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ON A PROBLEM OF STOLL CONCERNING A COHOMOLOGY MAP FROM A FLAG MANIFOLD INTO A GRASSMANN MANIFOLD

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Introduction and summary. The main purpose of this article is to answer a problem of W. Stoll which arose in his recent study [6] of value distribution of Schubert zeros.

We denote by A a p-tuple of non-negative integers $\{a_1, a_2, \cdots, a_p\}$ such that $0\leqq a_1\leqq a_2\leqq\cdots\leqq a_p\leqq q$ and let $m=p+q$. A flag F of type A is a sequence $\{V_{a_1+1}, V_{a_2+2}, \cdots, V_{a_p+p}\}$ of linear subspaces in $\boldsymbol{C}^m$, where the subscript indicates the dimension of each subspace, such that $V_{a_1+1}\subset V_{a_2+2}\subset\cdots\subset V_{a_p+p}$. The totality $F(A)$ of flags of type A is called the flag manifold of type A. It is a projective algebraic manifold on which the unitary group $U(m)$ acts transitively. For each flag F of type A, there is associated a Schubert variety $(A; F)$ which is an irreducible algebraic subvariety of the Grassmann manifold $Gr(p,m)$ of p-planes in $\boldsymbol{C}^m$. The Schubert variety $(A;F)$ consists of all p-planes V such that $\dim V\cap V_{a_i+i}\geqq i$ for all $i=1, 2, \cdots, p$. We denote by $S(A)$ the subset of $Gr(p,m)\times F(A)$ consisting of all pairs (V, F) such that $V\in(A; F)$. It is proved that $S(A)$ is an irreducible analytic subvariety of $Gr(p,m)\times F(A)$ [2]. The projection maps of $Gr(p,m)\times F(A)$ onto $Gr(p,m)$ and $F(A)$ induce holomorphic surjective maps π and σ of $S(A)$ onto $Gr(p,m)$ and $F(A)$ respectively. Then $S(A)$ is a holomorphic fiber bundle over $F(A)$ of projection σ whose typical fibre is the Schubert variety. $S(A)$ is also a holomorphic fibre bundle over $Gr(p,m)$ [2] and the operation π_* of fibre integration is defined.

Let λ be a differential form of type $(f-r, f-r)$ on the flag manifold $F(A)$, where f denotes the complex dimension of $F(A)$ and r is a non-negative integer. We assume that λ is invariant by the action of the unitary group $U(m)$ on $F(A)$. The group $U(m)$ acts also on $S(A)$ in a natural way and the pullback $\sigma^*\lambda$ is invariant by $U(m)$. The projection π: $S(A)\rightarrow Gr(p,m)$ commutes with the action of $U(m)$ on $S(A)$ and $Gr(p,m)$ and the fibre integration π_* is functorial. Hence $\pi_*(\sigma^*\lambda)$ is also invariant by the action of $U(m)$ on $Gr(p,m)$. Since $Gr(p,m)$ is a symmetric space, a differential form on $Gr(p,m)$ which is invariant by $U(m)$ is harmonic with respect to any invariant Kaehler metric.

Hence $\pi_*(\sigma^*\lambda)$ represents a cohomology class of type $(pq-|A|-r, pq-|A|-r)$, where $|A|=\Sigma a_i$ and pq is the complex dimension of $Gr(p, m)$.

We take the standard basis $\{e_1, \cdots, e_m\}$ of C^m and denote by V_k^0 the k-plane spanned by $\{e_1, \cdots, e_k\}$ and call $F^0=\{V^0_{a_1+1}, \cdots, V^0_{a_p+p}\}$ the standard flag of type A. We also call $(A; F^0)$ the standard Schubert variety and denote this by (A). The standard Schubert varieties (B), where $B=\{b_1, \cdots, b_p\}$ and $\Sigma b_i=r+\Sigma a_i$, form a basis of the homology group of dimension $2(|A|+r)$ of $Gr(p, m)$ and the Poincaré duals $C(B)$ of (B) form a basis of the cohomology group of type $(pq-|A|-r, pq-|A|-r)$. Therefore $\pi_*(\sigma^*\lambda)$ is a linear combination of these Poincaré duals $C(B)$ with the condition $|B|=\Sigma b_i=|A|+r$.

In the paper [6], Stoll defines a differential form $\hat{\lambda}$ of type $(f-1, f-1)$ on $F(A)$ which is invariant by $U(m)$ and which corresponds to the integral average of so-called Levine form. He asked the question that in the expression of $\pi_*(\sigma^*\hat{\lambda})$ by the Poincaré duals $C(B)$, for what kind of B the Poincaré dual $C(B)$ can appear with non-zero coefficient?

In this paper we prove the following theorem which answers the question of Stoll.

Theorem. *Let λ be a differential form of type $(f-r, f-r)$ on the flag manifold $F(A)$ which is invariant by the action of the unitary group $U(m)$.*
Let

$$\pi_*(\sigma^*\lambda) = \Sigma d_B C(B)\,,$$

where the summation extends over all $B=\{b_1, \cdots, b_p\}$ such that $\Sigma b_i=r+\Sigma a_i$. If the coefficient d_B is non-zero, then B verifies the following condition: there exist s indices $j_1, \cdots, j_s (1\leqq j_1<j_2<\cdots<j_s\leqq p,\ s\leqq r)$ and s positive integers $n_1, \cdots, n_s$ with the condition $\Sigma n_k=r$ such that

$$b_{j_k} = a_{j_k}+n_k \qquad \text{for } k=1, 2, \cdots, s$$

and

$$b_j = a_j \qquad \text{for } j\neq j_k\,.$$

An application of the theorem is discussed in the paper of Stoll [6]. We discuss here special cases. *The case $r=0$.* In this case λ is a volume element on $F(A)$ invariant by $U(m)$ and $\pi_*(\sigma^*\lambda)$ is a scalar multiple of $C(A)$. Hence there is an invariant volume element on $F(A)$ such that $\pi_*(\sigma^*\lambda)=C(A)$. *The case $r=1$.* In this case λ is of type $(f-1, f-1)$ and $\pi_*(\sigma^*\lambda)$ is a linear combination of those $C(B)$ with non-zero coefficient verifying the condition that $b_j=a_j$ except for one index j_1 and $b_{j_1}=a_{j_1}+1$. This condition on B means that the Schubert variety (B) of dimension $|B|$ contains the Schubert variety (A) of dimension $|B|-1$ as a "boundary component" with respect to the

Schubert cell decomposition of $Gr(p, m)$.

The proof of Theorem is group theoretical and we use the results of Kostant [3, a, b] on the de Rham duals of Schubert varieties. The papers of Kostant deal with more general case of homogeneous compact Kaehler manifolds and the proof in the general case is very complex. Therefore we have included in this paper proofs of these results in our special case of the Grassmann manifold in §3 and in an appendix. The proof of our theorem will be completed in §4.

1. Fibre bundle structures of $S(A)$.

In this section we introduce several notations which we use throughout this paper. By $A, B, \cdots$ we shall denote p-tuples $\{a_1, a_2, \cdots, a_p\}$, $\{b_1, b_2, \cdots, b_p\} \cdots$ of integers such that

$$0 \leqq a_1 \leqq a_2 \leqq \cdots \leqq a_p \leqq q,\ 0 \leqq b_1 \leqq b_2 \leqq \cdots \leqq b_p \leqq q, \cdots$$

where q is a fixed positive integer and we set

$$m = p+q .$$

For each A we denote by $|A|$ the sum Σa_i:

$$|A| = \Sigma a_i , \tag{1.1}$$

and we define k_i by

$$k_i = a_i + i \qquad (i = 1, 2, \cdots, p) . \tag{1.2}$$

We have

$$1 \leqq k_1 < k_2 < \cdots < k_p \leqq m .$$

A flag F of type A is an increasing sequence $F = \{V_{k_1}, \cdots, V_{k_p}\}$ of linear subspaces of C^m, where the subscript indicates the dimension of each linear subspace. The totality $F(A)$ of flags of type A is called the flag manifold of type A. We denote by $Gr(p,m)$ the Grassmann manifold of p-dimensional linear subspaces of C^m. We denote by $\{e_1, e_2, \cdots, e_m\}$ the standard basis of C^m and denote by V_k^0 the k-dimensional subspace spanned by $\{e_1, \cdots, e_k\}$. We call $F^0 = \{V^0{}_{k_1}, \cdots, V^0{}_{k_p}\}$ the *standard flag of type A*. We denote by o the point in $Gr(p, m)$ represented by V_p^0 and call the point o the origin of $Gr(p, m)$.

The groups $GL(m, C)$ and $U(m)$ act transitively on $F(A)$ and we can represent $F(A)$ as homogeneous spaces of these two groups:

$$\begin{aligned} F(A) &= GL(m, C)/P_A = U(m)/H_A , \\ H_A &= P_A \cap U(m) , \end{aligned} \tag{1.3}$$

where P_A (resp. H_A) consists of all $h \in GL(m, C)$ (resp. $h \in U(m)$) such that

$h\cdot F^0=\{hV^0_{k_1}, \cdots, hV^0_{k_p}\}=F^0$. We shall denote by π_F the projection

$$\pi_F\colon U(m) \to F(A) = U(m)/H_A\,. \tag{1.4}$$

The groups $GL(m, \boldsymbol{C})$ and $U(m)$ act transitively also on $Gr(p, m)$ and we have

$$\begin{gathered} Gr(p, m) = GL(m, \boldsymbol{C})/P = U(m)/U(p)\times U(q) \\ U(p)\times U(q) = P\cap U(m)\,, \end{gathered} \tag{1.5}$$

where P consists of all $h\in GL(m, \boldsymbol{C})$ such that $h\cdot o=o$. The subgroup P consists of all $h\in GL(m, \boldsymbol{C})$ of the form

$$h = \begin{pmatrix} h_1 & * \\ 0 & h_2 \end{pmatrix},\ h_1\in GL(p, \boldsymbol{C}),\ h_2\in GL(q, \boldsymbol{C}) \tag{1.6}$$

and $U(p)\times U(q)=P\cap U(m)$ is the subgroup of $U(m)$ of all the unitary matrices h of the form

$$h = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix},\ h_1\in U(p),\ h_2\in U(q)\,. \tag{1.7}$$

We denote by π_G the projection

$$\pi_G\colon GL(m, \boldsymbol{C}) \to Gr(p, m) = GL(m, \boldsymbol{C})/P\,. \tag{1.8}$$

For each flag F of type A let

$$(A; F) = \{V\in Gr(p, m) \mid \dim V\cap V_{k_i}\geqq i,\ i = 1, \cdots, p\} \tag{1.9}$$

$(A;F)$ is an irreducible algebraic subvariety of dimension $|A|$ of $Gr(p,m)$ and called the Schubert variety of type A (corresponding to F) [1]. We shall denote $(A;F^0)$ simply by (A) and we call (A) the *standard Schubert variety of type A*. If F is a flag of type A, then there exists $g\in GL(m, \boldsymbol{C})$ such that $g\cdot F^0=F$. Then we have

$$g(A) = (A; F)\,.$$

In particular, if $h\in P_A$, then $hF^0=F^0$ and we have

$$h(A) = (A)$$

for all $h\in P_A$. Thus the group P_A acts on the standard Schubert variety (A).

The standard Schubert varieties (A) with the condition $|A|=r$ form a basis of the homology group $H_{2r}(Gr(p, m), \boldsymbol{Z})$.

A flag $\widetilde{F}$ is an increasing sequence $\{V_1, V_2, \cdots, V_m\}$ of m linear subspaces of $\boldsymbol{C}^m$ and let

$$\rho(\widetilde{F}) = \{V_{k_1}, \cdots, V_{k_p}\} .$$

Then $\rho(\widetilde{F})$ is a flag of type A. Let $(A; \widetilde{F})^*$ be the set of all $V \in Gr(p,m)$ such that $\dim V \cap V_{k_i} \geqq i$ for all $i=1, 2, \cdots, p$ and $\dim V \cap V_{k_i-1} < i$ for all i such that $k_i - k_{i-1} = a_i - a_{i-1} + 1 > 1$, where we let $k_0 = 0$, $a_0 = 0$.

Obviously we have $(A; \widetilde{F})^* \subset (A; \rho(\widetilde{F}))$ and we have also

$$(A; \rho(\widetilde{F})) = (A; \widetilde{F})^* \cup (\bigcup_{a_i > a_{i-1}} (A_i; F_i))$$

where we put $A_i = \{a_1, \cdots, a_{i-1}, a_i - 1, \cdots, a_p\}$ for all i such that $a_i > a_{i-1} (a_0 = 0)$ and $F_i = \{V_{k_1}, \cdots, V_{k_{i-1}}, V_{k_i - 1}, \cdots, V_{k_p}\}$ [1]. Moreover $(A; \widetilde{F})^*$ is biholomorphic to $\boldsymbol{C}^{|A|}$ and hence it is a cell of (real) dimension $2|A|$ and $(A; \widetilde{F})^*$ is Zariski open in $(A; \rho(\widetilde{F}))$.

We denote by $\widetilde{F}^0$ the standard flag $\{V_1^0, V_2^0, \cdots, V_m^0\}$ and denote $(A; \widetilde{F}^0)^*$ by $(A)^*$ and call $(A)^*$ the standard Schubert cell of type A. Then we have

$$Gr(p, m) = \bigcup_A (A)^* \quad \text{(disjoint union)}$$

and this gives a cell decomposition of $Gr(p, m)$.

Let

$$S(A) = \{(V, F) \in Gr(p, m) \times F(A) \mid V \in (A; F)\} . \tag{1.10}$$

$S(A)$ is an irreducible analytic subvariety of $Gr(p, m) \times F(A)$ [2]. The group $U(m)$ acts on $Gr(p, m) \times F(A)$ by $g(V, F) = (gV, gF)$ and since we have $g\cdot(A; F) = (A; gF)$, we also have $g\cdot S(A) = S(A)$. Thus $U(m)$ acts on $S(A)$.

Let $\tilde{\pi}$ and $\tilde{\sigma}$ be the projection maps from $Gr(p, m) \times F(A)$ onto $Gr(p, m)$ and $F(A)$ respectively. These maps are equivariant with respect to the action of $U(m)$. Then $\tilde{\pi}$ and $\tilde{\sigma}$ induce holomorphic and surjective maps π and σ of $S(A)$ to $Gr(p, m)$ and $F(A)$.

If $F \in F(A)$, then $\sigma^{-1}(F) = (A; F) \times \{F\}$ and the fibres of σ is biholomorphic to the Schubert variety (A). $S(A)$ is a holomorphic P_A-bundle over $F(A)$ associated with the principal P_A-bundle $GL(m, \boldsymbol{C}) \to F(A)$ [2].

Let

$$f = \dim_C F_A .$$

Since $\dim_C (A) = |A|$, we have

$$\dim_C S(A) = f + |A| .$$

$S(A)$ is also a holomorphic P-bundle over $Gr(p, m)$ associated with the principal P-bundle $GL(m, \boldsymbol{C}) \to Gr(p, m)$ [2] and the operator π_* of fibre integration is defined. The operator π_* sends a form of type (u, v) on $S(A)$ to a form of type $(u-t, v-t)$ on $Gr(p, m)$, where

$$t = f + |A| - pq$$

and

$$p \cdot q = \dim_C Gr(p, m) .$$

The operator π_* commutes with the action of $U(m)$ on $S(A)$ and $Gr(p,m)$. Hence, if η is a form on $S(A)$ invariant by $U(m)$, so is $\pi_* \eta$.

Let λ be a form of type $(f-r, f-r)$ on $F(A)$ invariant by $U(m)$. The pullback $\sigma^*\lambda$ is a form of type $(f-r, f-r)$ on $S(A)$ and is also invariant by $U(m)$. Then $\pi_*(\sigma^*\lambda)$ is a form of type $(pq-|A|-r, pq-|A|-r)$ on $Gr(p, m)$ invariant by $U(m)$. However a form on $Gr(p,m)$ is invariant by $U(m)$ if and only if it is harmonic with respect to a Kaehler metric invariant by $U(m)$.

For a standard Schubert variety (B) we shall denote by $C(B)$ the Poincaré dual of (B). $C(B)$ is a form of type $(pq-|B|, pq-|B|)$ invariant by $U(m)$ such that

$$\int_{(B)} \psi = \int_{Gr(p,m)} C(B) \wedge \psi$$

for any closed form ψ of type $(|B|, |B|)$.

The Poincaré duals $C(B)$ with the condition $|B| = u$ form a basis of the vector space of all invariant forms of degree $2u$. Since $\pi_*(\sigma^*\lambda)$ is an invariant form of type $(pq-|A|-r, pq-|A|-r)$, we have

$$\pi_*(\sigma^*\lambda) = \sum_{|B|=|A|+r} d_B\, C(B) . \tag{1.11}$$

The de Rham dual (or simply the dual) of a standard Schubert variety (B) is the invariant form ξ_B of type $(|B|, |B|)$ such that

$$\int_{(D)} \xi_B = \delta_{B,D}$$

for all D satisfying $|D| = |B|$. Then we have

$$\int_{Gr(p,m)} C(D) \wedge \xi_B = \delta_{B,D}$$

for all D such that $|D| = |B|$. Then we get

$$d_B = \int_{Gr(p,m)} \pi_*(\sigma^*\lambda) \wedge \xi_B, \ |B| = |A| + r .$$

However from a well-known property of fibre integration, the right hand side is equal to the integral

$$\int_{S(A)} \sigma^*\lambda \wedge \pi^*\xi_B$$

and we obtain

$$d_B = \int_{S(A)} \sigma^*\lambda \wedge \pi^*\xi_B,\ |B| = |A|+r \tag{1.12}$$

We are going to transform the integral to an integral over $U(m)\times(A)$.

