdots+\alpha_l)$.

$p_j=2$ when j is odd, $p_j=1$ when j is even.

Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2, 4\}$ in case $l\geq 3$ and $\mathrm{ind}(\mathfrak{G})=4$. And $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2\}$ in case $l=2$ and $\mathrm{ind}(\mathfrak{G})=2$.

(c) G is of type C_l

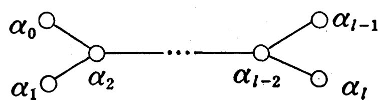


The highest root is $-\alpha_0=2(\alpha_1+\cdots+\alpha_{l-1})+\alpha_l$.

$p_l=2$ and $p_j=1$ when $j<l$.

Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2\}$ and $\mathrm{ind}(\mathfrak{G})=2$.

(d) G is of type D_l

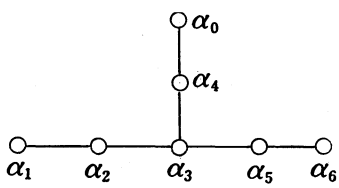


The highest root is $-\alpha_0=\alpha_1+2(\alpha_2+\cdots+\alpha_{l-2})+\alpha_{l-1}+\alpha_l$.

Case 1. When l is even, $p_j=2$ if $j\leq l-2$ is odd or $j=l-1$, l; $p_j=1$ otherwise. Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2\}$ and $\mathrm{ind}(\mathfrak{G})=2$.

Case 2. When l is odd, $p_j=2$ if $j\leq l-2$ is odd, $p_{l-1}=p_l=4$; $p_j=1$ otherwise. Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2, 4\}$ and $\mathrm{ind}(\mathfrak{G})=4$.

(e) G is of type E_6

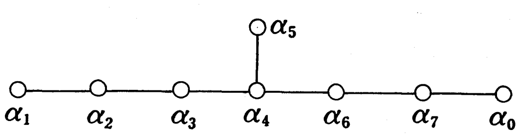


The highest root is $-\alpha_0=\alpha_1+2\alpha_2+3\alpha_3+2\alpha_4+2\alpha_5+\alpha_6$.

$p_1=p_2=p_5=p_6=3$ and $p_3=p_4=1$.

Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{1, 2, 3, 6\}$ and $\mathrm{ind}(\mathfrak{G})=6$.

(f) G is of type E_7



The highest root is $-\alpha_0=\alpha_1+2\alpha_2+3\alpha_3+4\alpha_4+2\alpha_5+3\alpha_6+2\alpha_7$.
$p_1=p_3=p_5=2$ and $p_j=1$ otherwise.
Hence $\{\mathrm{ind}(g);\, g\in\mathfrak{G}\}=\{\text{factors of } 12\}$ and $\mathrm{ind}(\mathfrak{G})=12$.
Note that $p_j=1$ for any j in case G is of type E_8, F_4, or G_2.

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