We now define a map

$$\mu\colon U(m)\times(A) \to S(A)$$

by

$$\mu(g, V) = (gV, gF^0),\ g\in U(m),\ V\in(A)\,.$$

As $V\in(A)$, $gV\in g(A)=(A; gF^0)$ and hence $gV\in(A; gF^0)$ and $\mu(g, V)\in S(A)$. The map μ is surjective. For, let $(W, F)\in S(A)$. Then there exists $g\in U(m)$ such that $gF^0=F$ and as $W\in(A; F)$, $V=g^{-1}W\in g^{-1}(A; F)=(A; F^0)=(A)$ and we get $\mu(g, V)=(W, F)$. Moreover we see also that for any $(g, V)\in U(m)\times(A)$, $\mu^{-1}(\mu(g, V))=\{(gh, h^{-1}V)\,|\,h\in H_A\}$.

The group H_A acts on $U(m)\times(A)$ from the right by

$$(g, V)\cdot h = (gh, h^{-1}V),\ h\in H_A\,,$$

and the action of H_A on $U(m)\times(A)$ is free.

We show that $U(m)\times(A)$ is a principal fibre bundle over $S(A)$ with structure group H_A. To see this we first choose an open covering $\{\mathcal{U}_\alpha\}$ of $F(A)$ such that over each open set $\mathcal{U}_\alpha$ a section $s_\alpha\colon \mathcal{U}_\alpha\to U(m)$ of the fibre bundle $\pi_F\colon U(m)\to F(A)$ exists. Let

$$\mathcal{V}_\alpha = S(A)\cap(Gr(p, m)\times\mathcal{U}_\alpha)\,.$$

Then $\{\mathcal{V}_\alpha\}$ is an open covering of $S(A)$. We define $\tau_\alpha\colon \mathcal{V}_\alpha\to U(m)\times(A)$ as follows. Let $q=(V, F)\in\mathcal{V}_\alpha$. Then $\sigma(q)\in\mathcal{U}_\alpha$ and $s_\alpha(\sigma(q))\in U(m)$ is defined. Let

$$\tau_\alpha(q) = (s_\alpha(\sigma(q)),\ s_\alpha(\sigma(q))^{-1}\pi(q))\,.$$

Since $\pi_F(g)=gF^0$ for $g\in U(m)$ and $\pi_F(s_\alpha(\sigma(q))=\sigma(q)=F$, we have $F=s_\alpha(\sigma(q))F^0$. We have also $s_\alpha(\sigma(q))^{-1}\pi(q)=s_\alpha(\sigma(q))^{-1}V$ and $V\in(A; F)$ and hence $s_\alpha(\sigma(q))^{-1}\pi(q)\in(A; F^0)=(A)$. This shows that $\tau_\alpha(q)\in U(m)\times(A)$ and also that

$$\mu(\tau_\alpha(q)) = (\pi(q), \sigma(q)) = q\,.$$

There is a map $g_{\beta\alpha}\colon \mathcal{U}_\alpha\cap\mathcal{U}_\beta\to H_A$ such that $s_\alpha(x)=s_\beta(x)\cdot g_{\beta\alpha}(x)$ for $x\in\mathcal{U}_\alpha\cap\mathcal{U}_\beta$. Then

$$g_{\beta\alpha}\circ\sigma\colon \mathcal{V}_\alpha\cap\mathcal{V}_\beta \to H_A$$

is defined and we have

$$\tau_\alpha(q) = \tau_\beta(q)g_{\beta\alpha}(\sigma(q)) \qquad \text{for}\ \ q\in\mathcal{V}_\alpha\cap\mathcal{V}_\beta\,.$$

We define

$$\phi_\alpha\colon H_A \times \mathcal{V}_\alpha \to \mu^{-1}(\mathcal{V}_\alpha) \tag{1.14}$$

by

$$\phi_\alpha(h, q) = \tau_\alpha(q)\cdot h\,.$$

It is easy to see that $U(m)\times(A)$ is a principal fibre bundle over $S(A)$.

We have to notice here that the maps μ, τ_α, $g_{\beta\alpha}\circ\sigma$, ϕ_α defined above are not only continuous but also differentiable at every simple points of the domains of these maps. Hence the pullbacks of differential forms by these maps are defined.

To simplify the notation we put

$$\eta = \sigma^*\lambda \wedge \pi^*\xi_B\,.$$

We prove now the following lemma.

Lemma 1.1. *Let θ_A be a left invariant form on $U(m)$ such that its restriction to the subgroup H_A is a left invariant volume element on H_A with*

$$\int_{H_A} \theta_A = 1\,.$$

Let $\omega_A = s_1^*\theta_A$ be the pullback of θ_A to $U(m)\times(A)$ by $s_1\colon U(m)\times(A)\to U(m)$. Then we have

$$\int_{S(A)} \eta = \int_{U(m)\times(A)} \omega_A \wedge \mu^*\eta\,. \tag{1.15}$$

Proof. It is easy to see that there exists a left invariant form θ_A satisfying our condition. We denote by θ_A' the restriction of θ_A to H_A. Since H_A is a compact connected Lie group, every left invariant volume element is also right invariant (see [4]). The fibre E of $U(m)\times(A)$ over a point $q\in S(A)$ is the orbit of a point $u\in E$ by the right action of H_A. Let $i_E\colon E\to U(m)\times(A)$ be the inclusion map and $i_u\colon H_A\to E$ a map defined by $i_u(h)=uh$. Then $(s_1\circ i_E\circ i_u)(h)=s_1(uh)=s_1(u)\cdot h=L_{s_1(u)}h$, where L_g ($g\in U(m)$) denotes the left translation of $U(m)$ by g. Pulling back θ_A by the map $s_1\circ i_E\circ i_u\colon H_A\to U(m)$ we get $i_u^*(i_E^*\omega_A)$. On the other hand pulling back θ_A by the map $H_A\to U(m)$ defined by $L_{s_1(u)}\circ i_H$, i_H being the inclusion map of H_A into $U(m)$, we get θ_A', because θ_A is left invariant. However these two maps $H_A\to U(m)$ are equal and we get

$$i_u^*(i_E^*\omega_A) = \theta_A'\,. \tag{1.16}$$

We now choose an open covering $\{\mathcal{V}_\alpha\}$ of $S(A)$ as we did before. To prove (1.15) it is enough to show

$$\int_{\mathcal{V}_\alpha} \eta = \int_{\mu^{-1}(\mathcal{V}_\alpha)} \omega_A \wedge \mu^* \eta \tag{1.17}$$

for each $\mathcal{V}_\alpha$. From now on we fix $\mathcal{V}_\alpha$ and drop the index α. We shall prove later that

$$\phi^*(\omega_A \wedge \mu^* \eta) = p_H^* \theta_A \wedge p_{\mathcal{V}}^* \eta\,, \tag{1.18}$$

where $\phi = \phi_\alpha$ is defined by (1.14) and p_H and $p_{\mathcal{V}}$ are the projections of $H_A \times \mathcal{V}$ onto H_A and $\mathcal{V}$. We have the commutative diagram

$$\begin{tikzcd} H_A \times \mathcal{V} \arrow[rr, "\phi"] \arrow[dr, "p_{\mathcal{V}}"'] & & \mu^{-1}(\mathcal{V}) \arrow[dl, "\mu"] \\ & \mathcal{V} & \end{tikzcd} \tag{1.19}$$

The integral on the right hand side of (1.17) is equal to

$$\int_{H_A \times \mathcal{V}} p_H^* \theta_A \wedge p_{\mathcal{V}}^* \eta$$

by (1.18). Then by Fubini theorem this integral is equal to

$$\int_{\mathcal{V}} \{\int_{H_A} \theta_A\} \eta = \int_{\mathcal{V}} \eta$$

and this proves (1.17). Hence to prove the lemma it remains to prove (1.18). To prove (1.18) we show that the restrictions of $\phi^* \omega_A$ and $p_H^* \theta_H$ on $H_A \times \{q\}$ on each $q \in \mathcal{V}$ are equal. For, let $u = \tau(q)$ and let E be the fibre $\mu^{-1}(q)$ over q. Then $u \in E$ and the following diagram is comutative:

$$\begin{tikzcd} H_A \times \{q\} \arrow[rr, "\phi_q"] \arrow[dr, "p_{H,q}"'] & & E \arrow[dl, "i_u"] \\ & H_A & \end{tikzcd}$$

where ϕ_q and $p_{H,q}$ are restrictions of ϕ and p_H on $H_A \times \{q\}$. The restriction of $\phi^* \omega_A$ to $H_A \times \{q\}$ is equal to $\phi_u^*(i_E^* \omega_A)$ and this is equal to $p_{H,q}^*(i_u^*(i_E^* \omega_A))$. However by (1.16), $i_u^*(i_E^* \omega_A)$ is equal to θ_A', the restriction of θ_A on H_A. Now $p_{H,q}^* \theta_A'$ is the restriction of $p_H^* \theta_A'$ to $H_A \times \{q\}$ and this proves our assertion.

Let $q \in \mathcal{V}$ and $h \in H_A$ and let $\{x_1, \cdots, x_k\}$ and $\{y_1, \cdots, y_l\}$ be local coordinates around q and h respectively. We express $p^* \eta$, $p_H^* \theta_A$ and $\phi^* \omega_A$ locally in the form $p^* \eta = a(x) dx_1 \wedge \cdots \wedge dx_k$, $p_H^* \theta_A' = b(y) dy_1 \wedge \cdots \wedge dy_l$ and $\phi^* \omega_A = c(x, y)$ $dy_1 \wedge \cdots \wedge dy_l + \omega'$, where ω' denotes the sum of the terms involving dx. Since

$p_H^*\theta_A=\phi^*\omega_A$ on $H\times\{q'\}$ for every $q\in\mathcal{CV}$ we have $c(x,y)=b(y)$ and $\phi^*\omega_A=p_H^*\theta_A+\omega'$. However $\omega'\wedge p^*\eta=0$ and hence $\phi^*\omega_A\wedge p_{\mathcal{CV}}^*\eta=p_H^*\theta_A\wedge p_{\mathcal{CV}}^*\eta$. However $p_{\mathcal{CV}}=\mu\circ\phi$ by (1.19) and we have $p^*\eta=\phi^*(\mu^*\eta)$ and $\phi^*\omega_A\wedge p_{\mathcal{CV}}^*\eta=\phi^*(\omega_A\wedge\mu^*\eta)$ and (1.18) is proved. From Lemma 1.1 and (1.12) we get

$$d_B=\int_{U(m)\times(A)}\omega_A\wedge\mu^*\sigma^*\lambda\wedge\mu^*\pi^*\xi_B\,.$$

However $\sigma\circ\mu=\pi_F\circ s_1$ and hence $\omega_A\wedge\mu^*\sigma^*\lambda=s_1^*(\theta_A\wedge\pi_F^*\lambda)$. Let $\beta=\pi\circ\mu$. Then $\mu^*\pi^*\xi_B=\beta^*\xi_B$ and β: $U(m)\times(A)\to Gr(p,m)$ is defined by

$$\beta(g,V)=gV,\ V\in(A),\ g\in U(m)\,. \tag{1.20}$$

Since the Schubert cell $(A)^*$ is a Zariski open set in (A), we can replace an integral over $U(m)\times(A)$ by an integral over $U(m)\times(A)^*$.

Summing up we get

Lemma 1.2. *Let λ be a $U(m)$-invariant form of type $(f-r,f-r)$ on $F(A)$ and let*

$$\pi_*(\theta^*\lambda)=\sum_{|B|=|A|+r}d_B\cdot C(B)\,.$$

Then the coefficient d_B is given by the integral

$$d_B=\int_{U(m)\times(A)^*}s_1(\theta_A\wedge\pi_F^*\lambda)\wedge\beta^*\xi_B\,, \tag{1.21}$$

where s_1: $U(m)\times(A)^*\to U(m)$ is the projection, θ_A is a form on $U(m)$ difined in Lemma 1.1, *the map β: $U(m)\times(A)^*\to Gr(p,m)$ is defined by* (1.20) *and ξ_B is the de Rham dual of the Schubert variety* (B).

In the next section we transform the integral (1.21) into an integral over $U(m)\times N(A)$, where $N(A)$ is a complex simply connected abelian Lie subgroup of $GL(m,\boldsymbol{C})$.

2. The abelian complex Lie group acting simply transitively on a Schubert cell.

We prove first the following elementary lemma.

Lemma 2.1. *A p-dimensional linear subspace V of $\boldsymbol{C}^m$ belongs to the Schubert cell $(A)^*$ if and only if V has a basis of the form $\{e_{k_1}+v_1, e_{k_2}+v_2,\cdots,e_{k_p}+v_p\}$, where $v_i\in V^0{}_{k_i-1}$ for $i=1,2,\cdots,p$; here $A=\{a_1,\cdots,a_p\}$, $k_i=a_i+i$, $k_0=a_0=0$ and $V^0{}_k$ is the subspace spanned by $\{e_1,e_2,\cdots,e_k\}$ for $k\geqq 1$ and $V^0{}_0=\{0\}$.*

Proof. Suppose that V has a basis of the from $\{e_{k_1}+v_1,\cdots,e_{k_p}+v_p\}$. Then $e_{k_j}+v_j\in V^0{}_{k_i}$ for $j\leqq i$ and hence $\dim V\cap V^0{}_{k_i}\geqq i$. If $\dim V\cap V^0{}_{k_i}>i$, $V\cap V^0{}_{k_i}$ would contain a non-zero vector v which is a linear combination of

$e_{k_l}+v_l$ with $l>i$. On the other hand since $v\in V^0{}_{k_i}$, v is a linear combination of $e_1, \cdots, e_{k_i}$. So we have $v=a_1e_1+\cdots+a_{k_i}e_{k_i}=b_{i+1}(e_{k_{i+1}}+v_{i+1})+\cdots+b_p(e_{k_p}+v_p)$. It follows from this that $b_p=0$ and inductively $b_{p-1}=0, \cdots, b_{i+1}=0$ and hence $v=0$, a contradiction. Hence we must have $\dim V\cap V^0{}_{k_i}=i$ for all $i=1,2,\cdots,p$. Suppose now that $k_i-k_{i-1}>1$. Then $k_i-1>k_{i-1}$ and $e_{k_j}+v_j$ with $j<i$ belongs to $V^0{}_{k_i-1}$ and $v_i\in V^0{}_{k_i-1}$. Suppose that $\dim V\cap V^0{}_{k_i-1}\geqq i$. Since $V^0{}_{k_i-1}\subset V^0{}_k$ and $\dim V\cap V^0{}_{k_i}=i$, we would have $\dim V\cap V^0{}_{k_i-1}=i$ and so $V\cap V^0{}_{k_i-1}=V\cap V^0{}_{k_i}$. Then $e_{k_i}+v_i\in V\cap V^0{}_{k-1}$ and hence $e_{k_i}=(e_{k_i}+v_i)-v_i$ belongs to $V^0{}_{k_i-1}$ and this is a contradiction. Hence we have $\dim V\cap V^0{}_{k_i-1}<i$ and this shows that $V\in(A)^*$.

Suppose now that $V\in(A)^*$. Then we have $\dim V\cap V^0{}_{k_1}\geqq 1$ and if $k_1-k_0=k_1>1$, we have also $\dim V\cap V^0{}_{k_1-1}=\{0\}$. If v is a non-zero vector in $V\cap V^0{}_{k_1}$ we can write $v=ae_{k_1}+w$, $w\in V^0{}_{k_1-1}$ and $a\neq 0$, for if $a=0$, then $v=w\in V\cap V^0{}_{k_1-1}=\{0\}$ and so $v=0$. From this we see that $V\cap V^0{}_{k_1}$ is one-dimensional and has a basis u_1 of the form $u_1=e_{k_1}+v_1$, $v_1\in V^0{}_{k_1-1}$. Suppose that we have already shown that $\dim V\cap V^0{}_{k_j}=j$ and $V\cap V^0{}_{k_j}$ has a basis $\{u_1, \cdots, u_j\}$, where u_s is of the form $u_s=e_{k_s}+v_s$, $v_s\in V^0{}_{k_s-1}$, for $j=1, 2, \cdots, i-1$. Consider now $V\cap V_{k_i}$. We have $\dim V\cap V_{k_i}\geqq i$ and if $k_i-k_{i-1}>1$, we have also $\dim V\cap V_{k_i-1}<i$. Then we have $\dim V\cap V_{k_i}>\dim V\cap V_{k_i-1}$, for, this is trivial in the case $k_i-k_{i-1}>1$ and when $k_i-k_{i-1}=1$, we have $V^0{}_{k_i-1}=V^0{}_{k_{i-1}}$ and $\dim V\cap V^0{}_{k_{i-1}}=i-1$ by our assumption of induction and so we have $\dim V\cap V^0{}_{k_i}>\dim V\cap V^0{}_{k_i-1}$ also in this case. Then there is a vector $v\in V\cap V^0{}_{k_i}$, $v\notin V^0{}_{k_i-1}$ and we can write $v=ae_{k_i}+w$, $w\in V^0{}_{k_i-1}$, $a\neq 0$. Let $u_i=a^{-1}v$. Then $u_i=e_{k_i}+v_i$. $v_i\in V^0{}_{k_i-1}$ and $V\cap V^0{}_{k_i}$ is spanned by u_i and $V\cap V^0{}_{k_i-1}$. As we have $k_i-1\geqq k_{i-1}$, $V^0{}_{k_i-1}\supset V^0{}_{k_{i-1}}$ and so $V\cap V^0{}_{k_i-1}\supset V\cap V^0{}_{k_{i-1}}$ and $\dim V\cap V^0{}_{k_i-1}\geqq \dim V\cap V^0{}_{k_{i-1}}=i-1$ and the equality holds when $k_i-1=k_{i-1}$. However $V\in(A)^*$ and we have $\dim V\cap V^0{}_{k_i-1}<i$ when $k_i-1>k_{i-1}$. Hence we have always $\dim V\cap V^0{}_{k_i-1}=i-1$ and so we get $\dim V\cap V^0{}_{k_i}=i$ and $V\cap V^0{}_{k_i-1}=V\cap V^0{}_{k_{i-1}}$. Then $V\cap V^0{}_{k_i}$ is spanned by $\{u_1, \cdots, u_{i-1}, u_i\}$. Proceeding in this way we see that $\dim V\cap V_{k_p}=p$ and so $V=V\cap V^0{}_{k_p}$ and V has a basis $\{u_1, \cdots, u_p\}$, where each u_i is of the form $u_i=e_{k_i}+v_i$.

Subtracting a suitable linear combination of $e_{k_j}+v_j$ with $j<i$ from $e_{k_i}+v_i$ we may assume that v_i is a linear combination of e_s with the condition $s<k_i$, $s\neq k_j$ for every $j=1, 2, \cdots, i-1$.

Let $M(A)$ be the set of all $m\times m$ complex matrices $u=(u_{st})$ $(1\leqq s,\ t\leqq m)$ satisfying the following conditions.

1) u is upper triangular, and unipotent, that is, $u_{st}=0$ for $s>t$ and $u_{tt}=1$;

2) if $t\neq k_i$ for $i=1, 2, \cdots, p$, then the entries of the t-th column vectors of u is zero except $u_{tt}=1$, that is, $u_{st}=\delta_{st}$.

3) $u_{k_jk_i}=0$ for $j<i$.

It is verified easily that $M(A)$ is a closed connected, simply connected abelian subgroup of $GL(m,\boldsymbol{C})$. The Lie algebra $\mathfrak{m}(A)$ of $M(A)$ is the abelian subalgebra of $\mathfrak{gl}(m,\boldsymbol{C})$ spanned by $e_{sk_i}(i=1, 2, \cdots, p)$, where s satisfies the condition $s<k_i$ and $s\neq k_j$ for $j=1, 2, \cdots, i-1$. Here e_{sk} denotes the matrix whose (s, k)-entry is 1 and others are 0. We see then that $\mathfrak{m}(A)$ is a complex abelian Lie algebra of dimension $|A|=\Sigma a_i$ and so $M(A)$ is a simply connected complex abelian Lie group of complex dimension $|A|$ and $M(A)$ is isomorphic to $\boldsymbol{C}^{|A|}$ as complex Lie group.

Lemma 2.2. *The group $M(A)$ acts holomorphically and simply transitively on the Schubert cell $(A)^*$.*

Proof. Let W be the p-dimensional subspace of $\boldsymbol{C}^m$ spanned by $\{e_{k_1}, \cdots, e_{k_p}\}$. By Lemma 2.1, W belongs to $(A)^*$. Moreover if $u\in M(A)$, then $u\cdot W=\{ue_{k_1}, \cdots, ue_{k_p}\}$ and $u\cdot e_{k_i}$ is the $k_i th$ column vector of the matrix u. It follows from the definition of $M(A)$ that $u\cdot e_{k_i}$ is of the form $e_{k_i}+v_i$, where v_i is a linear combination of e_s with $s<k_i$ and $s\neq k_j$, $j<i$. By Lemma 2.1, uW belongs to $(A)^*$. Conversely let $V\in(A)^*$. Then V has a basis of the form $\{e_{k_1}+v_1, \cdots, e_{k_p}+v_p\}$, where v_i is a linear combination of e_s with $s<k_i$ and $s\neq k_j$, $j=1, \cdots, i-1$. Let u be the $m\times m$ matrix whose t-th column vector is $e_{k_i}+v_i$ for $t=k_i$ $(i=1,2,\cdots,p)$ and is e_t for $t\neq k_i$. Then $u\in M(A)$ and $V=uW$. Let $u'\in M(A)$. Then $u'V=(u'u)\cdot W\in(A)^*$ and so $u'(A)^*=(A)^*$. These show that the group $M(A)$ acts transitively on $(A)^*$ and it is clear that $M(A)\times(A)^*\rightarrow(A)^*$ defined by $(u, V)\rightarrow uV$ is holomorphic. To show that $M(A)$ acts simply transitively on $(A)^*$, it is enough to show that, if $uW=W$, then u is the unit matrix. This is easy to show and the lemma is proved.

Let now τ a permutation of $\{1, 2, \cdots, m\}$. We associate to τ an $m\times m$ matrix u_τ by the condition

$$u_\tau\cdot e_i = e_{\tau(i)},\quad i = 1, 2, \cdots, m\,.$$

Then u_τ is a unitary matrix and $\tau\rightarrow u_\tau$ is a representation of the permutation group.

We associate to each $A=\{a_1, \cdots, a_p\}$ a permutation σ_A of $\{1, 2, \cdots, m\}$ as follows. Let $\{l_1, \cdots, l_q\}=\{1, 2, \cdots, m\}-\{k_1, \cdots, k_p\}$ and let $l_1<l_2<\cdots<l_q$. We define σ_A by the condition

$$\begin{aligned}\sigma_A^{-1}(i) &= k_i,\quad 1\leqq i\leqq p\\ \sigma_A^{-1}(p+s) &= l_s,\quad 1\leqq s\leqq q\,.\end{aligned}\tag{2.1}$$

We then have

$$u_{\sigma_A}(e_{k_i}) = e_i$$

and hence

$$u_{\sigma_A} W = V_p^0 = \{e_1, \cdots, e_p\} .$$

We define a subgroup $N(A)$ of $GL(m, \boldsymbol{C})$ by

$$N(A) = u_{\sigma_A} M(A) u_{\sigma_A}^{-1} .$$

The Lie algebra $\mathfrak{n}(A)$ is then given also by

$$\mathfrak{n}(A) = u_{\sigma_A} \mathfrak{m}(A) u_{\sigma_A}^{-1} .$$

Since $N(A)$ and $M(A)$ are conjugate, $N(A)$ is also a complex, simply connected closed subgroup of $GL(m, \boldsymbol{C})$.

Lemma 2.3. *The group $N(A)$ consists of all the complex $m \times m$ matrices $n = (n_{a,b})$ of the form*

$$n = \begin{pmatrix} 1 & 0 \\ n' & 1 \end{pmatrix}$$

where $n' = (n_{p+s,i})$ $(1 \leqq s \leqq q,\ 1 \leqq i \leqq p)$ is a $q \times p$ matrix satisfying $n_{p+s,i} = 0$ for $s > a_i$ $(i = 1, \cdots, p)$.

The Lie algebra $\mathfrak{n}(A)$ is the complex abelian Lie subalgebra of $\mathfrak{gl}(m, \boldsymbol{C})$ spanned by $e_{p+s,i}$ $(1 \leqq s \leqq q,\ 1 \leqq i \leqq p)$ such that $s \leqq a_i$.

Proof. To simplify the notation put $\sigma = \sigma_A$. We have

$$u_\sigma e_{ab} u_\sigma^{-1} = e_{\sigma(a)\sigma(b)}$$

and hence we have

$$u_\sigma e_{l_s k_i} u_\sigma^{-1} = e_{p+s,i} .$$

The Lie algebra $\mathfrak{m}(A)$ is spanned by e_{ak_i} $(i = 1, \cdots, p)$ such that $a < k_i$ and $a \neq k_j$ for $j = 1, \cdots, p-1$. Since $\{1, 2, \cdots, m\} - \{k_1, \cdots, k_p\} = \{l_1, \cdots, l_q\}$, we have $a = l_s$ for some s and $l_s < k_i$. As $\mathfrak{n}(A) = u_\sigma \mathfrak{m}(A) u_\sigma^{-1}$, $\mathfrak{n}(A)$ is spanned by $e_{p+s,i}$ such that $\sigma^{-1}(p+s) < \sigma^{-1}(i)$. We show now that the condition $\sigma^{-1}(p+s) < \sigma^{-1}(i)$ is equivalent to the condition that $s \leqq a_i$.

We see that for any s and i such that $1 \leqq s \leqq q$, $1 \leqq i \leqq p$, we have either

$$\sigma^{-1}(i) < i + s \leqq \sigma^{-1}(p+s)$$

or

$$\sigma^{-1}(p+s) < i + s \leqq \sigma^{-1}(i) .$$

Suppose that $\sigma^{-1}(p+s) < \sigma^{-1}(i)$. Then we have the second case and so $s \leqq \sigma^{-1}(i) - i = k_i - i = a_i$. Suppose now that $s \leqq a_i$. Then $i + s \leqq k_i = \sigma^{-1}(i)$ and hence $\sigma^{-1}(p+s) < i + s \leqq \sigma^{-1}(i)$ and so $\sigma^{-1}(p+s) < \sigma^{-1}(i)$ and our assertion is proved. It follows from this that $\mathfrak{n}(A)$ is spanned by $e_{p+s,i}$ satisfying the condition

$s \leqq a_i$. Then it is easy to see that if $X \in \mathfrak{n}(A)$, then $X^2=0$. Now $N(A)$ is a simply connected abelian Lie group and every matrix n in $N(A)$ is written uniquely in the form $n=\exp X$ with $X \in \mathfrak{n}(A)$. However $X^2=1$ and so $\exp X=1+X$. Thus n is the form $n=1+X$, $X \in \mathfrak{n}(A)$, and hence n is of the form stated in Lemma 2.3.

The gruop $M(A)$ is simply transitive on the Schubert cell $(A)^*$. It follows that $N(A)$ is then simply transitive on the Schubert cell $u_{\sigma_A}(A)^*=(A;\tilde{F})^*$, where $\tilde{F}$ is the flag $\{u_{\sigma_A}V_1^0, \cdots, u_{\sigma_A}V_m^0\}$. As we have $W=\{e_{k_1}, \cdots, e_{k_p}\} \in (A)^*$ and $V_p^0=u_{\sigma_A}W$, we have $V_p^0 \in (A;\tilde{F})^*$. We have denoted earlier the point of $Gr(p, m)$ represented by V_p^0 by o. So we have $o \in (A;\tilde{F})^*$ and this is one of the reasons why we replace $M(A)$ by $N(A)$.

We have $(A)^* \approx (A;\tilde{F})^*$ and as $N(A)$ is simply transitive on $(A;\tilde{F})^*$, $N(A)$ and $(A;\tilde{F})^*$ are also biholomorphic. Hence we can identify $U(m)\times(A)^*$ and $U(m)\times N(A)$ by identifying $(g, n) \in U(m)\times N(A)$ to the point $(g, u_{\sigma_A}^{-1}\cdot n\cdot o)$ of $U(m)\times(A)^*$.

On the other hand we have defined the map β: $U(m)\times(A)^* \to Gr(p, m)$ by $\beta(g, V)=gV$. Then, identifying $U(m)\times N(A)$ and $U(m)=(A)^*$ as above, the map β is identified with the map

$$\beta\colon\ U(m)\times N(A) \to Gr(p, m)$$

such that

$$\beta(g, n) = gu_{\sigma_A}^{-1}no\,. \tag{2.2}$$

We now define a map

$$\tilde{\gamma}\colon\ U(m)\times N(A) \to Gr(p, m)$$

by

$$\tilde{\gamma}\cdot(g, n) = u_{\sigma_A}gu_{\sigma_A}^{-1}no,\ g \in U(m),\ n \in N(A)\,. \tag{2.3}$$

Then $\tilde{\gamma}=t_u\circ\beta$, where t_u is the transformation of $Gr(p, m)$ induced by the action of $u=u_{\sigma_A} \in U(m)$.

The integral (1.21) is transformed to the integral over $U(m)\times N(A)$ and the integrand is $s_1^*(\theta_A \wedge \pi_F^*\lambda)\wedge\beta^*\xi_B$ and β is defined now by (2.2). We have $\tilde{\gamma}^*\xi_B=\beta^*t_u^*\xi_B$ and $t_u^*\xi_B=\xi_B$ because ξ_B is invariant by $U(m)$. Hence $\beta^*\xi_B=\tilde{\gamma}^*\xi_B$. We now define a map

$$\gamma\colon\ U(m)\times N(A) \to GL(m, \boldsymbol{C})$$

by

$$\gamma(g, n) = u_{\sigma_A}gu_{\sigma_A}^{-1}n,\ g \in U(m),\ n \in N(A) \tag{2.4}$$

and let π_G be the projection of $GL(m, \boldsymbol{C})$ onto $Gr(p, m)$ defined by (1.8). Then we have $\tilde{\gamma}=\pi_G\circ\gamma$ and $\tilde{\gamma}^*\xi_B=\gamma^*\pi_G^*\xi_B$. Then from Lemma 1.2 we obtain

Lemma 2.4. *We have*

$$d_B = \int_{U(m)\times N(A)} s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*\pi_G^*\xi_B\,, \tag{2.5}$$

where γ: $U(m)\times N(A)\to Gr(p, m)$ *is defined by* (2.4) *and* π_G: $GL(m, \boldsymbol{C})\to Gr(p, m)$ *is defined by* (1.8).

In the next section we shall get information about ξ_B and $\pi_G^*\xi_B$.

3. Invariant differential forms on the Grassmann manifold and the de Rham dual of a Schubert variety. We write the Grassmannian $Gr(p, m)$ in the form

$$Gr(p, m) = GL(m, \boldsymbol{C})/P$$

where P is the subgroup of $GL(m, \boldsymbol{C})$ consisting of the matrices of the form

$$\begin{pmatrix} a & c \\ 0 & b \end{pmatrix}, \; a\in GL(p, \boldsymbol{C}),\; b\in GL(q, \boldsymbol{C})\,.$$

The Lie algebra $\mathfrak{p}$ of P is the subalgebra of $\mathfrak{gl}(m, \boldsymbol{C})$ consisting of the matrices of the form

$$\begin{pmatrix} A & C \\ 0 & B \end{pmatrix}, \; A\in\mathfrak{gl}(p, \boldsymbol{C}),\; B\in\mathfrak{gl}(q, \boldsymbol{C})\,.$$

Let $\mathfrak{n}^+$ denote the abelian subalgebra of $\mathfrak{gl}(m, \boldsymbol{C})$ consisting of all matrices of the form

$$X = \begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix},$$

where D is a complex $q\times p$ matrix. Then we have

$$\mathfrak{gl}(m, \boldsymbol{C}) = \mathfrak{n}^+\oplus\mathfrak{p}\,.$$

Let o denote the origin of $Gr(m, p)$. The point o is the point represented by the p-dimensional subspace $V^0=\{e_1, \cdots, e_p\}$ of $\boldsymbol{C}^m$ and hence it is the coset P in our coset space representation of $Gr(p, m)$. Every tangent vector at o is the tangent vector at $t=0$ to an orbit: $t\to(\exp t\,X)\,(o)$ of the origin o by a 1-parameter subgroup $\exp t\,X$ $(X\in\mathfrak{gl}(m, \boldsymbol{C}))$ of $GL(m, \boldsymbol{C})$ and the tangent vector to the orbit at o is the zero vector if and only if $X\in P$. Hence by mapping $X\in\mathfrak{gl}(m, \boldsymbol{C})$ to the tangent vector to the orbit $(\mathrm{ext}\; t\,X)\,(o)$ at $t=0$ we get a real linear map from $\mathfrak{gl}(m, \boldsymbol{C})$ onto the (real) tangent space $T_0(Gr(p, m))$ whose kernel is $\mathfrak{p}$. Thus we can identify $T_0(Gr(p, m))$ with $\mathfrak{gl}(m, \boldsymbol{C})/\mathfrak{p}$ as *real vector space.*

On the other hand, $T_0(Gr(p, m))$ has the complex structure J_0 which comes from the complex structure of $Gr(p, m)$ and $\mathfrak{gl}(m, \boldsymbol{C})/\mathfrak{p}$ is also a complex vector space and we see easily that the above identification of $T_0(Gr(p,m))$ and $\mathfrak{gl}(m,\boldsymbol{C})/\mathfrak{p}$ is compatible with these complex structures, that is, the identification map is complex linear isomorphism of these two vector spaces regarded as complex vector spaces.

On the other hand we have $\mathfrak{gl}(m, \boldsymbol{C})=\mathfrak{n}^+\oplus\mathfrak{p}$ and we can identify canonically $\mathfrak{gl}(m, \boldsymbol{C})/\mathfrak{p}$ with $\mathfrak{n}^+$. Hence we identify $\mathfrak{n}^+$ with $T_0(Gr(p, m))$.

From now on the action of an element $g\in GL(m, \boldsymbol{C})$ on $Gr(p, m)$ will be denoted by t_g. Let $h\in P$. Then $t_h(o)=o$ and hence the differential t_h^T of t_h at o defines a non-singular linear transformation $\rho(h)$ of $T_0(Gr(p, m))$ and $h\rightarrow\rho(h)$ is a representation of the group P wihch we call the *isotropic representation* of P at o. Let us denote by φ_s the orbit $t_{\exp sX}(o)$, where $X\in\mathfrak{n}^+$ and denote by φ_0' the tangent vector of the orbit φ_s at $s=0$. Then $\rho(h)(\varphi_0')$ is the tangent vector to the curve $\psi_s=t_h(\varphi_s)$ at $s=0$. Since $h^{-1}\in P$, we have $t_{h^{-1}}(o)=o$ and hence $\psi_s=t_h(\varphi_s)=t_h(t_{\exp sX}(o))=t_h\cdot t_{\exp sX}\cdot t_{h^{-1}}(o)=t_{\exp s hXh^{-1}}(o)$. Hence ψ_0' is identified with the image of $hXh^{-1}\in\mathfrak{gl}(m,\boldsymbol{C})$ in $\mathfrak{n}^+$ by the projection $\mathfrak{gl}(m,\boldsymbol{C})\rightarrow\mathfrak{n}^+$. Let now

$$h=\begin{pmatrix} h_1 & h_3 \\ 0 & h_2 \end{pmatrix}\in P,\ X=\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix}\in\mathfrak{n}^+ .$$

Then hXh^{-1} is of the form

$$hXh^{-1}=\begin{pmatrix} * & * \\ h_2Dh_1^{-1} & * \end{pmatrix}$$

and hence the image of hXh^{-1} in $\mathfrak{n}^+$ is of the form

$$X'=\begin{pmatrix} 0 & 0 \\ h_2Dh_1^{-1} & 0 \end{pmatrix}.$$

Thus, identifying $\mathfrak{n}^+$ with $T_0(Gr(p, m))$, the isotropic representation ρ of P is given by

$$\rho(h)X=X', \qquad \text{where}$$

$$h=\begin{pmatrix} h_1 & * \\ 0 & h_2 \end{pmatrix},\ X=\begin{pmatrix} 0 & 0 \\ D & 0 \end{pmatrix},\ X'=\begin{pmatrix} 0 & 0 \\ h_2Dh_1^{-1} & 0 \end{pmatrix}. \tag{3.1}$$

We notice here that, as we see from (3.1), the isotropic representation ρ is essentially a representation of $GL(p, \boldsymbol{C})\times GL(q, \boldsymbol{C})$ which is a subgroup of P. When we regard $\mathfrak{n}^+$ as a vector space over $\boldsymbol{R}$, we denote this vector space by $\mathfrak{n}_R^+$. We regard also $T_0(Gr(p, m))$ as a vector space over $\boldsymbol{R}$ with complex structure J_0 defined by the complex structure of $Gr(p, m)$.

Consider the set F of all $\boldsymbol{R}$-linear maps of $\mathfrak{n}_R^+$ into $\boldsymbol{C}$. We consider F as a vector space over $\boldsymbol{C}$ and we can identify F with $(\mathfrak{n}_R^+)^* \otimes_R \boldsymbol{C}$. If $f \in F$, we denote by $\bar{f}$ the map $X \to \overline{f(X)}$. The dual space $(\mathfrak{n}^+)^*$ of the complex vector space $\mathfrak{n}^+$ is the subspace of F consisting of all $f \in F$ such that $f(iX) = if(X)$ for all $X \in \mathfrak{n}_R^+$. We put $(\mathfrak{n}^+)^* = F^+$. We denote by F^- the complex subspace of F consisting of all $g \in F$ such that $g(iX) = -ig(X)$ for all $X \in \mathfrak{n}_R^+$. Then $F^- = \bar{F}^+ = \{\bar{f} \mid \in F^+\}$ and we have

$$F = F^+ \oplus F^- .$$

Notice that, identifying $\mathfrak{n}_R^+$ with $T_0(Gr(p, m))$, F is identified with $T_0^*(Gr(p, m)) \otimes_R \boldsymbol{C}$, the vector space of all complex 1-froms of $Gr(p, m)$ at o, and F^+(resp. F^-) corresponds to the vector space of 1-forms of type $(1, 0)$ (resp. type $(0,1)$). Analogously $\overset{r}{\Lambda} F$ is identified with the vector space of r-forms at o. If $h \in P$ and $\zeta \in \overset{r}{\Lambda} F$, we define $\rho^*(h)\zeta \in \overset{r}{\Lambda} F$ by

$$(3.2) \qquad (\rho^*(h)\zeta)(X_1, \cdots, X_r) = \zeta(\rho(h^{-1})X_1, \cdots, \rho(h^{-1})X_r)$$

where $X_i \in \mathfrak{n}_R^+$ and we regard ζ as an alternating r-linear form on $\mathfrak{n}_R^+$. We call ρ^* the *isotropic representation* of P on ΛF.

Let ω be an r-form on $Gr(p, m)$ which is invariant by the action of $U(m)$ on $Gr(p, m)$, i.e. $t_g^*\omega = \omega$ for all $g \in U(m)$. In particular $t_h^*\omega = \omega$ for $h \in U(p) \times U(q) = P \cap U(m)$. This implies that for any tangent vector $u_1, \cdots, u_r$ at o, we have $\omega_0(\rho(h)u_1, \cdots, \rho(h)u_r) = \omega_0(u_1, \cdots, u_r)$ for all $h \in U(p) \times U(q)$ and this is equivalent to the condition that

$$(3.3) \qquad \rho^*(h)\omega_0 = \omega_0, \; h \in U(p) \times U(q) .$$

Conversely let ω_0 be an r-form at o satisfying (3.3) and let $x \in Gr(p, m)$. There exists then $g \in U(m)$ such that $t_g(x) = o$. Then $t_g^*\omega_0 = \omega_x$ is an r-form at x and the condition (3.3) guarantees that ω_x is independent of the choice of g such that $t_g(x) = o$. Then we can define an r-from ω on $Gr(p, m)$ by $x \to \omega_x$ and ω is obviously $U(m)$-invariant. This establishes an isomorphism between the vector space of $U(m)$-invariant r-forms on $Gr(p, m)$ and the vector space of all elements $\zeta \in \overset{r}{\Lambda} F$ satisfying $\rho^*(h)\zeta = \zeta$ for all $h \in U(p) \times U(q)$.

We call an element ζ of ΛF an *invariant element* if $\rho^*(h)\zeta = \zeta$ for all $h \in U(p) \times U(q)$.

Thus a $U(m)$-invariant form on $Gr(p, m)$ is identified with an invariant element of ΛF.

Now let

$$h_t = \begin{pmatrix} e^{it}I_p & 0 \\ 0 & e^{-it}I_q \end{pmatrix}, \ t \in \boldsymbol{R}.$$

Then we have

$$\rho(h_t^{-1})X = e^{2it}X$$

for all $X \in \mathfrak{n}_R^+$. Let $\zeta \in F^+ = (\mathfrak{n}^+)^*$. Then ζ is a complex linear function on $\mathfrak{n}^+$ and hence

$$(\rho(h_t)\zeta)(X) = \zeta(e^{2it}X) = e^{2it}\zeta(X)$$

and hence

$$\rho^*(h_t)\zeta = e^{2it}\zeta, \ \zeta \in F^+ .$$

Analogously we get

$$\rho^*(h_t)\bar{\zeta} = e^{-2it}\bar{\zeta}, \ \zeta \in F^+$$

Let $\{\zeta_1, \cdots, \zeta_N\}$ $(N = pq = \dim_C n^+)$ be a basis of F^+. Then $\{\bar{\zeta}_1, \cdots, \bar{\zeta}_N\}$ is a basis of $F^- = \bar{F}^+$ and since $F = F^+ \oplus F^-$, $\{\zeta_1, \cdots, \zeta_N, \bar{\zeta}_1, \cdots, \bar{\zeta}_N\}$ is a basis of F and every element $\zeta \in \overset{r}{\Lambda} F$ is written uniquely in the form

$$\zeta = \sum_{u+v=r} \zeta_{u,v} ,$$

$$\zeta_{u,v} = \sum_{I,J} a_{I,J} \zeta_I \wedge \bar{\zeta}_J$$

where $I = \{i_1, \cdots, i_u\}$, $i < \cdots < i_u$, $J = \{j_1, \cdots, j_v\}$, $j_1 < \cdots < j_v$ and $\zeta_I = \zeta_{i_1} \wedge \cdots \wedge \zeta_{i_u}$, $\bar{\zeta}_J = \bar{\zeta}_{j_1} \wedge \cdots \wedge \bar{\zeta}_{j_v}$. Now $\rho^*(h)\zeta = \sum_{u,v} \sum_{I,J} a_{I,J} \rho^*(h)\zeta_I \wedge \rho^*(h)\bar{\zeta}_J$ and $\rho^*(h)\zeta_I = \rho^*(h)\zeta_{i_1} \wedge \cdots \wedge \rho^*(h)\zeta_{i_u}$, $\rho^*(h)\bar{\zeta}_J = \rho^*(h)\bar{\zeta}_{j_1} \wedge \cdots \wedge \rho^*(h)\bar{\zeta}_{j_r}$. Hence we have $\rho^*(h_t)\zeta = \sum_{u+v=r} e^{i(u-v)t}\zeta_{u,v}$. If ζ is an invariant element, then we have $e^{i(u-v)t} = 1$ for all t and for, u, v such that $\zeta_{u,v} \neq 0$. Hence we have $\zeta_{u,v} = 0$ for $u \neq v$. Hence, if ζ is an invariant, then r is even and

$$\zeta = \zeta_{u,u}, \ 2u = r .$$

Thus we have proved that, if $\zeta \in \overset{r}{\Lambda} F$ is invariant, then $r = 2u$ and ζ is of type (u, u). It follows in particular that if ω is an invariant r-form on $Gr(p, m)$, then ω is of type (u, u) with $r = 2u$.

Let us denote by $F_{u,v}$ the subspace of ΛF consisting all elements of type (u,v) and by I the subspace of all invariant elements of ΛF. Then we have

$$I = \sum_u I_{u,u}, \ I_{u,u} = F_{u,u} \cap I .$$

To investigate the space $I_{u,u}$ we proceed as follows. We identify $\overset{u}{\Lambda} F^+$ with the subspace $F_{u,0}$ of ΛF and hence ΛF^+ with $\sum_{u} F_{u,0}$. Then $\overset{u}{\Lambda}F^+$ and ΛF^+ are invariant subspaces of ΛF, i.e. $\rho^*(h)\cdot\overset{u}{\Lambda}F^+\subset\overset{u}{\Lambda}F^+$ for all $h\in U(p)\times U(q)$. Analogously we identify $\overset{u}{\Lambda}F^-$ with the subspace $F_{0,u}$ of ΛF and ΛF^+ with $\Sigma F_{0,u}$. Then $\overset{u}{\Lambda}F^-$ and ΛF^- are also invariant subspaces of ΛF. The conjugate $\boldsymbol{C}$-linear isomorphism $\zeta\to\bar{\zeta}$ from F^+ onto F^+ is extended to a conjugate $\boldsymbol{C}$-linear isomorphism $\overset{u}{\Lambda}F^+\to\overset{u}{\Lambda}F^-$. Moreover, if $\eta\in\overset{u}{\Lambda}F^+$ and $h\in U(p)\times U(q)$, then we have

$$\overline{\rho^*(h)\eta}=\rho^*(h)\bar{\eta}\,.$$

Since $U(p)\times U(q)$ is compact, there is a positive definite invariant hermitian inner product $\langle\cdot,\cdot\rangle$ on F^+ such that

$$\langle\rho^*(h)\zeta,\rho^*(h)\eta\rangle=\langle\zeta,\eta\rangle$$

for all $h\in U(p)\times U(q)$. We can extend the inner product to a positive definite invariant hermitian inner product on $\overset{u}{\Lambda}F^+$. We then define a non-degenerate $\boldsymbol{C}$-bilinear function $(\cdot,\cdot)$ on $(\overset{u}{\Lambda}F^+)\times(\overset{u}{\Lambda}F^-)$ by

$$(\zeta,\bar{\eta})=\langle\zeta,\eta\rangle\,.$$

Then we have

$$(\rho^*(h)\zeta,\rho^*(h)\bar{\eta})=(\zeta,\bar{\eta})$$

for all $h\in U(p)\times U(q)$.

Using this bilinear function, we define a complex linear isomorphism from $\mathrm{Hom}\,(\overset{u}{\Lambda}F^+,\overset{u}{\Lambda}F^+)$ onto $F_{u,u}=(\overset{u}{\Lambda}F^+)\wedge(\overset{u}{\Lambda}F^-)$ in the following way. Let $\{\zeta_I\}$ be a basis of $\overset{u}{\Lambda}F^+$ and $S\in\mathrm{Hom}\,(\overset{u}{\Lambda}F^+,\overset{u}{\Lambda}F^+)$. For any $\zeta\in\overset{u}{\Lambda}F^-$ we have $S(\zeta)=\Sigma S_I(\zeta)\zeta_I$ and S_I is a linear function on $\overset{u}{\Lambda}F^+$. Then there is a unique $\bar{\eta}_I\in\overset{u}{\Lambda}F^-$ such that $S_I(\zeta)=(\zeta,\bar{\eta}_I)$ for all $\zeta\in\overset{u}{\Lambda}F^+$. We define

$$\varphi(S)=\sum_I\zeta_I\wedge\bar{\eta}_I\,.$$

The map φ: $\mathrm{Hom}(\overset{u}{\Lambda}F^+,\overset{u}{\Lambda}F^+)\to F_{u,u}$ is an isomorphism of complex vestor spaces and the definition is independent of the choice of the basis $\{\zeta_I\}$.

Moreover we have

$$\varphi(\rho^*(h)\cdot S\cdot\rho^*(h^{-1}))=\rho^*(h)\varphi(S)\tag{3.4}$$

for all $h\in U(p)\times U(q)$.

It follows from (3.4) that the space $I_{u,u}$ of invariant elements in $F_{u,u}$ is the image by φ of the subspace of Hom $(\overset{u}{\Lambda}F^+, \overset{u}{\Lambda}F^+)$ consisting of all S such thrt

$$\rho^*(h)S = S\cdot\rho^*(h)$$

for all $h\in U(p)\times U(q)$.

To study these endomorphisms S of $\overset{u}{\Lambda}F^+$ we decompose $\overset{u}{\Lambda}F^+$ into direct sum of irreducible invariant subspaces and use the Schur's Lemma. As we shall see later $\overset{u}{\Lambda}F^+$ decomposes into direct sum of irreducible invariant subspaces in the following way. There is a 1-1 correspondence between the set $\{A\}$ with the condition $|A|=u$ and the irreducible invariant subspaces $\{F_A\}$ of $\overset{u}{\Lambda}F^+$ and if $A\neq A'$, then F_A and $F_{A'}$ are not isomorphic as $U(p)\times U(q)$-module and we have

$$\overset{u}{\Lambda}F^+ = \sum_{A,|A|=u} F_A$$

and F_A and $F_{A'}$ are orthogonal for $A\neq A'$. The irreducible invariant subspace F_A is characterized as follows. The matrices $e_{p+s,1}(1\leqq i\leqq p,\ 1\leqq s\leqq q)$ form a basis of $\mathfrak{n}^+$ over $\boldsymbol{C}$. Let $\{\zeta_{p+s,i}\}$ be the dual basis of $F^+=(\mathfrak{n}^+)^*$. Let

$$\zeta_A = \bigwedge_{s\leqq a_i} \zeta_{p+s,i}\,, \tag{3.5}$$

where the exterior product extends over the pairs (i, s) such thrt $a_i>0$ and $s\leqq a_i$. Since $|A|=u$, ζ_A is the product of u elements $\zeta_{p+s,i}$ with $s\leqq a_i$ and hence $\zeta_A\in\overset{u}{\Lambda}F^+$ and in fact ζ_A is an element of F_A which is a weight vector for the *lowest weight* Λ_A of F_A and F_A is completely determined by ζ_A (see Theorem 2, Appendix). Now let S be an endomorphism of $\overset{u}{\Lambda}F^+$ such that $\rho^*(h)S=S\rho^*(h)$ for all h. Then the kernel of $S|F_A$ and the image $S(F_A)$ are both invariant subspaces of F_A and since F_A is irreducible, we have either $S(F_A)=\{0\}$ or else $S(F_A)\neq\{0\}$ and $S|F_A$ is an isomorphism of F_A onto $S(F_A)$ as $U(p)\times U(q)$-module. In the second case, $S(F_A)=F_{A'}$ for some A' with $|A'|=u$ and as $F_A\cong F_{A'}$ and we have $A=A'$. Thus for each A, we have either $S(F_A)=\{0\}$ or $S(F_A)=F_A$ and, in the case $S(F_A)=F_A$, by Schur's Lemma, $S|F_A=c_A\cdot 1$, where $c_A\in\boldsymbol{C}$ and 1_A is the identity map of F_A. Thus we have

$$S = \sum_{A,|A|=r} c_A\cdot P_A,\ c_A\in\boldsymbol{C}\,,$$

where P_A is the projection operator of $\overset{u}{\Lambda}F^+$ with respest to the direct sum decomposition $\overset{u}{\Lambda}F^+=\Sigma F_A$.

Let $\{\zeta_i(A)\,|\,1\leqq i\leqq m_A,\ m_A=\dim_C F_A\}$ be an orthonormrl basis of F_A. Since $F_A\perp F_{A'}$ for $A\neq A'$, $\{\zeta_i(A)\}_{i,A}$ is an orthonormal basis for $\overset{u}{\Lambda}F^+=\Sigma F_A$ and we have $P_A(\zeta)=\sum_i\langle\zeta,\zeta_i(A)\rangle\zeta_i(A)=\sum_i(\zeta,\overline{\zeta_i(A)})\zeta_i(A)$. Hence we get $\varphi(P_A)=\sum_i\zeta_i(A)\wedge\bar{\zeta}_i(A)$ and

$$\varphi(S)=\sum_{A,|A|=r}c_A\sum_i\zeta_i(A)\wedge\bar{\zeta}_i(A)$$

and we have proved the following lemma.

Lemma 3.1. *The complex vector space $I_{u,u}$ of invariant elements of type (u,u) is spanned by ω_A^0 with $|A|=u$, where*

$$\omega_A^0=\sum_{i=1}^{m_A}\zeta_i(A)\wedge\bar{\zeta}_i(A)$$

and $\{\zeta_1(A),\cdots,\zeta_{m_A}(A)\}$ is an orthonormal basis of the invariant irreducible susbpace F_A of $\overset{u}{\Lambda}F^+$. The invariants ω_A^0 are linearly independent.

We now discuss the decomposition of $\overset{u}{\Lambda}F^+$. The matrices $\{e_{p+s,i}\,|\,1\leqq i\leqq p, 1\leqq s\leqq q\}$ form a basis of $\mathfrak{n}^+$ and $\{\zeta_{p+s,i}\}$ is the dual basis of $F^+=(\mathfrak{n}^+)^*$. Let $\zeta\in F^+$ and let

$$m_{i,s}=\zeta(e_{p+s,i})\,.$$

Then $\zeta=\sum_{i,s}m_{i,s}\zeta_{p+s,i}$.

Let

$$h=\begin{pmatrix}h_1 & \\ & h_2\end{pmatrix}\in U(p)\times U(q)$$

and $h_1=(a_{ij})$ $(1\leqq i,j\leqq p)$ and $h_2^{-1}=(b_{p+t,p+s})$ $(1\leqq s,t\leqq q)$. Let

$$m'_{i,s}=(\rho^*(h)\zeta)\,(e_{p+s,i})\,.$$

Then $m'_{i,s}=\zeta(h^{-1}\cdot e_{p+s,i}h)=\sum_{t,j}a_{ij}b_{p+t,p+s}\zeta(e_{p+t,j})$ and we have

$$m'_{i,s}=\sum_{j,t}a_{ij}m_{j,t}b_{p+t,p+s}$$

Now let M be the complex vector space consisting of all $p\times q$ complex matrices. The group $U(p)\times U(q)$ operates on M by

$$T(h)m=h_1m\cdot h_2^{-1},\ m\in M$$

and $h\to T(h)$ is a representation of $U(p)\times U(q)$. Now the map $\zeta\to m=(m_{i,s})$ defines a vector space isomorphism of F^+ onto M. Moreover the above computation shows that this is an isomorphism of $U(p)\times U(q)$-module. The representation of $U(p)\times U(q)$ on the exterior algebra ΛM is discussed in the Appendix and since ΛF^+ and ΛM are isomorphic as $U(p)\times U(q)$-module, we obtain from Theorem 2 of Appendix the decomposition of $\overset{u}{\Lambda}F^+=\Sigma F_A$.

Finally we notice that $\zeta_{p+s,i}\in F^+$ corresponds to the matrix $e_{i,p+s}\in M$.

Let ω be a $U(m)$-invariant form of type (u,u) on $Gr(p,m)$. Identifying the vector space of (u,u)-forms at o with $F_{u,u}$, the value ω_0 of ω at o is identified with an invariant element $\in I_{u,u}$ and the map $\omega\to\omega_0$ is a vector space isomorphism of the space of $U(m)$-invariant (u,u)-forms onto $I_{u,u}$.

We denote by ω_B the $U(m)$-invariant (u,u)-form on $Gr(p,m)$ which corresponds to the basis element ω_B^0 of $I_{u,u}$, where $|B|=u$. We are going to show that ω_B is essentially the dual of the Schubert variety (B), that is, we show that

$$\int_{(C)}\omega_B=0 \qquad \text{for } B\neq C.$$

Now as $GL(m,\boldsymbol{C})$ is a holomorphic prinicpal bundle over $Gr(p,m)$ of projection π_G and group P, the pullbacks $\tilde{\omega}=\pi_G^*\omega$ of $U(m)$-invariant (u,u)-forms ω on $Gr(p,m)$ are characterized by the following properties:

1) $\tilde{\omega}$ is of type (u,u);

2) $i(Y)\tilde{\omega}=0$ for any left invariant vector field on $GL(m,\boldsymbol{C})$ belonging to the subalgebra $\mathfrak{p}$ of $\mathfrak{gl}(m,\boldsymbol{C})$;

3) $R_h^*\tilde{\omega}=\tilde{\omega}$ for all $h\in P$;

4) $L_g\tilde{\omega}=\tilde{\omega}$ for all $g\in U(m)$.

To simplify our notation we write $\mathfrak{gl}$ instead of $\mathfrak{gl}(m,\boldsymbol{C})$ and we denote this by $\mathfrak{gl}_{\boldsymbol{R}}$ when we regard $\mathfrak{gl}$ as a vector space over $\boldsymbol{R}$. Then $\mathfrak{gl}_{\boldsymbol{R}}^*\otimes_{\boldsymbol{R}}\boldsymbol{C}$ is regarded as the vector space of all $\boldsymbol{C}$-valued left invariant 1-forms on $GL(m,\boldsymbol{C})$. On the other hand we have $\mathfrak{gl}=\mathfrak{n}^+\oplus\mathfrak{p}$ and $F=(\mathfrak{n}_{\boldsymbol{R}}^+)^*\otimes_{\boldsymbol{R}}\boldsymbol{C}$. Hence we can identify F with the vector space of all left invariant 1-forms ζ on $GL(m,\boldsymbol{C})$ such that $i(Y)\zeta=0, Y\in\mathfrak{p}$. In particular F^+ (resp. F^-) is the space of holomorphic (resp. conjugate holomorphic) left invariant 1-forms ζ on $GL(m,\boldsymbol{C})$ satisfying $i(Y)\zeta=0,\ Y\in\mathfrak{p}$.

We choose an orthonormal basis $\{\zeta_i(A)\}$ of F_A. Then

$$\{\zeta_i(A)\wedge\bar{\zeta}_j(A')\}\ (1\leqq i\leqq m_A,\ 1\leqq j\leqq m_{A'};\ |A|=|A'|=u)$$

form an orthonormal basis of $F_{u,u}=(\overset{u}{\Lambda}F^+)\wedge(\overset{u}{\Lambda}F^-)$.

We identify $F_{u,u}$ with the vector space of all left invariant (u,u)-forms η on $GL(m,\boldsymbol{C})$ satsifying $i(Y)\eta=0,\ Y\in\mathfrak{p}$.

If ω is a $U(m)$-invariant form of type (u,u) on $Gr(p,m)$, then $\tilde{\omega}=\pi_G^*\omega$ satisfies $i(Y)\tilde{\omega}=0$ for all $Y\in\mathfrak{p}$ and we can write $\tilde{\omega}$ uniquely in the form

$$\tilde{\omega}=\Sigma a(i,A;j,A')\zeta_i(A)\wedge\bar{\zeta}_j(A') \tag{3.6}$$

where $a(i,A;j,A')$ are functions on $GL(m,\boldsymbol{C})$.

Since $\zeta_i(A)\wedge\bar{\zeta}_j(A')$ are left invariant and $L_y^*\tilde{\omega}=\tilde{\omega}$ for all $y\in U(m)$, we get

$$a(i,A;j,A')(yg)=a(i,A;j,A')(g) \tag{3.7}$$

for all $g\in GL(m, \boldsymbol{C})$ and $y\in U(m)$.

To study the effect of the left translation L_g by $g\in GL(m, \boldsymbol{C})$ on $\tilde{\omega}$, we need the following lemma.

Lemma 3.2. *Every $g\in GL(m, \boldsymbol{C})$ is written uniquely in the form*

$$g = u(g)\cdot b(g)\,, \tag{3.8}$$

where $u(g)$ is unitary and $b(g)$ is upper triangular. This decomposition is called the Iwasawa decomposition of g.

This lemma is a special case of a more general theorem of Iwasawa. However, in our special case the lemma is proved as follows. Let $g_1, \cdots, g_m$ be column vectors of g. Then we can construct an orthonormal basis $\{u_1, \cdots, u_m\}$ of $\boldsymbol{C}^m$ (by Schmidt method) such that

$$u_k = a_{1k}g_1+a_{2k}g_2+\cdots+a_{kk}g_k,\ a_{kk}\neq 0$$

for $k=1, 2, \cdots, m$, where a_{ij} $(i\leqq j)$ are complex numbers. Let $u(g)$ the unitary matrix whose column vectors are $u_1, \cdots, u_m$ and $b(g)^{-1}$ the upper triangular matrix whose (i, k)-entry $(i\leqq k)$ is a_{ik}. Then we have $u(g)=g\cdot b(g)^{-1}$ and hence $g=u(g)\cdot b(g)$ and $b(g)$ is also upper triangular. The uniqueness is easy to prove.

From (3.7) and (3.8) we get

$$a(i, A; j, A')\,(g) = a(i, A; j, A')\,(b(g))\,. \tag{3.9}$$

Thus these functions are completely determined by their values on B, B denoting the group of all non-singualr upper triangular matrices. Let $b\in B$. Then

$$L_b^*\tilde{\omega} = \Sigma(a(i, A; j, A')\circ L_b)\zeta_i(A)\wedge\bar{\zeta}_j(A')$$

and taking the value of both sides at the unit matrix 1, we get

$$(L_b^*\tilde{\omega})_1 = \Sigma a(i, A; j, A')\,(b)\zeta_i(A)_1\wedge\bar{\zeta}_j(A')_1\,. \tag{3.10}$$

We have $\pi_G\circ L_b=t_b\circ\pi_G$, where t_b denotes the action of $b\in B$ on $Gr(p, m)$. Then $L_b^*\tilde{\omega}=L_b^*(\pi_b^*\omega)=\pi_G^*(t_b^*\omega)$ and $\pi_G(1)=o$ and hence $(L_b^*\tilde{\omega})_1=(t_b^*\omega)_0\circ\pi_G^T$, where π_G^T denotes the surjective linear map $\Lambda T_1(GL(m, \boldsymbol{C}))\to\Lambda T_0(Gr(p, m))$ induced by the differential of π_G at 1. However $T_1(GL(m, \boldsymbol{C}))$ is canonically identified with $\mathfrak{gl}_R$ and the kernel of π_G^T: $\mathfrak{gl}_\times\to T_0(Gr(p, m))$ is equal to $\mathfrak{p}$ as discussed at the beginning of this session and π_G^T induces an isomorphism of $\mathfrak{n}^+$ onto $T_0(Gr(p, m))$. Therefore identifying $T_0(Gr(p, m))$ with $\mathfrak{n}_R^+$ as we did before, we have $\pi_G^T Z=X$, where $Z\in\mathfrak{gl}$ and X is the $\mathfrak{n}^+$-component of X with respect to the decomposition $\mathfrak{gl}=\mathfrak{n}^+\oplus\mathfrak{p}$. On the other hand, since B is a subgroup of P, we have $t_b(o)=o$ and hence $(t_b^*\omega)_0(X_1, \cdots, X_{2u})=\omega_0(\rho(b)X_1, \cdots, \rho(b)X_{2u})$ $(X_i\in\mathfrak{n}^+)$, where

ρ is the isotropic representation of P. Hence $(t_b^*\omega)_0=\rho^*(b^{-1})\omega_0$ by (3.2). Now let $\omega=\omega_B$, $|B|=u$. Then $\omega_0=\omega_B^0=\sum_i^{u}\zeta_i(B)\wedge\bar{\zeta}_i(B)$ by Lemma 3.1. Notice that we regard here $\zeta_i(B)$ as an element of ΛF^+ not as a differential form on $GL(m, \boldsymbol{C})$. Then $\rho^*(b^{-1})\omega_A^0=\sum_i\rho^*(b^{-1})\zeta_i(B)\wedge\rho^*(b^{-1})\bar{\zeta}_i(B)$. Thus, for any $Z_1, \cdots, Z_{2u}\in\mathfrak{gl}$, we have $(L_b^*\omega_B)_1(Z_1, \cdots, Z_{2u})=(t_*^b\omega_B)_0(X_1, \cdots, X_{2n})=\sum_i((\rho^*(b^{-1})\zeta_i(B))\wedge(\rho^{*i}(b^{-1})\bar{\zeta}_i(B)))\,(X_1, \cdots, X_{2u})$, where X_i is the $\mathfrak{n}^+$-component of $Z_i\in\mathfrak{gl}$. On the other hand from (3.10) we get $(L_b^*\omega_A)_1(Z_1, \cdots, Z_{2u})=\Sigma a(i, A; j, A')\,(b)\,(\zeta_i(A)_1\wedge\bar{\zeta}_j(A')_1)(Z_1, \cdots, Z_{2u})=\Sigma a(i, A; j, A')\,(b)\,(\zeta_i(A)\wedge\zeta_j(A')\,(X_1, \cdots, X_{2u})$. Hence we obtain

$$\text{(3.11)}\quad \Sigma a(i, A; j, A')\,(b)\,\zeta_1(A)\wedge\bar{\zeta}_j(A') = \Sigma\rho^*(b^{-1})\zeta_i(B)\wedge\rho^*(b^{-1})\bar{\zeta}_i(B)\,.$$

Now F_B is also invariant by the isotropic representation ρ^* of P on ΛF, because F_B is invariant by $\rho^*(g)$ for all $g\in GL(p, \boldsymbol{C})\times GL(q, \boldsymbol{C})$ and, as we see from (3.1) and (3.2), for $h\in\mathfrak{p}$, there is $g\in GL(p, \boldsymbol{C})\times GL(q, \boldsymbol{C})$ such that $\rho^*(h)=\rho^*(g)$. Therefore we can writ

$$\text{(3.12)}\qquad \rho^*(b^{-1})\zeta_i(B) = \sum_k \rho_{ki}(B; b^{-1})\zeta_k(B)$$

and since $\rho^*(b^{-1})\zeta_j(B))=\overline{\rho^*(b^{-1})\zeta_j(B)}$, we have also

$$\rho^*(b^{-1})\bar{\zeta}_i(B) = \sum_l \bar{\eta}_{li}(B; b^{-1})\bar{\zeta}_l(B)\,.$$

Hence

$$\begin{aligned}&\sum_i \rho^*(b^{-1})\eta_i(B)\wedge\rho^*(b^{-1})\bar{\zeta}_i(B)\\ &=\sum_{k,l}(\sum_i \rho_{ki}(B; b^{-1})\bar{\rho}_{li}(B; b^{-1}))\zeta_k(B)\wedge\bar{\zeta}_l(B)\end{aligned}$$

and it follows from (3.11) that

$$\begin{aligned}&a(i, A; j, A')\,(b)=0,\ \ if\ A\neq B \text{ or } A'\neq B\\ &a(i, B; j, B)\,(b) = \sum_k \rho_{ik}(B; b^{-1})\bar{\rho}_{jk}(B; b^{-1})\,.\end{aligned}$$

Hence we have proved the following lemma.

Lemma 3.3. *The pullback $\tilde{\omega}_B=\pi_G^*\omega_B$ is of the form*

$$\tilde{\omega}_B = \sum_{i,j} a_{ij}\zeta_i(B)\wedge\bar{\zeta}_j(B)\,,$$

where the functions a_{ij} on $GL(m, \boldsymbol{C})$ is given by

$$a_{ij}(g) = \sum_k \rho_{ik}(B; b(g)^{-1})\bar{\rho}_{jk}(B; b(g)^{-1})\,.$$

We now integrate ω_B over the Schubert variety (C) where $|C|=|B|=u$. This integral is equal to the integral of ω_B over the Schubert cell $(C)^*$. However as we have seen in §2 there is a biholomorphic map α from the simply connected complex abelian group $N(C)$ onto $(C)^*$ given by $\alpha(n)=u_\sigma^{-1}\cdot n\cdot o(\sigma=\sigma_C)$ and we have the following commutative diagram:

$$\begin{array}{ccc} GL(m,C) & \xrightarrow{\pi_G} & Gr(p,m) \\ {\scriptstyle j}\uparrow & & \uparrow{\scriptstyle i} \\ N(C) & \xrightarrow{\alpha} & (C)^* \end{array}$$

where $j=L_{u_\sigma^{-1}}\circ i_N$, i_N being the inclusion map of $N(C)$ into $GL(m,C)$. Hence we get $\int_{(C)^*}\omega_B=\int_{(C)^*}i^*\omega_B=\int_{N(C)}\alpha^*(i^*\omega_B)=\int_{N(C)}j^*\tilde{\omega}_B=\int_{N(C)}i_N^*L^*_{u_\sigma^{-1}}\tilde{\omega}_B$. However, we have $u_\sigma^{-1}\in U(m)$ and $L_x^*\tilde{\omega}_B=\tilde{\omega}_B$ for all $x\in U(m)$ and hence $L^*_{u^{-1}}\tilde{\omega}_B=\tilde{\omega}_B$ and we get

$$\int_{(C)^*}\omega_B=\int_{N(C)}i_N^*\tilde{\omega}_B\,. \tag{3.13}$$

Now the Lie algebra $\mathfrak{n}(C)$ $(C=\{c_1,\cdots,c_p\})$ is spanned by $e_{p+s,i}$ with the condition $s\leqq c_i$ (see §2). Hence $i_N^*\zeta_{p+s,i}=0$ for $s>c_i$ and $\{i_N^*\zeta_{p+s,i}, s\leqq c_i, i=1,2,\cdots,p\}$ form a basis of left invariant holomorphic 1-forms on the complex abelian Lie group $N(C)$. Every form belonging to $\overset{u}{\Lambda}F^+$ is a linear combination of forms of the type

$$\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_u,i_u} \tag{3.14}$$

and the pullback by i_N of these froms are all zero except for ζ_C, where ζ_C is defined by (3.5) and ζ_C is a weight vector for the lowest weight Λ_C of F_C. Suppose now that $i_N^*\tilde{\omega}_B\neq 0$. By Lemma 3.3, we must have $i_N^*\zeta_i(B)\neq 0$ for some i. Now $\zeta_i(B)\in F_B$ and F_B is a subspace of $\overset{u}{\Lambda}F^+$. Then $\zeta_i(B)$ is a linear combination of form of the type (3.14). Since $i_N^*(\zeta_i(B))\neq 0$, ζ_C must appear in the linear expression of $\zeta_i(B)$. We can conclude from this that $B=C$. For, we may assume that $\zeta_i(B)\in F_B$ is a weight vector for some weight Λ_1 of F_B. Then for any diagonal $m\times m$ matrix H, $\zeta_i(B)$ is an eigen-vector for the eigen value $\Lambda_1(H)$ of the linear transformation $\rho'^*(H)$ of $\overset{u}{\Lambda}F^+$, where ρ^* denotes the representation of the Lie algebra $\mathfrak{gl}(p,C)\times\mathfrak{gl}(p,C)$ induced by the representation ρ^* of $GL(p,C)\times GL(q,C)$. We see easily also that each element of the form of (3.14) is also an eigen-vector of $\rho'^*(H)$. Hence, when we express $\zeta_i(B)$ as a linear combination of elements of the type (3.13), only elements corresponding to the eigen-value $\Lambda_1(H)$ appears with non-zero coefficient and ζ_C appears with non-zero coefficient. However ζ_C is a weight vector for the weight Λ_C and hence ζ_C is an eigenvector of $\rho'^*(H)$ for the eigenvalue $\Lambda_C(H)$. Hence we have $\Lambda_1(H)=\Lambda_C(H)$ for any diagonal matrix H and this shows that $\Lambda_1=\Lambda_C$ and thus Λ_C is a weight of F_B. However, the eigenvector space for the weight Λ_C is a one-dimensional subspace of $\overset{u}{\Lambda}F^+$ (see Appendix) and contained in F_C. Since $\Lambda_1=\Lambda_C$ and $\zeta_1(B)$ is a weight vector for Λ_1, we get $\zeta_1(B)\in F_C$. Thus $F_C\cap F_B\neq(0)$ and hence $F_C=F_B$. This implies $B=C$ because $C\to F_C$ is bijective. Thus, if $i_N^*\tilde{\omega}_B\neq 0$, we get $B=C$. Hence, if $C\neq B$, we have $i_N^*\tilde{\omega}_B=0$ and from (3.13) it follows that

$$\int_{(C)}\omega_B = 0,\ C\neq B,\ |B| = |C| = u\,.$$

The Schubert varieties (C) with $|C|=u$ form a basis of the $2u$-dimensional homology group and ω_B is not cohomologous to zero. Then the value v_B of the integral of ω_B over (B) can not be zero and $\xi_B=v_B^{-1}\omega_B$ is the dual of the Schubert variety (B).

Thus we have proved th following

Lemma 3.4. *Let ω_B be the invariant (u, u)-form on $Gr(p, m)$ corresponding to the invariant element ω_B^0 of type (u, u) in $F_{u,u}$. Then*

$$\xi_B = v_B^{-1}\omega_B$$

is the dual of the Schubert variety (B), where v_B is the value of the integral of ω_B over (B).

REMARK. Lemma 3.4 is a special case of a more general result of Kostant [3,*b*]. We can express v_B explicitly by an integral of a certain function on $\boldsymbol{C}^u$ using Lemma 3.3 and (3.13).

4. The final step of the proof of Theorem. In Lemma 2.4 we have an expression of the number d_B by the integral (2.5) and the integrand involves the form $\gamma^*\pi_G^*\xi_B$. By Lemma 3.4 we have $\xi_B=v_B^{-1}\omega_B$ and hence $\gamma^*\pi_G^*\xi_B=v_B^{-1}\gamma^*\tilde{\omega}_B$, $\tilde{\omega}_B=\pi_G^*\omega_B$ and we have an information about $\tilde{\omega}_B$ by Lemma 3.3. We study now $\gamma^*\tilde{\omega}_B$ using Lemma 3.3; the map γ: $U(m)\times N(A)\to GL(m, \boldsymbol{C})$ is defined by

$$\gamma(g, n) = u_{\sigma_A} g u_{\sigma_A}^{-1} n,\ g\in U(m),\ n\in N(A)\,.$$

To simplify our notation we put $\sigma=\sigma_A$. We define two maps $I(u_\sigma)$: $U(m)\times N(A)\to U(m)\times N(A)$ and ν: $U(m)\times N(A)\to GL(m, \boldsymbol{C})$ by

$$I(u_\sigma)(g, n) = (u_\sigma g u_\sigma^{-1}, n)$$

and

$$\nu(g, n) = gn\,.$$

Then we have

$$\gamma = \nu \circ I(u_\sigma) .$$

To study the differentials of these maps at $(1, n)$, where 1 is the unit matrix, we identify the tangent vactor space of $U(m) \times N(A)$ at $(1, n)$ with $\mathfrak{n}(m) \times \mathfrak{n}(A)$, $\mathfrak{u}(\mathfrak{m})$ and $\mathfrak{n}(A)$ denoting the Lie algebra of $U(m)$ and $N(A)$ respectively. The elements of these Lie algebras will be regarded as left invariant vector fields on $GL(m, \boldsymbol{C})$ in a canonical way. Then a tangent vector at $(1, n)$ is a pair (X_1, Y_n) where $X \in \mathfrak{u}(m)$ and $Y \in \mathfrak{n}(A)$. We see easliy that

$$I(u_\sigma)^T(X_1, Y_n) = ((Ad(u_\sigma)X)_1, Y_n)$$

and

$$\nu^T(X_1, Y_n) = (Ad(n^{-1})X_n) + Y_n$$

Since $\gamma^T = \nu^T \circ I(u_\sigma)^I$, we get

$$\gamma^T(X_1, Y_n) = (Ad(n^{-1})Ad(u_\sigma)X)_n + Y_n \tag{4.1}$$

Let $\{\theta_\alpha\}$ $(\alpha = 1, 2, \cdots, m^2)$ be a basis of left invariant real 1-forms on $U(m)$. Let $\zeta_{p+s,i}$ be the left invariant 1-form on $GL(m, \boldsymbol{C})$ defined in §3. We have seen that if $s > a_i$, $i_N^*\zeta_{p+s,i} = 0$ and that $i_N^*\zeta_{p+s,i}$ and $i_N^*\bar{\zeta}_{p+s,i}$ with the condition $s \leqq a_i$ form a basis of left invariant complex 1-forms on $N(A)$. We denote by s_1 and s_2 the projections of $U(m) \times N(A)$ onto $U(m)$ and $N(A)$ respectively. Then $\{s_1^*\theta_\alpha, s_2^*(i_N^*\zeta_{p+s,i}), s_2^*(i_N^*\bar{\zeta}_{p+s,i})\}$ $(a = 1, 2, \cdots, m^2, s \leqq a_i, i = 1, 2, \cdots, p)$ form a bsis of left invariant 1-forms on the the group $U(m) \times N(A)$.

Let now ζ be a left invariant 1-form on $GL(m, \boldsymbol{C})$. Then we can write the pullback $\gamma^*\zeta$ in the form

$$\gamma^*\zeta = \sum_\alpha f_\alpha \cdot s_1{}^*\theta_\alpha + \Sigma g_{s,i} s_2^* i_N^* \zeta_{p+s,i} + \Sigma h_{s,i} s_2^* i_N^* \bar{\zeta}_{p+s,i} \tag{4.2}$$

where f_α, $g_{s,i}$ and $h_{s,i}$ are complex valued functions on $U(m) \times N(A)$. We shall show that $g_{s,i}$ and $h_{s,i}$ are constant and that

$$\begin{cases} f_\alpha(g, n) = f_\alpha(1, n), \text{ for all } g \in U(m); \\ f_\alpha(1, n) = \zeta_n(Ad(n^{-1})Ad(u_\sigma)X_\alpha) , \end{cases} \tag{4.3}$$

where $\{X_\alpha\}$ is the basis of the Lie algebra $\mathfrak{u}(m)$ such that $\theta_\alpha(X_\beta) = \delta_{\alpha\beta}$ and in the above formula we regard X_α as a left invariant vector field on $GL(m, \boldsymbol{C})$.

To see these, we consider the left translation $L_{(g,1)}$ of $U(m) \times N(A)$, where $g \in U(m)$. Then we have $s_1 \circ L_{(g,1)} = L_g \circ s_1$, $s_2 \circ L_{(g,1)} = s_2$ and $\gamma \circ L_{(g,1)} = L_{g'} \circ \gamma$, with $g' = u_\sigma g u_\sigma^{-1}$. Since ζ and θ_α are left invariant, we get $L^*{}_{(g,1)}(\gamma^*\zeta) = \gamma^*\zeta$, $L^*{}_{(g,1)}(s_1^*\theta_\alpha) = s_1^*\theta_\alpha$ and also $L^*{}_{(g,1)} s_2^* i_N^* \zeta_{p+s,i} = s_2^* i_N^* \zeta_{p+s,i}$. Then from (4.2) we get $f_\alpha \circ L_{(g,1)} = f_\alpha$, $g_{s,i} \circ L_{(g,1)} = g_{s,i}$ and $h_{s,i} \circ L_{(g,1)} = h_{s,i}$ and these mean that we have

$$f_\alpha(g, n) = f_\alpha(1, n),\ g_{s,i}(g, n) = g_{s,i}(1, n),\ h_{s,i}(g, n) = h_{s,i}(1, n) \tag{4.4}$$

for any $g \in U(m)$.

We get from (4.1) that

$$(\gamma^*\zeta)_{(1,n)}(X_1, Y_n) = \zeta_n(Ad(n^{-1})Ad(u_\sigma)X) + \zeta(Y)$$

and we also have $(s_1^*\theta_\alpha)_{(1,n)}(X_1, Y_n) = \theta_\alpha(X)$ and $s_2^* i_N^* \zeta_{p+s,i}(X_1, Y_n) = \zeta_{p+s,i}(Y)$. Then we get from (4.2) that

$$\begin{aligned}\sum_\alpha f_\alpha(1, n)\theta_\alpha(X) + \Sigma g_{s,i}(1, n)\zeta_{p+s,i}(Y) + \Sigma h_{s,i}(1, n)\bar{\zeta}_{p+s,i}(Y) \\ = \zeta_n(Ad(n^{-1})Ad(u_\sigma)X) + \zeta(Y)\,.\end{aligned} \tag{4.5}$$

Notice that since X and Y are left invariant vector fields and θ_α, $\zeta_{p+s,i}$ and $\bar{\zeta}_{p+s,i}$ are also left invariant 1-forms, $\theta_\alpha(X)$, $\zeta_{p+s,i}(Y)$ and $\bar{\zeta}_{p+s,i}(Y)$ are constant.

Letting $X=0$ in (4.5) we get

$$\zeta(Y) = \Sigma g_{s,i}(1, n)\zeta_{p+s,i}(Y) + \Sigma h_{s,i}(1, n)\bar{\zeta}_{p+s,i}(Y) \tag{4.6}$$

for all $Y \in \mathfrak{n}(A)$. Since $e_{p+s,j}$ and $ie_{p+s,j}$ ($i^2=-1$) with the condition $s \leqq a_j$ ($j=1, 2, \cdots, p$) form a basis over $\boldsymbol{R}$ of $\mathfrak{n}(A)$, letting $Y=e_{p+s,i}$ and $Y=ie_{p+s,i}$ respectively in (4.6), we get $\zeta(e_{p+s,j}) = g_{s,j}(1, n) + h_{s,j}(1, n)$ and $\zeta(ie_{p+s,j}) = ig_{s,j}(1, n) - ih_{s,j}(1, n)$ and hence $g_{s,j}(1, n) = \{\zeta(e_{p+s,j}) - i\zeta(ie_{p+s,j})\}/2$ and $h_{s,j}(1, n) = \{\zeta(e_{p+s,j}) + i\zeta(ie_{p+s,j})\}/2$ and hence combined with (4.4), we see that $g_{s,j}$ and $h_{s,j}$ are constant. Then since (4.6) holds for any $Y \in \mathfrak{n}(A)$ we obtain also

$$i_N^*\zeta = \Sigma g_{s,i} i_N^* \zeta_{p+s,i} + \Sigma h_{s,i} i_N^* \bar{\zeta}_{p+s,i}$$

and hence the second term on the right hand side of (4.2) is equal to $s_2^*(i_N^*\zeta)$.

Now letting $Y=0$ and $X=X_\alpha$ in (4.5) we get $f_\alpha(1, n) = \zeta_n(Ad(n^{-1})Ad(u_\sigma)X_\alpha)$ and this, together with (4.4) proves (4.3). Thus we have shown that, for any left invariant 1-form ζ on $GL(m, \boldsymbol{C})$, we have

$$\gamma^*\zeta = \sum_\alpha f_\alpha s_1^*\theta_\alpha + s_2^*(i_N^*\zeta) \tag{4.7}$$

and the function f_α satisfies (4.3).

We consider now the pullback by γ of a left invariant u-form η on $GL(m, \boldsymbol{C})$ of the form

$$\eta = \zeta_{p+s_1,i_1} \wedge \cdots \wedge \zeta_{p+s_u,i_u}\,. \tag{4.8}$$

We say that a form on $U(m) \times N(A)$ is of type (a, b) if it is a linear combination of forms of the type $s_1^*(\theta) \wedge s_2^*(\xi)$, where θ is a left invariant a-form on $U(m)$ and ξ is a left invariant b-form on $N(A)$. The exterior product of a form of type (a, b) and a form of type (a', b') is a form of type $(a+a', b+b')$.

It follows from (4.7) that if $s > a_i$, then $\gamma^*\zeta_{p+s,i}$ is of type $(1, 0)$, because

$i_N^*\zeta_{p+s,i}=0$, and that if $s\leqq a_i$, then $\gamma^*\zeta_{p+s,i}$ is a sum of a form of type $(1, 0)$ and a form of type $(0, 1)$.

We denote by u_1 (resp. u_2) the numbers of factors ζ_{p+s_k,i_k} in (4.8) such that $s_k>a_{i_k}$ (resp. $s_k\leqq a_{i_k}$). Then

$$u=u_1+u_2\,.$$

Assume that $u\geqq|A|$ and let

$$u=|A|+r, \qquad r\geqq 0\,.$$

Since $|A|$ is equal to the number of 1-forms $\zeta_{p+s,i}$ satisfying $s\leqq a_i$, we have $u_2\leqq|A|$ and since $u_1+u_2=|A|+r$, we have also

$$u_1\geqq r$$

and the equality holds if and only if $u_2=|A|$.

We have then

$$\gamma^*\eta=\sum_{a+b=u_2}\eta_{(a+u_1,b)}\,, \tag{4.9}$$

where $\eta_{(a+u_1,b)}$ is a form of type $(a+u_1, b)$.

Analogously if $\bar{\eta}'$ is of the form

$$\bar{\eta}'=\bar{\zeta}_{p+t_1,j_1}\wedge\cdots\wedge\bar{\zeta}_{p+t_u,j_u} \tag{4.10}$$

we have

$$\gamma^*\bar{\eta}'=\sum_{a'+b'=u_2'}\bar{\eta}'_{(a'+u_1',b')}\,, \tag{4.11}$$

where $\bar{\eta}'_{(a'+u_1',b')}$ is of type $(a'+u_1', b')$ and u_1' (resp. u_2') is the number of the factors in (4.11) satisfying the condition $t_k>a_{j_k}$ (resp. $t_k\leqq a_{j_k}$). Then we have also $u=u_1'+u_2'$, $u_1'\geqq r$, $u_2'\leqq|A|$.

We now consider the pullback $\gamma^*\tilde{\omega}_B$ and the integrand $s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*\pi_G^*\xi_B$ of the integral (2.5) which is equal to $v_B^{-1}s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*\tilde{\omega}_B$. Here λ is a $U(m)$-invariant form of type $(f-r, f-r)$ on $F(A)=U(m)/H_A$ with $f=\dim_C F(A)$ and θ_A is a left invariant form on $U(m)$ defined in Lemma 1.1 and the degree of θ_A is equal to $\dim H_A$. Since $2f=m^2-\dim H_A$, $\theta_A\wedge\pi_F^*\lambda$ is a left invariant form on $U(m)$ of degree m^2-2r, where $m^2=\dim U(m)$. Hence $s_1^*(\theta_A\wedge\pi_F^*\lambda)$ is a form of type $(m^2-2r, 0)$ on $U(m)\times N(A)$. The form $\tilde{\omega}_B$ on $GL(m,\boldsymbol{C})$ is of type (u,u) and

$$u=|B|=|A|+r\,.$$

Lemma 4.1. *Let η and $\bar{\eta}'$ be left invariant forms on $GL(m,\boldsymbol{C})$ of the form* (4.8) *and* (4.10) *respectively. Suppose that $s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*\eta\wedge\gamma^*\bar{\eta}'\neq 0$. Then we have $u_1=u_1'=r$ and $u_2=u_2'=|A|$.*

Proof. Put $\xi=s_1^*(\theta_A\wedge\pi_F^*\lambda)$. By (4.9) and (4.11), $\xi\wedge\gamma^*\eta\wedge\gamma^*\bar{\eta}'$ is a sum of $\xi\wedge\eta_{(a+u_1,b)}\wedge\bar{\eta}'_{(a'+u_1,b)}$ and these forms are of type (c, d), where

$$c = m^2-2r+a+a'+u_1+u_1',\ d = b+b'.$$

and these are non-zero only if $c=m^2=\dim U(m)$ and $d=2|A|=2\dim_C N(A)$. Since one of these forms is non-zero, we have $-2r+a+a'+u_1+u_1'=0$. However we have $u_1\geqq r$ and $u_1'\geqq r$ and a and a' are non-negative. Hence we get $u_1=u_1'=r$ and $a=a'=0$. However $u=u_1+u_2=u_1'+u_2'=|A|+r$ and so we have $u_2=u_2'=|A|$.

If η verifies the condition $u_1=|A|$, η is of the form

$$\eta = \pm\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A, \tag{4.12}$$

with $s_k>a_{i_k}$ for $k=1,\cdots,r$, where ζ_A is defined by (3.5).

Analogously, if $\bar{\eta}'$ verifies the condition $u_1'=|A|$, $\bar{\eta}'$ is of the form

$$\bar{\eta}' = \pm\bar{\zeta}_{p+t_1,j_1}\wedge\cdots\wedge\bar{\zeta}_{p+t_r,j_r}\wedge\zeta_A. \tag{4.13}$$

with $t_k>j_k$ for $k=1,\cdots,r$. Hence we can state Lemma 4.1 in the following form.

Lemma 4.2. *Let η and $\bar{\eta}'$ be left invariant u-forms on $GL(m, C)$ defined by* (4.8) *and* (4.10) *respectively. If $s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*(\eta\wedge\bar{\eta}')$ is non-zero, then η and $\bar{\eta}'$ are of the form* (4.12) *and* (4.13) *respectively.*

Now by Lemma 3.3, $\tilde{\omega}_B$ is of the form

$$\tilde{\omega}_B = \Sigma a_{ij}\zeta_i(B)\wedge\bar{\zeta}_j(B), \tag{4.14}$$

where a_{ij} are functions on $GL(m, C)$ defined in Lemma 3.3 and $\{\zeta_i(B)\}$ is an orthonormal basis of F_B. However we don't need to assume here that $\{\zeta_i(B)\}$ is orthonormal, because if we replace $\{\zeta_i(B)\}$ by another basis, then the matrix (a_{ij}) is simply multiplied by constant matrices and this does not disturb our following study. We choose here a basis $\{\zeta_i(B)\}$ in the following way. Since F_B is an irreducible $GL(p, C)\times GL(q, C)$-module with respect to the isotropic representation ρ^* of $GL(p, C)\times GL(q, C)$ and since ζ_B is the weight vector for the lowest weight Λ_B of F_B, F_B is spanned by ζ_B and elements of the form

$$\rho'^*(e_{\alpha_1})\cdots\rho'^*(e_{\alpha_l})\zeta_B \qquad (l\geqq 0), \tag{4.15}$$

where ρ'^* is the representation of the Lie algebra $\mathfrak{gl}(p, C)\times\mathfrak{gl}(q, C)$ defined by ρ^* and $\alpha_1, \cdots, \alpha_l$ are simple roots and $e_{\alpha_i}\in\mathfrak{gl}(p, C)\times\mathfrak{gl}(q, C)$ is a root vector for the simple root α_i (see Appendix).

Let $\zeta_1(B)=\zeta_B$ and let $\zeta_2(B), \cdots, \zeta_{m_B}(B)$, $m_B=\dim_C F_B$ be the linearly independent elements of the form (4.15). Each $\zeta_k(B)$ is then a weight vector for a weight Λ_k of F_B and we number $\zeta_2(B), \zeta_3(B), \cdots$ in such a way that we have $\Lambda_B=\Lambda_1<\Lambda_2\leqq\Lambda_3\leqq\cdots$. Since each $\zeta_k(B)$ is of the form (4.15) for $k\geqq 2$, $\zeta_k(B)$ is of the form

$$\zeta_k(B)=\rho'^*(e_\alpha)\xi\,, \tag{4.16}$$

where $\xi=\rho'^*(e_{\alpha_2})\cdots\rho'^*(e_{\alpha_l})\zeta_B$ and ξ is a weight vector for the weight $\Lambda_k-\alpha$, where α is a simple root (see Lemma 1 of Appendix).

From now on we assume that

$$s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*\tilde{\omega}_B\neq 0\,.$$

Then we see from (4.14) that

$$s_1^*(\theta_A\wedge\pi_F^*\lambda)\wedge\gamma^*(\zeta_k(B)\wedge\bar{\zeta}_l(B))\neq 0 \tag{4.17}$$

for some k and l. Since $\zeta_k(B)$ and $\zeta_l(B)$ are elements of F_B and F_B is a subspace of $\overset{u}{\Lambda}F^+$ with $u=|B|$, $\zeta_k(B)$ and $\bar{\zeta}_l(B)$ are linear combinations of u-forms η and $\bar{\eta}'$ respectively, where η and $\bar{\eta}'$ are defined by (4.8) and (4.10). From (4.17) and Lemma 4.2 it follows that $\zeta_k(B)$ is of the form

$$\zeta_k(B)=c\cdot\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A+\cdots\,, \tag{4.18}$$

where $s_k>a_{i_k}$ for $k=1,2,\cdots,r$ and c is a non-zero constant.

We prove the following lemma.

Lemma 4.3. *Assume that* (4.17) *holds for some k and l. Then there exist s indices $j_1,\cdots,j_s$ $(1\leqq j_1<j_2<\cdots<j_s\leqq p,\ s\leqq r)$ and s positive integers $n_1,n_2,\cdots,n_s$ with the condition $n_1+n_2+\cdots+n_s=r$ such that*

$$b_{j_c}=a_{j_c}+n_c \qquad \textit{for}\ \ c=1,2,\cdots,s$$

and

$$b_j=a_j \qquad \textit{for}\ \ j\neq j_a\,.$$

To prove Lemma 4.3, we first assume $k=1$. From our choice of the basis $\{\zeta_i(B)\}$ we have

$$\zeta_1(B)=\zeta_B$$

and ζ_B is the exterior product of $\zeta_{p+s,i}$ satisfying the condition $s\leqq b_i$ $(i=1,2,\cdots,p;\ B=\{b_1,\cdots,b_p\})$. By (4.18) we have $\zeta_B=c\cdot\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A+\cdots$, where $c\neq 0$. Since ζ_B and $\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A$ are both of the form (4.8) and distinct elements of the form (4.8) are linearly independent, we get $c=\pm 1$ and

$$\zeta_B=\pm\zeta_{p+s_1,i_1}\wedge\cdots<\zeta_{p+s_r,i_r}\wedge\zeta_A\,. \tag{4.19}$$

Let j be an index, $1\leqq j\leqq p$, which is distinct from each of i_l, $l=1,2,\cdots,r$. The number of factors of the form $\zeta_{p+t,j}$ on the right hand side of (4.19) is equal to a_j and it is equal to b_j on the left hand side. Hence we get $b_j=a_j$. Let $j_1,\cdots,j_s$ be the distinct indices among $i_1,\cdots,i_r$ and let n_c be the number of the indices i_l which are equal to j_c. Then we have $n_1+\cdots+n_s=r$, $n_c>0$. Then the number of factors of the form ζ_{p+t,j_c} on the right hand side of (4.19) is equal to $n_c+a_{j_c}$ and on the left hand side it is equal to b_{j_c} and hence $b_{j_c}=a_{j_c}+n_c$. The lemma is thus proved in the case $k=1$.

Assume now $k>1$. We may assume that k is the smallest index $k>1$ satisfying (4.17). By our choice of $\{\zeta_i(B)\}$, $\zeta_k(B)$ is a weight vector for a weight Λ_k with $\Lambda_B<\Lambda_k$ and by (4.16) $\zeta_k(B)=\rho'^*(e_\alpha)\xi$, where α is a simple root of $\mathfrak{gl}(p,\boldsymbol{C})\times\mathfrak{gl}(q,\boldsymbol{C})$ and ξ is a weight vector for a weight $\Lambda_k-\alpha$. Since $\xi\in F_B$ and F_B is a subspace of $\overset{u}{\Lambda}F^+$, $\xi=\Sigma c_\eta\,\eta$, where η are elements of the form (4.8) and $c_\eta\in\boldsymbol{C}$. We have $\rho'^*(e_\alpha)\xi=\Sigma c_\eta\rho'^*(e_\alpha)\eta$ and $\rho'^*(e_\alpha)\eta=\sum_{l=i}^{n}\zeta_{p+t_1,j_1}\wedge\cdots\wedge\rho'^*(e_\alpha)\zeta_{p+t_l,j_l}\wedge\cdots\wedge\zeta_{p+t_u,j_u}$. We notice here that simple roots are of the form $\alpha=\lambda_i-\lambda_{i+1}(1\leqq i\leqq p-1)$ or $\alpha=\lambda_{p+s}-\lambda_{p+s+1}(1\leqq s\leqq q-1)$ and hence $e_\alpha=e_{i,i+1}$or $e_\alpha=e_{p+s,p+s+1}$. We also have

$$\rho'^*(e_{j,l})\zeta_{p+s,i}=\delta_{li}\zeta_{p+s,j}\qquad(1\leqq i,j,\ l\leqq p,\ 1\leqq s\leqq q)$$

and

$$\rho'^*(e_{p+t,p+u})\zeta_{p+s,i}=-\delta_{t,s}\zeta_{p+u,i}\qquad(1\leqq i\leqq p,\ 1\leqq s,\ t,\ u\leqq q)$$

(see §3 and Appendix).

It follows in particular that the term $\zeta_{p+s_1,i_1}\wedge\cdots\rho'^*(e_\alpha)\zeta_{p+s_l,i_l}\wedge\cdots\wedge\zeta_{p+s_u,i_u}$ is either zero or of the form (4.8). We have $\zeta_k(B)=\Sigma c_\eta\rho'^*(e_\alpha)\eta$ and (4.19). We see then that $\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A=\pm\zeta_{p+t_1,j_1}\wedge\cdots\wedge\rho'^*(e_\alpha)\zeta_{p+t_l,j_l}\wedge\cdots\wedge\zeta_{p+t_u,j_u}$ for some $\eta=\zeta_{p+t_1,j_1}\wedge\cdots\wedge\zeta_{p+t_u,j_u}$ and for some l, where η appears in the expression $\xi=\Sigma c_\eta\eta$ with $c_\eta\neq 0$. Renumbering, if necessary, we may assume $l=1$ and hence we have

$$\text{(4.20)}\qquad \begin{aligned}&\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A\\ &=\pm(\rho'^*(e_\alpha)\zeta_{p+t_1,j_1})\wedge\zeta_{p+t_2,j_2}\wedge\cdots\wedge\zeta_{p+t_u,j_u}\,.\end{aligned}$$

The simple root α is either $\alpha=\lambda_i-\lambda_{i+1}$ and $e_\alpha=e_{i,i+1}$ or $\alpha=\lambda_{p+s}-\lambda_{p+s+1}$ and $e_\alpha=e_{p+s,p+s+1}$. We first assume $\alpha=\lambda_i-\lambda_{i+1}$ and $e_\alpha=e_{i,i+1}$. Then, since $\rho'^*(e_{i,i+1})\zeta_{p+t_1,j_1}=\delta_{i+1,j_1}\zeta_{p+t_1,i}\neq 0$ we have

$$\text{(4.21)}\qquad i+1=j_1$$

and

$$\begin{aligned}(4.22)\qquad & \zeta_{p+t_1,i}\wedge\zeta_{p+t_2,j_2}\wedge\cdots\wedge\zeta_{p+t_u,j_u}\\ &=\pm\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A\end{aligned}$$

Suppose that $\zeta_{p+t_1,i}=\zeta_{p+s_l,i_l}$ for some l, $1\leqq l\leqq r$ i.e. $i=i_l$ and $t_1=s_l$. We may assume without loss of generality $l=1$, hence $i=i_1$ and $t_1=s_1$. Then we have $\zeta_{p+t_2,j_2}\wedge\cdots\wedge\zeta_{p+t_u,j_u}=\pm\zeta_{p+s_2,i_2}\wedge\cdots\wedge\zeta_{p+s_r,i_r}\wedge\zeta_A$. Then we may assume $\zeta_{p+t_l,j_l}=\zeta_{p+s_l,i_l}$ for $l=2,\cdots,r$ and $\zeta_{p+t_{r+1},j_{r+1}}\wedge\cdots\wedge\zeta_{p+t_u,j_u}=\pm\zeta_A$. Then $\eta=\zeta_{p+t_1,j_1}\wedge\cdots\wedge\zeta_{p+t_u,j_u}=\pm\zeta_{p+t_1,j_1}\wedge\cdots\wedge\zeta_{p+t_r,j_r}\wedge\zeta_A$ and $\xi=c_\eta\eta+\cdots$ with $c_\eta\neq0$. Since ξ is a weight vector for the weight $\Lambda_k-\alpha$ and $\Lambda_k-\alpha<\Lambda_k$, ξ is a linear combination of the basis elements $\zeta_i(B)$ with $\Lambda_i=\Lambda_k-\alpha$. Since we have numbered $\{\zeta_i(B)\}$ in such a way that $\Lambda_B=\Lambda_1(B)<\Lambda_2(B)\leqq\cdots$, ξ is a linear combination of $\zeta_i(B)$ with $i<k$. On the other hand $\xi=c_\eta\eta+\cdots=\pm c_\eta(\zeta_{p+t_1,j_1}\wedge\cdots\wedge\zeta_{p+t_r,j_r}\wedge\zeta_A)+\cdots$ and $\xi=\sum_{i<k}d_i\zeta_i(B)$. It follows then that when we express $\zeta_i(B)$ as a linear combination of the elements of the form (4.8), at least one of $\zeta_i(B)$ with $d_i\neq0$ must be of the form (4.18). Since $i<k$ and since k is the least index >1 such that $\zeta_k(B)$ has the form (4.18), we must have $i=1$. Then we have $\Lambda_k-\alpha=\Lambda_1=\Lambda_B$ and since the space of weight vectors for the lowest weight Λ_B is one dimensional, we obtain $\xi=d\cdot\zeta_B=d\cdot\zeta_1(B)$. On the other hand $\xi=\pm c_\eta(\zeta_{p+t_1,j_1}\wedge\cdots\wedge\zeta_{p+t_r,j_r}\wedge\zeta_A)+\cdots$ and hence $\zeta_1(B)$ is also of the form (4.18) and in this case the lemma is already proved.

We assume now $\eta_{p+t_1,i}\neq\zeta_{p+s_l,i_l}$ for all $l=1,2,\cdots,r$. We see then from (4.22) that $\zeta_{p+t_1,i}$ is a factor of ζ_A and hence

$$(4.23)\qquad t_1\leqq a_i\,.$$

We may assume $\zeta_{p+t_2,j_2}\wedge\cdots\wedge\zeta_{p+t_{r+1},j_{r+1}}=\pm\zeta_{p+s_1,i_1}\wedge\cdots\wedge\zeta_{p+s_r,i_r}$. Then we have

$$\zeta_{p+t_1,i}\wedge\zeta_{p+t_{r+2},j_{r+2}}\wedge\cdots\wedge\zeta_{p+t_u,j_u}=\pm\zeta_A$$

and hence

$$\eta=\pm\zeta_{p+t_1,j_1}\wedge(\zeta_{p+t_2,j_2}\wedge\cdots\wedge\zeta_{p+t_{r+1},j_{r+1}})\wedge(\zeta_A/\zeta_{p+t_1,i})$$

where $\zeta_A/\zeta_{p+t_1,i}$ means that the factor $\zeta_{p+t_1,i}$ is deleted from the product ζ_A. By (4.21) and (4.23) $a_{j_1}=a_{i+1}\geqq a_i\geqq t_1$ and hence ζ_{p+t_1,j_1} is a factor of $\zeta_A/\zeta_{p+t_1,i}$. Then we have $\zeta_{p+t_1,j_1}\wedge(\zeta_A/\zeta_{p+t_1,i})=0$ and we get $\eta=0$. This is a contradiction because $\eta\neq0$. Therefore the case $\zeta_{p+t_1,i}\neq\zeta_{p+s_l,i_l}(l=1,\cdots,r)$ can not occur.

The last case we have to consider is the case $\alpha=\lambda_{p+s}-\lambda_{p+s+1}$, $e_\alpha=e_{p+s,p+s+1}$. This case is treated quite analogously as in the case $\alpha=\lambda_i-\lambda_{i+1}$ and the proof of Lemma 4.3 is completed.

Suppose now that the integral (2.5)

$$d_B = \int_{U(m)\times N(A)} s_1^*(\theta_A \wedge \pi_F^*\lambda) \wedge \gamma^* \pi_G^* \xi_B$$

is non-zero. Then the integrand is certainly non-zero and as $\pi_G^*\xi_B = v_B^{-1}\tilde{\omega}_B$, v_B^{-1} being a non zero constant, $s_1^*(\theta_A \wedge \pi_F^*\lambda) \wedge \gamma^*\tilde{\omega}_B$ is non-zero and (4.17) holds. Then by Lemma 4.3, $B=\{b_1, \cdots, b_p\}$ verifies the conditions in Lemma 4.3. Summing up we have proved the following theroem which is stated in the introduction.

Theorem. *Let λ be a form of type $(f-r, f-r)$ on the flag manifold $F(A)$ which is invariant by the action of the unitary group $U(m)$, where $f=dim_C F(A)$ and r is a non-negative integer. Then $\pi_*(\rho^*\lambda)$ is a form of type $(pq-|A|-r, pq-|A|-r)$ on the Grassmann manifold $Gr(p, m)$ $(p\cdot q=dim_C Gr(p, m))$ which is also invariant by the action of $U(m)$. Then $\pi_*(\sigma^*\lambda)$ is a linear combination of the Poincaré dual $C(B)$ of the Schubert varieties (B), where $B=\{b_1, b_2, \cdots, b_p\}$ verifies the following conditions: Let $A=\{a_1, a_2, \cdots, a_p\}$. There exist s indices $j_1, j_2, \cdots, j_s$ $(1\leqq j_1 < j_2, \cdots, < j_s \leqq p; s\leqq r)$ and s positive integers $n_1, \cdots, n_s$ with the condition $n_1+n_2+\cdots+n_s=r$ such that $b_{j_c}=a_{j_c}+n_c$ for $c=1, \cdots, s$ and $b_j=a_j$ for $j\neq j_c$.*

Appendix. In this appendix we discuss the representation of the direct product $GL(p, \boldsymbol{C})\times GL(q, \boldsymbol{C})$ of complex general linear groups on the exterior algebra ΛM, where M is the complex vector space of all $p\times q$ complex matrices. The problem is to decompose ΛM into direct sum of irreducible invariant subspaces. This had been done in the last section of the paper of Kostant [3, *a*] as an application of his more general theory and he attributed the main result in this special case to Ehresmann. The purpose of this appendix is to formulate the main theorem of Ehresmann and Kostant in a form suitable for our purpose and to introduce a few notions in the representation theory which are needed in §3 and §4.

Let

$$G = GL_p \times GL_q\,,$$

where $GL_k = GL(k, \boldsymbol{C})$ for any positive integer k, and we identify G with the subgroup of GL_m, $m=p+q$ in the usual way.

The diagonal matrices in G form an abelian subgroup H of G which we call a Cartan subgroup of G. The Lie algebra $\mathfrak{g}$ of G consists of all $m\times m$ complex matrices X of the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \quad X_1 \in \mathfrak{gl}_p,\ X_2 \in \mathfrak{gl}_q \tag{1}$$

and the subalgebra $\mathfrak{h}$ corresponding to H consists of all diagonal matrices. $\mathfrak{h}$

is called a *Cartan subalgebra* of $\mathfrak{g}$. Let

$$K = U(p) \times U(q) .$$

Then K is a maximal compact subgroup of G and the Lie algebra $\mathfrak{k}$ of K is a real Lie subalgebra of $\mathfrak{g}$ consisting of matrices of the from (1) satisfying the condition ${}^t\bar{X}_i = -X_i$ $(i=1, 2)$. Every matrix $X \in \mathfrak{g}$ is written uniquely in the form $X = Y_1 + iY_2$ with Y_1, $Y_2 \in \mathfrak{k}$, and we can identify $\mathfrak{g}$ with the complexification of the real Lie algebra $\mathfrak{k}$.

Let L be a holomorphic representation of G in a complex vector space W. L is a holomorphic homorphism of G into $GL(W)$. The representation L of G defines a representation L' of the complex Lie algebra $\mathfrak{g}$ in W such that

$$(2) \qquad \exp tL'(X) = L(\exp tX)$$

for all $t \in \boldsymbol{R}$.

The restriction M of L to K is a representation of K in W and since K is a compact group, W decomposes into direct sum $W = W_1 + \cdots + W_k$ of simple K-modules and the representation M_j of K on W_j induced by M is irreducible. Each W_j is a $\mathfrak{g}$-module and also a G-modlue, i.e. $L'(X)W_j \subset W_j$ and $L(g)W_j \subset W_j$ for $X \in \mathfrak{g}$ and $g \in G$. In fact, let M_j' the representation of the Lie algebra $\mathfrak{k}$ defined by the rperesentation M_j of K. Then $M_j'(Y) = L'(Y)$ for all $Y \in \mathfrak{k}$, bacause M is the restriction of L to K. Every $X \in \mathfrak{g}$ is written uniquely as $X = Y_1 + iY_2$, Y_1, $Y_2 \in \mathfrak{k}$ and so $L'(X) = M_j'(Y_1) + iM_j'(Y_2)$. Then as $M_j'(Y)W_j \subset W_j$ for $Y \in \mathfrak{k}$, we have also $L'(X)W_j \subset W_j$ and W_j is a $\mathfrak{g}$-module. It follows then from (2) that $L(\exp tX)W_j \subset W_j$ for all $X \in \mathfrak{g}$. Then we have $L(g)W_j \subset W_j$ for all $g \in G$, because G is generated by 1-parameter subgroups. We show that W_j is a simple $\mathfrak{g}$-module. Let W_j' be a subspace of W_j such that $L(X)W_j' \subset W_j'$ for all $X \in \mathfrak{g}$. Then we have $L(g)W_j' \subset W_j'$ for all $g \in G$ and in particular $M(g)W_j' \subset W_j'$ for all $g \in K$. Since W_j is simple as K-module, we have either $W_j' = W_j$ or $W_j' = \{0\}$ and this shows that W_j is simple as $\mathfrak{g}$-module.

Conversely let $W = V_1 + \cdots + V_k$ be a decomposition of W into direct sum of simple $\mathfrak{g}$-modules. We can show in a similar way that each V_j is also a simple K-module. Thus a decomposition of W into simple K-modules and into simple $\mathfrak{g}$-modules is the the same thing and as W is always semi-simple (or completely reducible) as K-module, it is so as well as $\mathfrak{g}$-module.

A linear function Λ on the Cartan subalgebra $\mathfrak{h}$ is called a *weight* of the holomorphic representation L of G in W, if there exists $w \in W$, $w \neq 0$, such that

$$(3) \qquad L'(X)w = \Lambda(X)w$$

for all $X \in \mathfrak{h}$. A vector w satisfying (3) is called a *weight vector* for the weight Λ. The weight vectors for Λ form a subspace of W, the *eigen space* for Λ, and the dimension of the eigen space is called the *multiplicity* of Λ. The Cartan subalgebra $\mathfrak{h}$ is spanned by the matrices e_{kk} $(k=1, 2, \cdots, m)$ and we have

$$\Lambda(X) = n_1\lambda_1+\cdots+n_m\lambda_m ,$$

where $X=\lambda_1 e_{11}+\cdots+\lambda_m e_{mm}$ and $n_k=\Lambda(e_{kk})$. It follows from (2) and (3) that $L(\exp X)w=(\exp \Lambda(X))w$ for $X\in\mathfrak{h}$. Let $X=2\pi i e_{kk}$. Then $\exp \Lambda X=I$ and $\Lambda(X)=2\pi i n_k$ and hence $\exp 2\pi i n_k=1$. This proves that n_k is an integer for $k=1, \cdots, m$ and a weight Λ is an *integral linear form* of $\lambda_1, \cdots, \lambda_m$.

Let now $X\in\mathfrak{g}$ and $x\in G$. Then the matrix xXx^{-1} belongs also to $\mathfrak{g}$ and we define the adjoint representation Ad of G in the vector space $\mathfrak{g}$ by

$$Ad(x)X = xXx^{-1} .$$

The representation Ad' of $\mathfrak{g}$ is denoted by ad and we have

$$ad(Y)X = [Y, X] .$$

A weight (weight vector) of the adjoint representation is called a *root* (*root vector*). The Lie algebra $\mathfrak{g}$ is spanned by the matrices e_{ij} and $e_{p+s,p+t}$, where $1\leqq i, j\leqq p$ and $1\leqq s, t\leqq q$, $p+q=m$, and we have $ad(X)e_{ij}=(\lambda_i-\lambda_j)\cdot e_{ij}$ and $ad(X)e_{p+s,p+t}=(\lambda_{p+s}-\lambda_{p+t})e_{p+s,p+t}$, where $X=\sum_{k=1}^{m}\lambda_k e_{kk}\in\mathfrak{h}$. Hence $\alpha_{ij}=\lambda_i-\lambda_j$ and $\alpha_{p+s,p+t}=\lambda_{p+s}-\lambda_{p+t}$ $(1\leqq i, j\leqq p, 1\leqq s, t\leqq q)$ are roots of $\mathfrak{g}$ and e_{ij} and $e_{p+s,p+t}$ are the corresponding root vectors. Moreover these roots exhaust the roots of $\mathfrak{g}$. If α is a root of $\mathfrak{g}$, e_α denotes the matrix e_{ij} or $e_{p+s,p+t}$ according as $\alpha=\alpha_{ij}$ or $\alpha=\alpha_{p+s,p+t}$.

We mention here the following simple result as Lemma 1.

Lemma 1. *Let Λ be a weight of a representation L of G in W and w a weight vector for Λ. If α is a root and $L'(e_\alpha)w\neq 0$, then $\Lambda+\alpha$ is also a weight of L and $L'(e_\alpha)w$ is a weight vector for $\Lambda+\alpha$.*

In fact, let $X\in\mathfrak{h}$. Then $L'(X)L'(e_\alpha)=L'(e_\alpha)L'(X)+L'([X, e_\alpha])$ and $[X, e_\alpha] =ad(X)e_\alpha=\alpha(X)e_\alpha$. Hence $L'(X)L'(e_\alpha)=L'(e_\alpha)L'(X)+\alpha(X)L'(e_\alpha)$. Then $L'(X)(L'(e_\alpha)w)=L'(e_\alpha)(\Lambda(X)w)+\alpha(X)L'(e_\alpha)w=(\Lambda+\alpha)(X)L'(e_\alpha)w$ and this proves that $\Lambda+\alpha$ is a weight and $L'(e_\alpha)w$ is a weight vector.

We now introduce the lexicographic order on the $\boldsymbol{Z}$-module of integral linear forms of the variables $\lambda_1, \cdots, \lambda_m$. Let $\Lambda=\sum n_k\lambda_k$ and $\Lambda'=\sum n_k'\lambda_k$. Then $\Lambda>\Lambda'$ if there exists an index $k_0\geqq 1$ such that $n_k=n_k'$ for $k<k_0$ and $n_{k_0}>n_{k_0}'$.

Since weights of a representation L are integral forms we have an order relation among weights of L. A weight Λ (root α) is *positive,* if $\Lambda>0$ $(\alpha>0)$. The *lowest weight* (*highest weight*) of L is a weight Λ of L such that $\Lambda<\Lambda'(\Lambda>\Lambda')$ for any weight Λ' of L distinct from Λ.

We also have the notion of simple roots of $\mathfrak{g}$. A root α is said to simple, if α is not a sum of two positive roots.

We can see easily that $\{\alpha_{12}, \alpha_{22}, \cdots, \alpha_{p-1,p}, \alpha_{p+1,p+2}, \cdots, \alpha_{p+q-1,p+q}\}$ is a maximal set of simple roots of $\mathfrak{g}$.

We have the following theorem which we have used in §3 and §4.

Theorem 1. *Let L be an irreducible holomorphic representation of G in a complex vector space W. Then*

1) *The multiplicity of the lowest (or the highest weight) is one.*

2) *Let w $(\neq 0)$ be a weight vector for the lowest weight Λ. Then W is spanned by w and by the vectors of the form*

$$L'(e_{\alpha_1})\cdots L'(e_{\alpha_k})w\,,$$

where $\alpha_1, \cdots, \alpha_k$ are simple roots of $\mathfrak{g}$.

3) *The irreducible representation L is completely determined by the lowest weight Λ of L. This means that, if L_1 is another irreducible holomorphic representation of G with the same lowest weight Λ, then L and L_1 are equivalent.*

For the proof of this theorem, see [5, Chapter VII].

Let M be the complex vector space consisting of all $p\times q$ complex matrices. We identify M with the subspace of $\mathfrak{gl}_m$ $(m=p+q)$ consisting of all matrices Q of the form

$$Q=\begin{pmatrix}0 & D\\ 0 & 0\end{pmatrix},\ D\colon p\times q\text{-matrix}\,. \tag{4}$$

We define a holomorphic representation T of G in M by

$$T(g)Q=gQg^{-1}\,.$$

If

$$g=\begin{pmatrix}g_1 & 0\\ 0 & g_2\end{pmatrix}$$

and Q is of the form (4), then

$$T(g)Q=\begin{pmatrix}0 & g_1Dg_2^{-1}\\ 0 & 0\end{pmatrix}.$$

We have

$$T'(X)\cdot Q=[X, Q],\ X\in\boldsymbol{g}\,,$$

and if X is of the form (1), then

$$T'(X)Q=\begin{pmatrix}0 & X_1D-DX_2\\ 0 & 0\end{pmatrix}. \tag{5}$$

The vector space M is spanned by the matrices $e_{i,p+s}(1\leqq i\leqq p, 1\leqq s\leqq q)$ and

$$T'(X)e_{i,p+s}=(\lambda_i-\lambda_{p+s})e_{i,p+s} \tag{6}$$

for all $X=\sum_{i=1}^{p}\lambda_i e_{ii}+\sum_{s=1}^{q}\lambda_{p+s}e_{p+s,p+s}$ in $\mathfrak{h}$. Hence $\lambda_i-\lambda_{p+s}$ is a weight and $e_{i,p+s}$ is a weight vector.

We extend T to a representation of G in the exterior algebra ΛM by defining

$$T(g)(Q_1\wedge\cdots\wedge Q_r)=T(g)Q_1\wedge\cdots\wedge T(g)Q_r\,.$$

Then

$$T'(X)(Q_1\wedge\cdots\wedge Q_r)=\sum_{t=1}^{r}Q_1\wedge\cdots\wedge T'(X)Q_t\wedge\cdots\wedge Q_r\,. \tag{7}$$

The elements of the form

$$E=e_{i_1,p+s_1}\wedge\cdots\wedge e_{i_r,p+s_r} \tag{8}$$

with $(i_1, s_1)<\cdots<(i_r, s_r)$ form a basis of ΛM, where $<$ means the lexicographic order for the double indices (i, s) with $1\leqq i\leqq p$, $1\leqq s\leqq q$. It follows from (6) and (7) that E is a weight vector for the weight

$$\Lambda_E=\sum_{t=1}^{r}\lambda_{i_t}-\sum_{t=1}^{r}\lambda_{p+s_t}\,. \tag{9}$$

It can be proved easily that a weight Λ of T is always of the form Λ_E for some E and that the multiplicity of Λ is equal to the number of the basis elements E such that $\Lambda=\Lambda_E$. Clearly $\overset{r}{\Lambda}M$ is a G-module and $\Lambda M=\Sigma\overset{r}{\Lambda}M$ is a direct sum of G-modules.

Let $A=\{a_1, a_2, \cdots, a_p\}$ be a p-tuple of integers such that $0\leqq a_1\leqq a_2\leqq\cdots\leqq a_p\leqq q$ (see §1). We define E_A by

$$E_A=\bigwedge_{s\leqq a_i} e_{i,p+s}\,, \tag{10}$$

where the exterior product extends over all $e_{i,p+s}$ such that $s\leqq a_i$, $i=1, 2, \cdots, p$ and the order of the product is the lexicographic order as in (8). Then E_A is one of the basis element (8) and as there are $|A|(=\Sigma a_i)$ elements in the product (10). E_A belongs to $\overset{|A|}{\Lambda}M$. We dentoe Λ_{E_A} by Λ_A. Then

$$\begin{aligned}\Lambda_A&=\sum_i\sum_{s,s\leqq a_i}(\lambda_i-\lambda_{p+s})\\&=a_1\lambda_1+\cdots+a_p\lambda_p-b_1\lambda_{p+1}-\cdots-b_q\lambda_{p+q}\,,\end{aligned}$$

where b_s is the number of the a_i satisfying $a_i\geqq s$. Hence we have

$$0\leqq a_1\leqq\cdots\leqq a_p\leqq q,\qquad 0\leqq b_q\leqq b_{q-1}\leqq\cdots\leqq b_1\leqq p\,.$$

It follows then that, if $A \neq A'$, then

$$\Lambda_A \neq \Lambda_{A'} .$$

We formulate the main theorem of Ehresmann and Kostant (see [3, *a*, §8]) in the following form.

Theorem 2.

1) *The multiplicity of the weight Λ_A in ΛM is one.*

2) *The subspace M_A of ΛM spanned by E_A and by elements of the form $T'(e_{\alpha_1})\cdots T'(e_{\alpha_k})E_A$, where $\alpha_1, \cdots, \alpha_k$ are simple roots of $\mathfrak{g}$, is a simple $\mathfrak{g}$-module and Λ_A is the lowest weight of the representation of G in M_A induced from T.*

3) *Two simple $\mathfrak{g}$-modules M_A and $M_{A'}$ are isomorphic only if $A=A'$.*

4) *ΛM is the direct sum of simple $\mathfrak{g}$-modules M_A; in particular $\overset{r}{\Lambda} M = \sum_{A,|A|=r} M_A$, where the summation extends over all A satisfying the condition $|A|=r$.*

REMARK. If N is a simple $\mathfrak{g}$-submodule of ΛM, then $N=M_A$ for a unique A. For N is isomorphic to one of M_A by 4) and by 3), A is unique. Then the lowest weight of N is Λ_A. Since the multiplicity of Λ_A is one by 1), E_A is then contained in N together with $T'(e_{\alpha_1})\cdots T'(e_{\alpha_k})E_A$. Then M_A is contained in N by 2) and as N is simple, we get $N=M_A$.

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